Neutral magic-angle bilayer graphene: Condon instability and chiral resonances

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(Dated: September 5, 2022)

We discuss the full optical response of twisted bilayer graphene at the neutrality point close to the magic angle within the continuum model. (i) First, we define the full optical response consistent with the underlying $D_3$ symmetry, yielding the total, magnetic, and chiral response that transform according to the irreducible representations $A_1$, $A_2$, and $E$, respectively. Then, we numerically calculate the dissipative and reactive response for twist angles around the magic angle $\theta_m$ and comment on the possibility of a Condon instability. (ii) Second, we numerically calculate the full optical response almost at $\theta_m$. The total response is characterized by three universal plateaus which can be obtained from an analytical calculation. The magnetic and the chiral response, however, is given by corresponding non-universal plateaus depending on the twist angle $\theta$ via the dimensionless parameter $\alpha \sim \theta_m - \theta$. We observe an algebraic behavior for $\omega \rightarrow 0$ with $\text{Re} \sigma_\omega \sim \alpha^{-\nu}$, $\gamma \geq 0$, where $\nu$ stands for magnetic and chiral responses. Via the Kramers-Kronig relation, we conclude that the ground-state of the continuum model is unstable with respect to current fluctuations, a so-called Condon instability. (iii) Following the discussion on the large magnetic response, we calculate the plasmonic excitations at the neutrality point inside the optical gap of relaxed twisted bilayer graphene. We find that acoustic plasmons extend over almost the whole optical gap and carry the largest oscillator strength. Furthermore, they display chiral character and their energy density can be strongly enhanced around a certain frequency giving rise to a chiral resonance. (iv) Finally, we discuss symmetry relations for the response functions as function of the chemical potential and highlight the consequences of the approximate particle-hole symmetry of the continuum model for twisted bilayer graphene. We then discuss a detailed balance relation where the chiral response at charge neutrality can be understood in terms of electron (hole) transitions for which the initial (final) states are energetically closer to charge neutrality than the final (initial) states.

I. INTRODUCTION

Twisted bilayer graphene\textsuperscript{1–9} has attracted much attention due to a plethora of new correlated phases such as correlated insulators,\textsuperscript{10} unconventional superconductivity\textsuperscript{11,12} or anomalous quantum Hall ferromagnetism.\textsuperscript{13,14} Most features are related to the emergence of a flat band which is related to the vanishing of the Fermi velocity at the so-called magic angle $\theta_m$. Also, a prominent counterflow can be found where the current has opposite direction with respect to the two layers and which becomes balanced at $\theta_m$.\textsuperscript{7} A third feature is the pronounced circular dichroism\textsuperscript{15,16} at frequencies close to the van Hove singularities.

Fermi velocity, counterflow and circular dichroism are related to the total (electric), magnetic and chiral response. These have first been introduced in Refs. 17 and 18 and discussed in detail for large twist angles. Here, we shall calculate these quantities for twist angles around the magic angle.

We will also comment on flat band plasmonics in twisted bilayer graphene\textsuperscript{19} that is related to localized collective modes,\textsuperscript{20} and which is an area of active interest.\textsuperscript{21–25} Finally, we give a general discussion and new insights on the chiral optical response and how it is related to symmetries.

The flat bands\textsuperscript{7,26–30} and the optical response\textsuperscript{22,31–34} of twisted bilayer graphene have been investigated in numerous articles so far. However, a detailed discussion on the scaling behavior of the response function for small frequencies, as $\omega \rightarrow 0$, and almost at the magic angle is missing until now. Another topic is related to the Condon instability\textsuperscript{35} that has recently been discussed in related systems.\textsuperscript{36–39} We will argue that the Condon instability arises in the continuum model\textsuperscript{1,7} of twisted bilayer graphene at the neutrality point, too, precisely at the magic angle.

Apart from that, also the large counterflow or magnetic response has been discussed in several papers.\textsuperscript{7,17,30} Nevertheless, the importance related to acoustic plasmonic modes has not received sufficient attention, so far, and we believe that our results will be relevant for flat-band plasmonics in twisted bilayer graphene, especially at the chiral resonance where ordinary optical plasmons cease to exist.

Finally, the chirality in graphene might be used to design novel cavities that lead to strong chiral light-
matter interaction. In order to understand the underlying physics we point out some new aspects related to particle-hole symmetry. This leads us to distinguish between electron and hole transitions where the initial states are energetically closer and further away from the neutrality point than the final states, respectively.

The paper is organized as follows. In Sec. II, we define the continuum model for twisted bilayer graphene. In Sec. III, we introduce the minimal model for the full linear response that defines the total, magnetic, and chiral responses. In Sec. IV, we present our numerical recipe of how to calculate the plasmonic properties around the magic angle. Finally, in Sec. VII, we outline the symmetry conditions for chiral response. We close with a summary and outlook. Appendices on the numerical recipe of how to calculate the dissipative response in the clean limit as well as on an analytical calculation of the optical conductivity almost at the magic angle are also provided.

II. HAMILTONIAN

The local Hamiltonian of a twisted bilayer graphene can be approximated by

\[
\mathcal{H} = \begin{pmatrix} H_0^{\theta/2} & V(r) \\ V^\dagger(r) & H_0^{\theta/2} \end{pmatrix},
\]

where \( H_0 = -i\hbar v_F \tau^3 \cdot \partial_r \) denotes the Hamiltonian of the separate layers with \( \tau^3_g, \tau^3_i = e^{i\gamma} \tau^3 / (2\tau^3) \) being the Pauli matrices. The interlayer coupling \( V(r) \) also denotes a 2 \times 2 matrix and defines the coupling between the layers. It explicitly depends on the stacking order, but a common approximation is that all components are defined by only one common function \( u(r) \).

Expanding \( u(r) \) into the first three Fourier-components of the moiré lattice and representing the Hamilton operator by plane waves, one arrives at the non-interacting Hamiltonian used for calculating the total fields

\[
\mathcal{H} = \hbar v_F \sum_{k,\alpha,\beta} c_k^{\dagger} \tau^{\alpha\beta}_{\alpha\beta} \cdot k^{\dagger} c_k^{\dagger} + \hbar v_F \sum_{k,\alpha,\beta} c_k^{\dagger} \tau^{\alpha\beta}_{\alpha\beta} \cdot k^{\dagger} c_k^{\dagger} + \frac{t_\perp}{3} \sum_{k,\alpha,\beta,\G} c_{k+G,\alpha,\1} T_{\alpha\beta}(\G) c_k^{\dagger} + H.c.,
\]

where the separation between twisted cones is \( \Delta K = 2|K| \sin(\theta/2) (0, 1) \) with \( K = \frac{4\pi}{\pi} (1, 0) \). Interlayer hopping is restricted to wavevectors \( \G = 0, G_1, G_2 \) with \( G_1 = |\Delta K| \left( \frac{\sqrt{3}}{2}, -\frac{3}{2} \right), G_2 = |\Delta K| \left( \frac{\sqrt{3}}{2}, \frac{3}{2} \right) \),

\[
T(0) = \begin{pmatrix} 1 & 1 \\ \kappa & 1 \end{pmatrix}, \quad T(G_1) = T^*(G_2) = \begin{pmatrix} \kappa e^{i2\pi/3} & 1 \\ e^{-i2\pi/3} & \kappa e^{i2\pi/3} \end{pmatrix}.
\]

Calculations are performed with \( t = 2.78 \text{ eV} \) and \( t_\perp = 0.33 \text{ eV} \), being \( \hbar v_F = \frac{\sqrt{3}}{2} \text{a}_g \) the Fermi velocity with graphene lattice constant \( \text{a}_g = 2.46 \text{ Å} \); the interlayer distance has been taken as \( a = 3.5 \text{ Å} \). In the first part of the work, we discuss the symmetric model with \( \kappa = 1 \) and in the second part of the work, the asymmetric model introduced in Ref. 29 with \( \kappa = 0.8 \) is used that accounts for out-of-plane relaxation, see also Ref. 43.

