MATH464, Sec. 0101: Transform Methods
Department of Mathematics, UMCP **Fall 2022**
Homework 10 **Posted**: Thursday, 12/01/22
Due: Thursday, 12/08/22 **IN CLASS**

Answer all questions. Show all your steps and justify your answers. **Total number of pts: 70**
Note: The use of Matlab, or any other software, is strictly NOT permitted.

63. [10pts] From the theory developed in class, determine the solution $u(x,t)$ for the following diffusion initial-value problem on the real line:

$$\frac{\partial u}{\partial t} = 4\,\frac{\partial^2 u}{\partial x^2}\ ,\ x \in \mathbb{R}\ ,\ t \geq 0; \qquad u(x,0) = e^{-x^2/2} \qquad (x \in \mathbb{R})\ .$$

64. [10pts] Solve the following periodic heat equation problem with $p = 1$ ($u(x+1,t) = u(x,t)$):

$$\frac{\partial u}{\partial t} = 9\,\frac{\partial^2 u}{\partial x^2}\ ,\ 0 \leq x \leq 1\ ,\ t \geq 0\ ; \qquad u(x,0) = \sin(6\pi x)\ .$$

65. [10pts] Consider the wave equation with Cauchy initial data in the real line, viz.,

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}\ ,\ x \in \mathbb{R},\ t \geq 0\ (c > 0)\ ; \qquad u(x,0) = f(x)\ ,\ \frac{\partial u}{\partial t}(x,0) = g(x)\ , \quad x \in \mathbb{R}\ .$$

By using the Fourier transform of u with respect to x, derive the D' Alembert solution:

$$u(x,t) = \frac{1}{2}\{f(x-ct) + f(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy\ .$$

66. [10pts] Find all generalized functions f that satisfy

$$\int_{-\infty}^{\infty} e^{-|x-u|}\,f(u)\,du = x\ .$$

67. [10pts] By use of Fourier series, find all functions $u(x,t)$ that are 1-periodic in x and satisfy

$$2\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \quad 0 \leq x \leq 1\ ,\ t \geq 0\ ; \quad u(x,0) = \cos(4x) \qquad (0 \leq x \leq 1)\ .$$

68. [10pts] Consider the window function $g = \Pi$, which is the Box function. Compute explicitly the windowed Fourier transform $V_g f$ of the following functions f:

(a)[5pts] $f(x) = e^{2\pi i x a} - e^{2\pi i x b}$, for some real constants a and b.

(b)[5pts] $f(x) = \Pi(ax + b)$, for some $a > 0$ and real b.

69. [10pts] Consider the window function $g(x) = e^{-\pi x^2/2}$. Let $F(t,\omega) = e^{-\pi(t^2+\omega^2)/2}$.

(a)[5pts] Suppose that F is the windowed Fourier transform of function f. Apply the reconstruction formula (inversion formula of windowed Fourier transform) to F to determine f.

(b)[5pts] Compute the windowed Fourier transform of the f found in part (a), with respect to the window g. Do you get F? Why?