MATH464, Sec. 0101: Transform Methods
Department of Mathematics, UMCP **Fall 2022**
Homework 7 **Posted**: Friday, 10/21/22
Due: Tuesday, 11/01/22 **IN CLASS**

Answer all questions. Explain all your steps and justify your answers. Each problem is worth 10 points, except Problem 50 which is worth 40 points. **Total number of points: 80**

Note: The use of Matlab, or any other software, is strictly NOT permitted.

46. [10pts] Let $f : \mathbb{R} \to \mathbb{C}$ be a square-integrable function whose Fourier transform is supported in the frequency band $[2 \text{ kHz}, 12 \text{ kHz}]$. This means that the Fourier transform $F(s)$ of $f(x)$ vanishes for any frequency $s = \omega$ outside the interval $[2000, 12000]$. Assume that we know the samples $\{f(nT)\ ;\ n \in \mathbb{Z}\}$ for $T = 100\mu s = 10^{-4}$sec. Show how to synthesize $f(x)$ from the above set of samples. **Note:** 1 kHz = 1,000 Hz = 1,000 sec^{-1}.

47. [10pts] What is the maximum sampling period of a 50 kHz bandlimited signal, so that we can perfectly compute the signal at any time t by using Shannon's formula? Explain in detail.

48. [10pts] A 10 kHz bandlimited signal is sampled at its Nyquist rate (that is, at the maximal rate where Shannon's formula holds true). The only nonzero samples are:

$$x(t) = \begin{cases} -2\ , & \text{if } t = -1.2\,\text{ms} \\ 1\ , & \text{if } t = 0.2\,\text{ms}\ . \end{cases}$$

Compute $x(t)$ for $t = 0$ms and $t = 1\mu$s. Note: 1 ms = 10^{-3}s and 1μa=10^{-6}s.

49. [10pts] An unknown signal $f : \mathbb{R} \to \mathbb{R}$ is sampled at the sampling frequency 1kHz. We do not know whether f is bandlimited; however, we know that its Fourier transform F has an upper bound according to

$$|F(s)| \leq e^{-|s|}\ , \text{ for all } s \in \mathbb{R}\ .$$

Estimate what is the maximal reconstruction error when using Shannon's formula with all samples $\{f(n/1000)\ ;\ n \in \mathbb{Z}\}$.

50. [40pts] Suppose that $f : \mathbb{R} \to \mathbb{C}$ is an Ω-bandlimited function, with $f \in B^2_\Omega \cap C^k(\mathbb{R})$ for some k ($k = 0, 1, \ldots$). Let $2\Omega = T^{-1}$. Show that

$$\left| f^{(k)}(x) - \sum_{n=M}^{N} \frac{f(nT)}{T^k} \operatorname{sinc}^{(k)}\left(\frac{x - nT}{T}\right) \right|^2 \leq \frac{1}{2k+1} \left(\frac{\pi}{T}\right)^{2k} \left\{ \sum_{n<M} |f(nT)|^2 + \sum_{n>N} |f(nT)|^2 \right\}\ ,$$

for any integers M, N with $M < N$. **Hint:** You may use the Cauchy-Schwarz inequality, viz.,

$$\left| \int_a^b \alpha(s)\beta(s)\, ds \right|^2 \leq \int_a^b |\alpha(s)|^2\, ds \cdot \int_a^b |\beta(s)|^2\, ds\ .$$

Note: Aspects of this problem will be discussed in class, before the due date.