Let us finally note that besides the parameter \( \kappa \), the above model is only characterized by one dimensionless parameter \( \alpha_i = \frac{\sqrt{\text{a}_i^3}}{2\pi} \) which combines \( t_\perp \) and the twist angle \( \theta_i \) parametrized by \( i \) via \( \text{A}_i = 3\text{a}_i^2 + 3i + 1 \) with \( \cos(\theta_i) = 1 - \frac{1}{4\pi} \). This can readily be seen from the Hamiltonian of Eq. (1) by introducing the dimensionless coordinates \( \mathbf{r} = |\Delta K| \mathbf{r} \) such that the new interlayer coupling between the layers is independent of the twist angle.\(^1\) In principle, \( i \in \mathbb{N} \) denotes a commensurate twist angle, but the expressions can be generalised to arbitrary real numbers \( i \in \mathbb{R} \).

III. LINEAR RESPONSE

To describe chiral effects without breaking time-reversal or rotational \((C_3)\) symmetry, we have to treat an effectively three-dimensional system. The minimal model thus consists of treating each layer of the twisted bilayer separately. With the Kubo formula \( j^e_{\alpha\beta} = -\chi^e_{\alpha\beta,\gamma\delta} A^\gamma_{\delta} \), where \( A^\alpha_{\gamma} \) denotes the gauge field and summation over repeated indices is implied, the \( 4 \times 4 \) local \((q = 0)\) conductivity tensor then is

\[
s_{\alpha\beta}(\omega) = \frac{\chi_{\alpha\beta}^e(\omega)}{\omega + i0^+},
\]

with axis indices \( \alpha, \beta = x, y \) and plane indices \( \ell, \ell' = 1, 2 \).

The retarded current-current response is given by
Here, \( g_s = g_v = 2 \) are the spin and valley degeneracies. The states \(|m, k\rangle\) are eigenstates of \( \mathcal{H} \) in subband \( m \) and of momentum \( k \) in the first Brillouin zone of the superstructure. Their eigenenergies are \( \epsilon_{n,k} \) and \( n_F \) is the Fermi function. For single layer graphene with the Hamiltonian of Eq. (2), the general current operator is independent of \( k \).

The full current response due to an applied in-plane electric field that satisfies rotational (or \( C_3 \)) and time-reversal symmetry reads

\[
\begin{pmatrix}
  j_{x1} \\
  j_{y1} \\
  j_{x2} \\
  j_{y2}
\end{pmatrix} =
\begin{pmatrix}
  \sigma_0 & 0 & \sigma_1 & \sigma_{xy} \\
  0 & \sigma_0 & -\sigma_{xy} & \sigma_1 \\
  \sigma_1 & -\sigma_{xy} & \sigma_0 & 0 \\
  \sigma_{xy} & \sigma_1 & 0 & \sigma_0
\end{pmatrix}
\begin{pmatrix}
  E_{x1} \\
  E_{y1} \\
  E_{x2} \\
  E_{y2}
\end{pmatrix}.
\]

The conductivities \( \sigma_{\mu} = i \chi_{\mu}(\omega) \) with \( \mu = 0, 1, xy \) are defined via the following current-current response functions:

\[
\begin{align*}
\chi_0 &= \chi_{x1,j_1} = \chi_{j_1,y2} \\
\chi_1 &= \chi_{x1,j_2} = \chi_{j_2,y1} = \chi_{j_2,y2} = \chi_{j_2,j_2} \\
\chi_{xy} &= \chi_{x1,j_y} = -\chi_{j_y,y1} = -\chi_{j_y,j_1} = \chi_{j_y,j_2} \\
\end{align*}
\]

Note that the electric field may be different at the two layers and that the above symmetries allow for different in-plane conductivities in layer 1 and 2 which may arise due to a perpendicular gate voltage. Also the influence of a perpendicular magnetic field can be included.

In the following we will discuss the response functions that transform with respect to the irreducible representations of the underlying lattice symmetry group \( D_3 \) which consists of two one-dimensional and one two-dimensional representation. The responses of the total (electronic) current \( j_{\text{tot}} = j^1 + j^2 \) and of the magnetic current \( j_{\text{mag}} = j^1 - j^2 \) transform as the one-dimensional representations \( A_1 \) and \( A_2 \), respectively. The chiral response involves the two current densities \( j^1 \) and \( j^2 \) which transform as the two-dimensional representation \( \bar{E} \). This defines the total, magnetic and chiral response, respectively, as

\[
\begin{align*}
\sigma_{\text{tot}} &= 2(\sigma_0 + \sigma_1), \\
\sigma_{\text{mag}} &= 2(\sigma_0 - \sigma_1), \\
\sigma_{\text{chi}} &= \sigma_{xy}.
\end{align*}
\]

This also defines the current-current response \( \chi_\nu = -i(\omega + i0^+)\sigma_\nu \) and the Drude weight \( D_\nu = \lim_{\omega \to 0} \omega \chi_\nu(\omega) \) with \( \nu = \text{tot, mag, chi} \). The subindices \( xy \) and \( chi \) can be used interchangeably.

The above definitions also allow to deduce the exact symmetry relations when the twist angle is reversed, i.e., when the opposite enantiomer is considered: \( \sigma_0(\theta) = \sigma_0(-\theta), \sigma_1(\theta) = \sigma_1(-\theta), \sigma_{xy}(\theta) = -\sigma_{xy}(-\theta) \).

It thus suffices to consider the response for one twist-direction.

### IV. OPTICAL RESPONSE AROUND THE MAGIC ANGLE

The current response consists of a dissipative (imaginary) and reactive (real) response. Numerically, the dissipative part is almost equivalent to the evaluation of a generalized density of states and we have

\[
\Im \chi_{\alpha\beta}^{\ell,\ell'}(\omega) = \pi \frac{g_s g_v}{A} \sum_{k} \sum_{m,n} O_{m,n,k;\alpha,\beta}^{\ell,\ell'} \left[ n_F(\epsilon_{m,k}) - n_F(\epsilon_{n,k}) \right] \delta(\hbar \omega - \epsilon_{n,k} + \epsilon_{m,k}),
\]

where we introduced the transition matrix element

\[
O_{m,n,k;\alpha,\beta}^{\ell,\ell'} = \langle m, k | j_{\alpha}^{\ell'} | n, k \rangle \langle n, k | j_{\beta}^{\ell} | m, k \rangle.
\]

Note that the matrix elements can always be considered as real since the final spectral density must be real when time-reversal symmetry is not broken (which is the case here).

Since the current response function is an analytic function in the upper \( \omega \)-complex plane, see Eq. (5), the real part is obtained from the Cauchy or Kramers-Kronig relation. Performing the principle value integral, this gives

\[
\Re \chi_{\alpha\beta}^{\ell,\ell'}(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\Im \chi_{\alpha\beta}^{\ell,\ell'}(\omega')}{\omega' - \omega},
\]

which can be written as an integral only over positive frequencies using \( \Im \chi_{\alpha\beta}^{\ell,\ell'}(\omega) = -\Im \chi_{\alpha\beta}^{\ell,\ell'}(-\omega) \). Since the integral extends over all frequencies and the continuum
model does not contain an intrinsic cut-off, we need to invoke the high-frequency behavior of the current response of decoupled graphene layers, as discussed in Ref. 31.

Also the complex conductivity can be obtained by first considering the dissipative (real) part and then the reactive (imaginary) part. The only difference is the Drude term that needs to be added to the dissipative part as follows:

\[
\Re\sigma_{\alpha\beta}(\omega) = \pi D_{\alpha\beta}^e(\delta(\omega)) + \frac{\pi g_ \sigma g_e}{\omega} \sum_k \sum_{n,m} g_{\mu}^\alpha \left[ n_F(\epsilon_{m,k}) - n_F(\epsilon_{n,k}) \right] \delta(\omega - \epsilon_{n,k} - \epsilon_{m,k}),
\]

where \( D_{\alpha\beta}^e = \lim_{\omega \to 0} \chi_{\alpha\beta}(\omega) \) denotes the Drude weight matrix. By this, also the total, magnetic, and chiral Drude weight can be defined according to Eqs. (10-12). In the Kramers-Kronig relation, though, only the interband (regular) term enters due to \( \Im\chi_{\alpha\beta}(\omega = 0) = 0 \).

### A. Dissipative response

In Fig. 1, the dissipative response of the electric, magnetic and chiral currents at the neutrality point is shown in terms of the real part of the conductivity for angles around the magic angle \( \theta \approx 1.03^\circ \). It is numerically obtained from Eq. (19) following the recipe outlined in Appendix A, see also Ref. 45.

As can be seen from the left panel of Fig. 1, the total optical response for \( \omega \to 0 \) is characterized for all twist angles by the universal conductivity of two uncoupled graphene layers, \( 2\sigma_G \), with \( \sigma_G = \frac{g_\sigma g_e e^2}{\hbar \pi} \). Additional universal plateaus can be seen that will be discussed in the next section.

In the center panel of Fig. 1, the dissipative magnetic conductivity is shown. As in the case of the total response, there are plateaus for \( \omega \to 0 \) that strongly increase around the magic angle, reaching values larger than \( 10^6 \sigma_G \). This might also be the origin of the large orbital \( g \)-factor seen experimentally in twisted bilayer graphene.46-48

In the right panel of Fig. 1, the dissipative chiral conductivity is shown. Again, there are plateaus marked by the Dirac regime whose values change sign at \( \theta \approx 1.08^\circ \). Interestingly, this is the angle where the spectrum displays an approximate \( C_6 \)-symmetry at each valley which renders the chiral Drude weight zero even for relatively large finite chemical potential \( |\mu| \lesssim 75 \text{meV} \) as discussed in Ref. 49.

### B. Reactive response

In Fig. 2, the reactive response of the total, magnetic and chiral current at the neutrality point is shown for angles around the magic angle. It is obtained from Eq. (16) via the Kramers-Kronig relation of Eq. (18). For this, the dissipative part needs to be determined up to a frequency \( \omega_L \) for which \( \Re\sigma_{\text{tot}}(\omega \geq \omega_L) \approx 2\sigma_G \), \( \Re\sigma_{\text{mag}}(\omega \geq \omega_L) \approx 2\sigma_G \), \( \Re\sigma_{\text{chi}}(\omega \geq \omega_L) \approx 0 \).\textsuperscript{31} These high-frequency limits represent the response of two uncoupled layers and are also a consequence of the optical sum-rule.\textsuperscript{50}

In the left panel of Fig. 2, the real part of the total current response is shown. It must be zero for \( \omega \to 0 \) as there is no excess charge in the system,\textsuperscript{49} and we can adjust small numerical errors.\textsuperscript{51} These shifts are also introduced to \( \chi_{\text{mag}} \) even though this does hardly have an effect as the absolute values are much higher.

In the center panel of Fig. 2, the real part of the magnetic current response is shown. We note that there is a non-monotonic behavior with respect to the twist angle, i.e., even though the dissipative magnetic response is peaked around the magic angle \( \theta_m \approx 1.03^\circ \), see center panel of Fig. 1, the reactive response is not peaked at magic angle, but reaches a maximum around \( \theta \approx 1.1^\circ \). This is due to the fact that for these angles, the magnetic response reaches very high values at finite frequencies with \( 0.1 \leq \omega \leq 1 \text{meV} \) that yield the large response due to the integration of Eq. (18). In Sec. V, though, we will argue that there is an extremely narrow window almost at the magic angle for which the magnetic current response becomes maximal and even diverges.

In the right panel of Fig. 2, the real part of the chiral current response is shown. It must be zero for \( \omega \to 0 \) as there is no excess charge in the system,\textsuperscript{49} and we can adjust small numerical errors.\textsuperscript{52} For \( \theta = 1.0^\circ \), the maximal values can be as large as \( \chi_{\text{chi}} = 0.15 \sqrt{e^2} \) at \( \hbar \omega \approx 1.9 \text{µeV} \).

### C. Discussion on the Condon instability

There has been considerable interest in finding systems with a symmetry-broken ground-state due to photon- condensation, the so-called Condon instability.\textsuperscript{36-38} In bilayer systems, this instability can also be discussed by calculating the magnetic response \( D_{\text{mag}} \). Within the random-phase approximation, the response must reach a critical value \( D_{\text{mag}}^C \) with

\[
\frac{\mu_0}{\sqrt{2\pi}} D_{\text{mag}}^C = -1,
\]

where \( \mu_0 \) denotes the magnetic permeability.\textsuperscript{39}
For AA-stacked graphene, this limit is reached due to the logarithmic divergence of the magnetic susceptibility.\(^{39}\) However, the response of twisted bilayer graphene is generally too weak to reach the instability, i.e., including damping, one obtains \(D_{\text{mag}} = -6.6 t_e^2 \frac{e^2}{h} \).\(^{17,18}\) Our refined calculations without damping now yield a significantly lower bound for \(\theta = 1.1^\circ\) with \(D_{\text{mag}} = -36 t_e^2 \frac{e^2}{h} \). In the above units, this translates to \(D_{\text{mag}} \approx 0.008(\mu_0 a)^{-1}\) and we have \(D_{\text{mag}}/D_{\text{mag}}^C \approx 0.002\). This is still far away from a possible Condon transition. However, in Sec. V, we will discuss the possibility of finding a Condon instability almost at the magic angle in more detail.

We can compare our results also with previously reported values for the magnetic susceptibility.\(^{38}\) The static magnetic susceptibility \(\chi_{\text{mag}}^0\) is directly related to the magnetic Drude weight at the neutrality point and given by \(\chi_{\text{mag}}^0 = \frac{e^2}{4} D_{\text{mag}}\).\(^{18}\) With \(D_{\text{mag}} = -6.6 t_e^2 \frac{e^2}{h}\), this yields \(\chi_{\text{mag}}^0 = 0.02 \frac{\mu_B}{\text{nm}^2}\) with \(\mu_B\) the Bohr magneton. This value, obtained for \(\kappa = 1\), is slightly larger than the one reported in Ref. 38 for the continuum model with \(\kappa = 0.2\).

With \(D_{\text{mag}} \approx -36 t_e^2 \frac{e^2}{h}\), see central panel of Fig. 2, we obtain for the static magnetic susceptibility an even larger value of \(\chi_{\text{mag}}^0 = 0.12 \frac{\mu_B}{\text{nm}^2}\). This amounts to 18\(\mu_B\) per moiré cell only due to the orbital motion of counter-propagating electrons. This purely quantum mechanical effect is remarkable as no charge excitations are involved.

\section{V. Optical Response Almost at the Magic Angle}

In this section, we discuss the optical response almost at the magic angle, i.e., we will highly zoom into this region. As we will see, for any angle one can find an energy regime which is still characterized by the Dirac cone, i.e., we will never be exactly at the magic angle. Furthermore, other plateaus develop which can be anticipated from the band structure which shall be discussed before we describe the scaling relations.

\subsection{A. Bands around the magic angle}

In Fig. 3, the band structure for the symmetric model is shown for different twist angles around the magic angle \(\theta_m \approx 1.032^\circ\) where the linear dispersion (Fermi velocity)
at the \( K \)-point vanishes\(^{27,28,53} \) Notice that this does not coincide with the smallest band-width condition which would yield a magic twist-angle of \( \theta_m^* \approx 1.11^\circ \). This can be appreciated on the left panel of Fig. 3 where a new regime starts with an accidental crossing on the \( \Gamma M \)-direction.

In the center and right panel of Fig. 3, we see the evolution towards the magic angle from above and below, respectively. Most notably, there is an avoided crossing moving outward and eventually inward again to form the second magic angle. The evolution in \( \theta \) around the second magic angle at \( \theta \approx 0.49^\circ \), however, is qualitatively different.

**B. Scaling almost at magic angle**

The universal conductivity of graphene for small frequencies, \( \sigma_G = \frac{e^2}{\pi^2} \), is due to the perfect cancellation between the transition-matrix element and the Fermi velocity.\(^{54-56} \) This is also the case for the total conductivity of twisted bilayer graphene for transitions around the Dirac cones. Considering different quantities such as the magnetic absorption related to \( \Re \sigma_{\text{mag}} \) or the chiral absorption related to \( \Re \sigma_{\text{chi}} \) will not show this cancellation and we expect the following relations for \( \omega \to 0 \):\(^7 \)

\[
\sigma_{\text{mag}}^0 = \sigma_G \left( \frac{v_{\text{mag}}}{v_F} \right)^2, \quad \sigma_{\text{chi}}^0 = \sigma_G \left( \frac{v_{\text{chi}}}{v_F} \right)^2
\]

Above, we defined suitable velocities that characterize the magnetic and chiral excitations.

As the Fermi velocity vanishes at the magic angle, Eq. (21) suggest that the magnetic and chiral absorption diverge. Our numerical calculations confirm precisely this, as can be seen in Fig. 4, where we show the three response functions for twist angles below the magic angle \( \theta_m \approx 1.032^\circ \). Whereas the absorption shows universal behavior, the magnetic as well as the chiral absorption diverge.

In order to discuss the scaling behavior of these quantities, we introduce the effective parameter

\[
\alpha = \frac{\alpha_{\theta} - \alpha_m}{\alpha_m} \approx \frac{\theta_m - \theta_i}{\theta_i}, \quad (22)
\]

with \( \alpha_{\theta} = \frac{\sqrt{A_i/3}}{2\pi} \) and \( \alpha_m \approx 0.605 \) (for \( i = 31.54 \)).

First, we investigate the scaling of the Dirac regime \( \epsilon_D \) that is defined by the abrupt increase of the absorption from \( 2\sigma_G \) to \( 12\sigma_G \). As is shown in the inset of the left panel of Fig. 4, there is a linear behavior of \( \ln \epsilon_D \) as function of \( \ln \alpha \), leading to \( \epsilon_D \approx 0.88\alpha^{0.5} \mu\text{eV} \) with \( \gamma_c \approx 1.35 \pm 0.04 \).

Along the same lines, we obtain the scaling behavior of the magnetic and chiral absorption plateau for \( \hbar\omega < \epsilon_D \) as

\[
\sigma_{\text{mag}}^0 = 570\sigma_G \alpha^{-\gamma_{\text{mag}}}, \quad \sigma_{\text{chi}}^0 = 0.48\sigma_G \alpha^{-\gamma_{\text{chi}}}. \quad (23)
\]

with \( \gamma_{\text{mag}} \approx 1.62 \pm 0.03 \) and \( \gamma_{\text{chi}} \approx 1.41 \pm 0.02 \).

It is generally argued that the Fermi velocity scales linearly in \( \alpha \).\(^{1,7,8,57-59} \) This implies that \( v_{\text{mag}} = 24v_F\alpha^{1-\gamma_{\text{mag}}/2} \). In addition, the chiral velocity must also tend to zero at the magic angle as \( v_{\text{chi}} = 0.7v_F\alpha^{1-\gamma_{\text{chi}}/2} \).

**C. Condon instability at the magic angle**

As mentioned in Sec. IV C, in AA-stacked bilayer graphene there is a Condon instability at \( T \sim 0 \). Since in twisted bilayer graphene, the electronic wave functions at the magic angle are highly localized around the AA-stacked islands,\(^6 \) there might be the possibility of a Condon instability in twisted bilayer graphene at \( \theta_m \). The imaginary part of the conductivity or magnetic Drude weight is obtained from the Kramers-Kronig relation which can be split into the following two contribu-
\begin{align}
D_{\text{mag}} &= \frac{2}{\pi} \left[ 2\omega_A \sigma_G - \int_{0}^{\omega_A} d\omega \Re \sigma_{\text{mag}}(\omega) \right] \\
&= \frac{2}{\pi} \left[ 2\omega_A \sigma_G - \left( \int_{0}^{\omega_{\text{mag}}^D} d\omega \Re \sigma_{\text{mag}}(\omega) + \int_{\omega_{\text{mag}}^D}^{\omega_A} d\omega \Re \sigma_{\text{mag}}(\omega) \right) \right] \\
&= D_{\text{mag}}^* + D_{\text{mag}}^{\text{reg}} ,
\end{align}

where \( \omega_A \) denotes the high-frequency cutoff and \( \omega_{\text{mag}}^D > 0 \) is the smallest frequency after the van Hove singularity for which \( \sigma_{\text{mag}}^D = \sigma_{\text{mag}}(\omega_{\text{mag}}^D) \), i.e., for \( \theta = 1.03^\circ \), this gives \( \hbar \omega_{\text{mag}}^D \approx 0.01 \mu eV \).

The second term \( D_{\text{mag}}^{\text{reg}} \) is assumed to be regular. The possible divergent contribution at the magic angle, \( D_{\text{mag}}^* \), can be estimated as follows:

\begin{align}
D_{\text{mag}}^* &= -\frac{2}{\pi} \int_{0}^{\omega_{\text{mag}}^D} d\omega \sigma_{\text{mag}}(\omega) \sim -\alpha^{-\gamma_{\text{mag}}^*}\omega_{\text{mag}}^D 
\end{align}

The exponent \( \gamma_{\text{mag}}^* \) is again obtained from a linear fit of a log-log plot and we obtain \( \gamma_{\text{mag}}^* \approx 1.41 \pm 0.04 \). We thus find a divergence at the magic angle that scales like \( D_{\text{mag}}^* \sim -\alpha^{-\delta_{\text{mag}}} \), where \( \delta_{\text{mag}} = 0.21 \pm 0.05 \). Since the Condon instability is marked by \( D_{\text{mag}} \sim D_{\text{mag}}^C = \frac{4}{\pi M} \), see Eq. (20), there will be a symmetry-broken ground-state with orbital magnetic domains at the magic angle.

The presence of an instability in the non-interacting model of Eq. (2) is a remarkable result and we are not aware of any other non-interacting model that exhibits a symmetry-broken ground-state. Let us finally note that the total chiral Drude weight \( D_{\text{chi}} \) has to vanish at the neutrality point due to gauge symmetry.  

\section{D. New universal absorption regimes}

The absorption spectrum almost at the magic angle can approximately be understood from the universal conductivity formula\(^{19}\) of a general dispersion \( \epsilon_k \propto |k|^\nu \)

\begin{align}
\sigma(\omega) = \frac{g_6 g_7 g_8 C_6}{16} \left( \frac{\nu}{\hbar} \right)^2 \sigma_G .
\end{align}

Above, we introduced the usual spin, valley, and layer-degree of freedom, but also a possible \( g_{C_5} \) degeneracy which takes the value 3 in case of an explicit 3-fold degeneracy (otherwise it is 1). In the following, we will discuss the results in units of the universal conductivity of graphene \( \sigma_G = g_6 g_7^2 e^2 / 16 \pi \). Notice that we introduce here explicitly the degeneracy factors which are usually set to \( g_s = g_v = 2 \).

At low frequencies, there will in principle always be a regime where the absorption is governed by the universal absorption of Dirac Fermions with \( \nu = 1 \) and we have \( \sigma(\omega) = 2\sigma_G \). For twist angles in the extreme vicinity of the magic angle, the plateau of a single quadratic dispersion relation with \( \nu = 2 \) is obtained with \( \sigma(\omega) = 4\sigma_G \), seen in the left panel of Fig. 4 for \( \theta = 1.03^\circ \) for \( 0.01 \mu eV \lesssim \epsilon \lesssim 1 \mu eV \).

Between these plateaus, a new plateau emerges with \( \sigma_0 = 12\sigma_G \), because a new absorption channel opens at the frequency of the avoided crossing as seen in the inset of the right panel of Fig. 3. Even though the band minima are elongated, as a first approximation they can be assumed to be a quadratic dispersion and due to the \( C_3 \)-symmetry, there are three of them for each Dirac point. We thus numerically obtain \( \sigma(\omega) = 12\sigma_G \).  

However, we have been neglecting the contribution of the central Dirac cone and the above qualitative discussion can be made quantitative by considering the following two-band model which was first introduced in Refs. 62 and 63:

\begin{align}
H_k = \frac{1}{2m} \begin{pmatrix} M & \varpi^2 + \eta \varpi^* \\ \varpi^2 + \eta \varpi & -M \end{pmatrix}
\end{align}

where \( \varpi = \hbar(k_x - i k_y) \). The model has eigenenergies \( 2m \epsilon_k = \pm \sqrt{M^2 + \varpi^4 + 2k^2 \cos(3\theta) \eta + k^2 \eta^2} \) displaying trigonal warping and zeros at \( |\varpi| = \eta \). For \( M = 0 \), there are three nodal points which lie in the directions \( \theta = \frac{2n\pi}{3} \) (\( \eta < 0 \)) and \( \theta = \pi - \frac{2n\pi}{3} \) (\( \eta > 0 \)) with \( n \in \mathbb{N} \). This transition can be also seen in the center and right panel of Fig. 3, where the avoided crossing changes from the \( K \)-direction (right from the \( K \)-point) to the \( \Gamma M \)-direction (left from the \( K \)-point), related by a 60\(^\circ\)-rotation.
As shown in Appendix B, the above model with $M = 0$ yields $\sigma = 12\sigma_G$ for small frequencies and $\sigma = 4\sigma_G$ for large frequencies. The reason for not obtaining the Dirac regime $\sigma = 2\sigma_G$ is because the model of Eq. (27) with $M = 0$ does not exhibit a gap at the three nodal points with $|k| = \eta$.

This can partially be remedied by introducing a $k$-dependent mass term with $M = |\hbar k|^2$ such that the gap or Dirac-regime energy is given by $\epsilon_D = \frac{k_0^2}{m} \eta^2$. From the numerical approach we obtain $\epsilon_D = 0.88\alpha^{\gamma_s} \mu eV$ which allows us to connect $\eta$ to $\alpha$ of Eq. (22):

$$h\eta = \sqrt{0.88m\mu eV\alpha^{\gamma_s}/2} \quad (28)$$

Notice that with the discussion of the dimensionless energy scale $\tilde{\omega}$ defined in Appendix B, we would obtain the same scaling relation. Eq. (28) provides a mapping between the continuum model of twisted bilayer graphene and the model of Eq. (27).

VI. FLAT-BAND PLASMONICS

Since twisted bilayer graphene consists of two layers, there will be two plasmonic modes. For layers far away, these modes hardly hybridize, but for an interlayer distance $a = 3.5 \AA$, anti-bonding and bonding modes emerge. Due to the long-ranged Coulomb interaction, the dispersions show square-root and linear behavior in the momentum $q$ and define the so-called optical (charge even) and acoustic (charge odd) branches, respectively.

In the local approximation, they are generally given by

$$\omega^2_+ = \frac{\chi_{\text{tot}}(\omega+)q}{2\alpha}, \quad (29)$$

$$\omega^2_- = \frac{\chi_{\text{mag}}(\omega-)aq^2}{2\alpha}, \quad (30)$$

which define self-consistent equations for the plasmonic frequencies $\omega_+$ and $\omega_-$ with momentum $q$, respectively. Note that the optical mode depends on the dielectric environment through $\epsilon = (\epsilon_{\text{up}} + \epsilon_{\text{down}})/2$, but the acoustic mode does not.  

The plasmon dispersion does not depend on the chiral Drude weight since the non-retarded approximation does not allow for a coupling of longitudinal and transverse modes. Nevertheless, the optical (acoustic) mode, usually defined by electric (magnetic) dipole oscillations, is now accompanied by parallel magnetic (electric) dipole oscillations. This is expressed by the following relations:

$$e_q \cdot m = \alpha \chi e_q \cdot j_{\text{tot}} , \quad (31)$$

$$ae_{q,\perp} \cdot j_{\text{tot}} = \bar{\chi} e_{q,\perp} \cdot m , \quad (32)$$

with $\chi = \chi_{\text{chi}}/\chi_{\text{tot}}$ and $\bar{\chi} = 4\chi_{\text{chi}}/\chi_{\text{mag}}$. As the total current is related to the electric dipole, $j_{\text{tot}} = -i\omega p$, we have $p \parallel m \parallel q$ for the optical mode and $p \parallel m \perp q$ for the acoustic mode.

The above relations are obtained from the transport equations of Eq. (6) and hold also in the static limit, i.e., the total Drude weight $D_{\text{tot}}$ and chiral Drude weight $D_{\text{chi}}$ are Fermi-line properties as discussed in Ref. 49. Similar conclusions have been drawn in Refs. 66 and 67.

Let us finally note, that Eqs. (29) and (30) can be generalized to a non-local approximation by the replacements $\chi_{\text{tot}}(\omega) \rightarrow \chi_{\text{tot}}(\omega, q)$ and $\chi_{\text{mag}}(\omega) \rightarrow \chi_{\text{mag}}(\omega, q)$ that leads to flat plasmonic bands.

A. Poynting vector

Even though the optical and acoustic plasmon dispersions only depend on $\chi_{\text{tot}}$ and $\chi_{\text{mag}}$, respectively, the Poynting vector depends also on the chiral response $\chi_{\text{chi}}$. To show this, let both modes be induced by the sheet current $j_{\parallel}$ parallel to the plasmon momentum $q$. We then get in the non-retarded limit, close to the sheet and up to second order in $qa$, the following expressions for the Poynting vectors of the optical (+) and acoustic (-) mode (see also Ref. 41):

$$P_+ = P_0 \left( \begin{array}{c} 1 + \gamma(qa)^2 \\ -2\chi qa \\ 0 \end{array} \right), \quad (33)$$

$$P_- = P_0 \left( \begin{array}{c} 4\chi^2 q^2 + \gamma(qa)^2 \\ -2\chi qa \\ 0 \end{array} \right), \quad (34)$$

with $P_0 = \frac{sgn(z)q^2}{2\epsilon_0\omega}$ and $\gamma = [1 + (4\chi^2 - 1)\frac{k_0^2}{q^2}]/4$, where $k_0 = \omega/\epsilon$ is the wavelength of light in free space and $\chi = \chi_{\text{chi}}/\chi_{\text{tot}}$. This shows that the chirality determines the energy flux and deflection the plasmonic energy flux. The Poynting vector of the acoustic mode is zero to lowest order in $qa$ in the non-chiral limit $\chi = 0$ since the mode consists of perfectly cancelling counterpropagating current densities.

From the definition of $\chi$, we infer that there is a diverging regime for $\chi_{\text{tot}} = 0$. This regime seems to be necessarily realized at the neutrality point for $\omega \rightarrow 0$, since the total Drude weight has to vanish, $D_{\text{tot}} = 0$. However, in the d.c. limit also the chiral Drude weight needs to vanish, again due to gauge invariance, $D_{\text{chi}} = 0$. At the neutrality point, no deflection is thus expected even for the acoustic mode. At finite chemical potential, though, Bloch electrons are deviated without a magnetic field as has recently been discussed by several authors. 

At finite frequencies, we expect sweet spots whenever $\chi_{\text{tot}}(\omega) = 0$. These frequencies will usually lead to $\chi \rightarrow \infty$ which we will denominate as chiral resonances. At these frequencies, the optical mode will eventually disappear. Still, the energy flux of the chiral acoustic mode will be highly enhanced and ultimately diverges. Let us finally note that chiral resonances also occur if $\chi_{\text{chi}} \rightarrow \infty$, but they might be limited to lower temperatures.
Re \( \chi(\omega) \) [in \( t(e/\hbar) \)]

\[ \Re \chi(\omega) \text{ in } t(e/\hbar) \]

\[ \nu = 0.02 \]

\[ \nu = 0.01 \]

\[ \nu = -0.02 \]

\[ \nu = -0.01 \]

\[ \nu = 0 \]

\[ \nu = 0.01 \]

\[ \nu = 0.02 \]

\[ \nu = 0.03 \]

\[ \nu = 0.04 \]

\[ \theta = 1.3^\circ \]

\[ \theta = 1.2^\circ \]

\[ \theta = 1.1^\circ \]

**FIG. 5.** The real part of the current susceptibility \( \Re \chi(\omega) \) with \( \nu = \text{tot}, \text{mag, chi} \) of the asymmetric continuum model with \( \kappa = 0.8 \) in Eq. (2) at the neutrality point in units of \( t_F^2 \) for temperatures \( T = 0.10 \text{K} \). The optical gap is indicated by the white area. Left panel: twist angle \( \theta = 1.3^\circ \). Center panel: twist angle \( \theta = 1.2^\circ \). Right panel: twist angle \( \theta = 1.1^\circ \).

### B. Chiral plasmons at the neutrality point

A Dirac system does not host plasmons at the neutrality point. Even though electron-hole transitions may lead to positive and negative charge densities, the charge response is always negative such that the RPA-condition for plasmonic excitations can never be fulfilled.

This changes in moiré systems, where flat bands emerge. The moiré potential that confines the electrons in the AA-stacked region then acts as restoring force such that the electronic and hole charge density can oscillate in-phase. From a technical point of view, this can be deduced from the highly peaked absorption due to the flat bands as this may lead to a positive charge response due to the Kramer-Kronig relation. As the collective motion is composed of localized electrons, also the plasmonic bands are usually flat.\(^{20-25}\)

One crucial condition for long-lived plasmons is the presence of an optical gap which emerges in the continuum model by considering relaxation effects.\(^{29}\) Now, if the absorption is sufficiently peaked, a positive reactive part of the charge excitations can leak inside the optical gap even though there are no nominal charges in the system. This implies the possibility of a mode ("plasmon") as a pole in RPA response. The resulting response functions are shown in Fig. 5 for different twist angles and temperatures with \( \kappa = 0.8 \).

The features of the plasmonic excitations can be summarized as follows: (i) Optical plasmons can exist right above the optical gap and persist for temperatures up to \( T \sim 50 \text{K} \) for \( \theta < 1.3^\circ \). This is similar to the optical plasmons in flat bands with excess charge.\(^{21,69}\) (ii) Acoustic plasmons can exist almost in the whole optical window. Most notably, the magnetic Drude weight carries by far the largest optical weight and we expect excitations with frequencies larger than that of the corresponding optical plasmon for \( qa \leq 100 \). At the chiral resonance, for which \( \chi_{\text{tot}}(\omega) = 0 \), these modes are characterized by an extremely high energy density \( w \sim \chi^2 \rightarrow \infty \) as can be deduced from the continuity equation and Eq. (34).

Let us finally highlight that both plasmon modes are intrinsically chiral since \( \chi_{\text{chi}} \) is finite throughout the protected window. This is due to the broken particle-hole symmetry as will be discussed in the Sec. VII.

### VII. CHIRAL RESPONSE AT THE NEUTRALITY POINT

Chiral response in twisted bilayer graphene has been observed experimentally in Ref. 15 and is thus manifested in misaligned van-der-Waals heterostructures. In Ref. 16, it was shown that neglecting the relative rotation of the pseudospin-orientation between the two layers renders the chiral response. The difference in pseudospin orientation, which is a consequence of the real space chiral symmetry, is thus responsible for the chiral response.

Here, we will relate this property solely to particle-hole symmetry and generalize the results. We then discuss why a slight particle-hole asymmetry will lead to a finite chiral response.

#### A. Symmetries of response functions

The continuum model displays particle-hole symmetry if the pseudo-spin rotation is neglected \( \tau_\alpha \rightarrow \tau_\alpha \).\(^{42}\) This can be seen by the following anti-unitary transformation \( \mathcal{U} = S \mathcal{P} \mathcal{K} \). The unitary operator \( S \) reverts the sign of \( k_x \), \( S|k_x, k_y, \alpha, \ell \rangle = | -k_x, k_y, \alpha, \ell \rangle \), the unitary operator \( \mathcal{P} \) adds a \( \pi \)-phase to states in layer \( \ell = 2 \), \( \mathcal{P}|k_x, k_y, \alpha, 2 \rangle = | -k_x, k_y, \alpha, 2 \rangle \), and the complex-conjugate \( \mathcal{K} \) effectively changes the sign of \( k_y \). We thus have \( \mathcal{U} \mathcal{H} \mathcal{U}^{-1} = -\mathcal{H} \).

We can now discuss the effect of \( \mathcal{U} \) on the general response function. For this, we suppress the index \( k \) and write

\[ \chi_{AB} = \sum_{m, m} \frac{n_F(\epsilon_m) - n_F(\epsilon_n)}{\omega + i0^+ - \epsilon_n + \epsilon_m} \langle m|\mathcal{A}|n\rangle \langle n|\mathcal{B}|m\rangle. \quad (35) \]

Using the eigenbasis \( \{|\tilde{n}\rangle\} \) of \( \mathcal{H} \), with \( |\tilde{n}\rangle = \mathcal{U}|n\rangle \) and \( \mathcal{H}|\tilde{n}\rangle = \epsilon_{\tilde{n}}|\tilde{n}\rangle \) where \( \epsilon_{\tilde{n}} = -\epsilon_n \), one can calculate any
response as
\[ \chi_{AB} = \sum_{\tilde{n},\tilde{m}} \frac{n_F(\epsilon_n) - n_F(\epsilon_m)}{\omega + i0^+ - \epsilon_n + \epsilon_m} \langle \tilde{m} | A | \tilde{n} \rangle \langle \tilde{n} | B | \tilde{m} \rangle. \]  
\[ (36) \]

We can then write
\[ \frac{n_F(\epsilon_m - \mu) - n_F(\epsilon_m - \mu)}{\omega + i0^+ - \epsilon_m + \epsilon_m} = \frac{n_F(\epsilon_n + \mu) - n_F(\epsilon_m + \mu)}{\omega + i0^+ - \epsilon_m + \epsilon_n}, \]
\[ (37) \]
where we have explicitly included the chemical potential \( \mu \) in the argument of the Fermi function. We now have for the antiunitary transformation \( \langle \tilde{n} | \phi \rangle = [\langle n | (U^\dagger | \phi \rangle)^* \].
Therefore, we have \( \langle \tilde{m} | A | \tilde{n} \rangle = \langle m | A | n \rangle^* = \langle n | A | m \rangle \) with \( \tilde{A} \) defined below. The particle-hole symmetry \( \tilde{U} \) thus leads to the following relation:
\[ \chi_{AB}(\mu) = \chi_{\tilde{A}B}(\mu), \]
\[ (38) \]
with \( \tilde{A} = UAU^{-1} \) and \( B = UBU^{-1} \). We now see, because of \( \tau_x = \tilde{\tau}_x \) and \( \tau_y = -\tilde{\tau}_y \), that the response obeys the following relations:
\[ \sigma_0(\mu) = \sigma_0(-\mu), \]
\[ (39) \]
\[ \sigma_1(\mu) = \sigma_1(-\mu), \]
\[ (40) \]
\[ \sigma_{xy}(\mu) = -\sigma_{xy}(-\mu). \]
\[ (41) \]
For \( \mu = 0 \), we thus have \( \sigma_{xy} = 0 \) for all temperatures and frequencies as claimed.

### B. Electron and hole transitions

To make the discussion more illustrative, we switch to the particle-hole picture by defining \( e_n^h = e_n \) if \( e_n > 0 \) and \( e_m^h = -e_m \) if \( e_m < 0 \). We only consider vertical transitions and a general transition \( n \to m \) at half-filling with \( \mu = 0 \) is now characterized by the initial and final momenta, \( e_n^h \to e_m^e \).

For the electron-hole symmetric model, there are transitions with \( e_n^h = e_m^e \). However, this symmetry is usually slightly broken and generally one finds \( e_n^h \neq e_m^e \). We can thus classify all (relevant) transitions by either electron transitions if \( e_n^h > e_m^e \) or by hole transitions if \( e_n^h < e_m^e \).

Let us now denote response functions consisting of electronic (hole) transitions as \( \chi^{(h)} \). The particle-hole transformation \( \tilde{U} \) further relates \( e_n^h = e_n = -e_n^h \) and \( e_m^h = e_m = -e_m^h \). We now see, because of \( \tau_x = \tilde{\tau}_x \) and \( \tau_y = -\tilde{\tau}_y \), that the response of electron transitions and hole transitions obeys the following relations:
\[ \chi_{xx}^e = \chi_{xx}^h, \chi_{xy}^e = -\chi_{xy}^h \]
\[ (42) \]
Numerically, we find that the dominant chiral electron (hole) transitions between different bands and with small energy denominators are negative (positive). However, for larger energy denominators, we also find chiral electronic (hole) transitions which have the opposite sign. Furthermore, the sign of the chiral response due to electron (hole) transitions between the same bands can change. The momenta of electron and hole transitions then normally form a well-defined boundary in the Brillouin-zone. For transitions within the flat bands, however, we also found fractal boundaries.

### C. Detailed balance

The transformation \( U \) links the momentum \( (k_x, k_y) \) to momentum \( (-k_x, k_y) \). Eq. (37) guarantees that the transition \( n \to m \) at momentum \( (k_x, k_y) \) from \( \epsilon_n \) to \( \epsilon_m \) and at chemical potential \( -\mu \) carries the same weight as the transition \( \tilde{m} \to \tilde{n} \) at momentum \( -(k_x, k_y) \) from \( \epsilon_n \) to \( \epsilon_m \) and at chemical potential \( \mu \). Since also the matrix elements have the same (absolute) value, we thus obtain a detailed balance relation for the above transitions at the neutrality point \( \mu = 0 \). This is illustrated in the left panel of Fig. 6.

With \( \chi_{\alpha\beta} = \sum_{m,n,k_x,k_y} \chi_{\alpha\beta}(m,n;k_x,k_y) \), we can link a single electron transition to a single hole transition as follows:
\[ \chi_{xx}(m,n;k_x,k_y) = \chi_{xx}(n,m;-k_x,k_y), \]
\[ (43) \]
\[ \chi_{xy}(m,n;k_x,k_y) = -\chi_{xy}(n,m;-k_x,k_y). \]
\[ (44) \]
This detailed balance between the electron transition at \( (k_x,k_y) \) and the corresponding hole transition at \( (-k_x,k_y) \) eventually leads to a vanishing chiral response at half-filling.

We can also define a different particle-hole transition as was proposed by Moon and Koshino.\(^{42} \) Together with time-reversal and rotational symmetry, this leads to
\[ \chi_{xx}(m,n;k_x,k_y) = \chi_{xx}(n,m;-k_x,-k_y), \]
\[ (45) \]
\[ \chi_{xy}(m,n;k_x,k_y) = -\chi_{xy}(n,m;-k_x,-k_y). \]
\[ (46) \]

### D. Dissipative chiral response close to the magic angle

We will now discuss the chiral response of the full model of Eq. (2) at the neutrality point. Crucially, the rotation in pseudospin-space needs to be included to break particle-hole symmetry as discussed before. However, the approximate electron-hole symmetry suggested by \( U \) will still relate sublattice and layer, leading to a coherence of the wave function between these two degrees of freedom which must not be related to the underlying lattice (spatial) symmetry.\(^{49,70} \)

Since electron-hole symmetry is slightly broken, we can label all transitions as either electron or hole transitions. The electronic wave function is not strongly affected by this small perturbation and due to continuity arguments, around certain regions in \( k \)-space, electron and hole transitions must still have well-defined, but opposite signs.
FIG. 6. Left panel: Illustration of the detailed balance relation of a particle-hole symmetric mode. Via the anti-unitary transformation $U$, the transitions from $\epsilon_n \rightarrow \epsilon_m$ at momentum $(k_x, k_y)$ are directly related to the transitions from $\epsilon_m \rightarrow \epsilon_n$ at momentum $(-k_x, k_y)$. Any hole transition ($\epsilon_n^h > \epsilon_m^h$) is automatically related to an electron transition ($\epsilon_n^e > \epsilon_m^e$) since $\epsilon_n^e = \epsilon_m^h$ and $\epsilon_n^h = \epsilon_m^e$. Center panel: Chiral response $\text{Re} \sigma_{\mu \nu}(\omega)$ of the asymmetric continuum model with $\kappa = 0.8$ in Eq. (2) at the neutrality point with twist angle $\theta = 1.1^\circ$ for temperatures $T = 0, 10, 300K$. The inset highlights the chiral response around $\hbar \omega = 95$meV. Right panel: Corresponding band structure and density of states (DOS) on logarithmic scale. The transitions related to the van Hove singularities around $\hbar \omega = 25$meV and $\hbar \omega = 95$meV are indicated by red (electronic transition) and blue (hole-like transition) arrows.

Apart from the transition-matrix element, the response is also determined by the electronic dispersion. In any Bloch-band, there is at least one van-Hove singularity and in principle, we expect an enhanced optical response if either the initial or final state is located at a singular $k$-point. However, the transition-matrix element might be suppressed due to symmetries and precisely the approximate particle-hole symmetry suppresses the optical transitions of the total current at the $M$-point. This is not the case, though, for the magnetic and chiral transitions and we thus expect a large response due to the large van-Hove singularity which can also be located around the $K$ or $\Gamma$-point.

In the electron-hole symmetric model, van Hove singularities necessarily appear in the occupied and unoccupied bands at $\epsilon_n^H = \epsilon_m^H$. Slightly breaking this symmetry will lead to a splitting with $\epsilon_n^H \neq \epsilon_m^H$. Possible transitions are now of electron and hole nature that have opposite chiral response, but do not cancel each other anyhow. Also the band-edges of the electronic and hole bands will slightly shift due to the broken symmetry, given rise to either pure electron or hole transitions. To conclude, we expect prominent features coming from singularities of the band structure, either discontinuities or logarithmic divergencies, where the electronic and hole transitions are not compensated by each other.

This can be seen in the center panel of Fig. 6 where the dissipative response of twisted bilayer with twist angle $\theta = 1.1^\circ$ and $\kappa = 0.8$ is shown. There are always two peaks that come in pairs, a negative peak and a positive peak associated with either electron or hole transitions.

The first pair originates from transitions within the flat bands and is strongly temperature dependent, i.e., practicable at room temperature. The second and third pair are related to transitions from the flat to the first remote band and associated to van Hove singularities located at the $\Gamma$ and $K$-point, respectively. They thus do not as strongly depend on temperature and in both cases, the negative (positive) response is related to electron (hole) transitions. The response of the third pair is highlighted in the inset of the center panel of Fig. 6 for the sake of clarity.

In the right panel of Fig. 6, the band structure is shown and the electron (red arrow) and hole (blue arrow) transitions are shown for the second and third pairs. Generally, we expect strong chiral response at energies involving a large density of states. These energies can be identified from the density-of-states (DOS), shown next to the band structure. However, the larger the transition energy becomes, the weaker the response is.

VIII. SUMMARY AND OUTLOOK

We have investigated the full optical response of magic angle graphene at the neutrality point including the total, magnetic and chiral response. The dissipative response is obtained without the usual damping term by analytically integrating the delta-function on a linearized grid. The reactive response is then obtained via the Kramers-Kronig relation applying a suitable cutoff for large frequencies. By this, we obtain accurate results close to the magic angle even for low energies.

Furthermore, we deduced a scaling law for the dissipative response functions in the low-frequency limit and concluded via the Kramers-Kronig relation that the ground state of the continuum model at the magic angle is unstable towards current fluctuations. This Condon instability is characterized by the formation of orbital magnetic domains.

From the experimental point of view, our main result is that acoustic plasmons should be dominating the plasmonic response at the neutrality point, leading to a "plasmonic band-inversion" for which the acoustic bonding-
modes are energetically higher than the optical antibonding modes. Since all excitations are intrinsically chiral, the plasmonic energy flux will be deflected. The frequency for which \( \chi_{\text{tot}}(\omega) = 0 \) should eventually lead to a chiral resonance with strongly enhanced energy density.

Finally, we discussed the chiral response in the particle-hole symmetric model and showed that it is necessarily zero at the neutrality. However, this symmetry is slightly broken in the continuum model and the resulting chiral response can be characterized by either electron or hole transitions that have opposite chirality.

**IX. ACKNOWLEDGMENTS**

This work was supported by the mobility program Salvador Madariaga under PRX19-00024 and by the projects No. PGC2018-096955-B-C42, No. PID2020-113164GB-I00, and No. CEX2018-000805-M financed by MCIN/ AE1/10.13039/501100011033. The access to computational resources of CESGA (Centro de Supercomputación de Galicia) is also gratefully acknowledged. The work of T.S. and of J.S. was further supported by Deutsche Forschungsgemeinschaft via SFB 1277. D.M. wishes to thank Dr. A. B. Watson for an inspiring discussion on the twisted bilayer graphene near the magic angle.

**Appendix A: Numerical integration of a generalized density of states**

In this appendix, we describe the numerical recipe how to obtain the optical response functions without introducing the usual damping term. The main numerical task in our approach is the numerical evaluation of two-dimensional integrals that involve a delta-function. If we wish to evaluate \( \chi_{\text{tot}}(\omega) \) numerically, the usual damping term. The main numerical task in our approach is the numerical evaluation of two-dimensional integrals that involve a delta-function. If we wish to evaluate \( \chi_{\text{tot}}(\omega) \) numerically, the usual damping term.

We will calculate the response \textit{without} disorder, i.e., we will take the delta-function literally and perform the integration \textit{analytically} after having discretized the Brillouin zone’s. This can be done by introducing a triangular grid on the Brillouin zone and assuming a linear interpolation. In Fig. 7, we show the discretization used for the calculations. We have checked that the final result does not crucially depend on the discretization. Another optimization is obtained by assuming a quadratic interpolation between the three base-points. We have checked that for large grids used here, this does not lead to significant improvements, also nicely explained in Ref. 71.

We discretize the Brillouin zone by \( N \) with \( n, m = 0, ..., N \) in the following way:

\[
k = \frac{n}{N} G_1 + \frac{m}{N} G_2 ,
\]

with the lattice vectors \( G_1 = |\Delta K| \left( \frac{-\sqrt{3}}{2}, -\frac{3}{2} \right) \), \( G_2 = |\Delta K| \left( \frac{\sqrt{3}}{2}, -\frac{3}{2} \right) \), see Fig. 7 A). In our calculations, we chose discretizations up to \( N \approx 10000 \); for twist angles \textit{almost} at the magic angle even as large as \( N \approx 20000 \).

We shall calculate the following generalized density of states with \( g \) denoting a degeneracy factor:

\[
\rho(\epsilon) = \frac{g}{A} \sum_{k} f_{k} \delta(\epsilon - \epsilon_{k})
\]

As we assume periodic boundary conditions, the sample area is given by \( A = N^2 a_1 \), where \( a_i \) is the area of the unit cell. In the case of twisted bilayer graphene, we have \( a_c = \frac{\sqrt{3}}{2} a_2^2 A_i \) as the area of the moiré supercell with \( a_g = 2.46 \text{Å} , A_i = 3i^2 + 3i + 1 \) and \( \cos(\theta) = \frac{1}{\pi^2} \).

We will now consider each of the \( N^2 \) mini-rhombii individually which are characterized by the vertices \( k_{i}, \epsilon_{i} \), and optionally \( f_{i} \) with \( i = A, B, C, D \). First, we will divide the mini-rhombs in two and consider first the triangle defined by \( i = A, B, C \) and afterwards the triangle defined by \( i = B, C, D \).

To outline the algorithm, we will now only consider the first triangle and further assume that \( \epsilon_{A} \leq \epsilon_{B} \leq \epsilon_{C} \) which can always be achieved by relabelling the vertices. We now interpolate linearly between the three vertices and any momentum \( k \) and energy \( \epsilon \) inside the triangle is parameterized by the two parameters \( t \) and \( s \):

\[
\begin{pmatrix}
k_{x} \\
k_{y} \\
\epsilon_{k}
\end{pmatrix}
= \begin{pmatrix}
k_{A,x} & -k_{A,x} & k_{B,x} - k_{A,x} \\
k_{B,y} - k_{A,y} & k_{C,x} - k_{A,x} & 0 \\
\epsilon_{k} - \epsilon_{k_{A}} & \epsilon_{k_{B}} - \epsilon_{k_{A}} & \epsilon_{k_{C}} - \epsilon_{k_{A}}
\end{pmatrix}
\begin{pmatrix}
t \\
s \\
1
\end{pmatrix}
\]

We can now write the integral that contains the contribution \( \rho_{A} \) to \( \rho \) over the triangle with respect to the two variables \( t \) and \( s \). The integration limits corresponding to the vertices \( [A, B, C] \) are now given with respect to the axis defined by \( t, s \), i.e., \( [(0,0), (1,0), (0,1)] \). Neglecting for the moment the weight function \( f_{k} \) and setting \( g = 1 \),
we arrive at the following expression:

\[
\rho_\Delta(\epsilon) = \frac{1}{(2\pi)^2} \int_\Delta d^2k \, \delta(\epsilon - \epsilon_k),
\]

(\ref{eq:rho_D})

\[
= \frac{\tilde{J}}{(2\pi)^2} \int_0^1 dt \int_0^{1-t} \delta(s - s(\epsilon, t)) ,
\]

(\ref{eq:rho_D_2})

where we introduced the Jacobian \( J = \frac{1}{(k_{B,x} - k_{A,x})(k_{C,y} - k_{A,y}) - (k_{C,x} - k_{A,x})(k_{B,y} - k_{A,y})} \) with \( \tilde{J} = J / (\epsilon_C - \epsilon_A) \) and \( s(t) = s(\epsilon, t) = \frac{\epsilon - \epsilon_A}{\epsilon_C - \epsilon_A} - \frac{\epsilon_B - \epsilon_A}{\epsilon_C - \epsilon_B} \).

The integral depends on the value of \( \epsilon \) relative to the energies \( \epsilon_i \) and we obtain

\[
\rho_\Delta(\epsilon) = \frac{\tilde{J}}{(2\pi)^2} \left[ \frac{\epsilon - \epsilon_A}{\epsilon_B - \epsilon_A} \theta(\epsilon - \epsilon_A) \theta(\epsilon_B - \epsilon) \right. \\
+ \frac{\epsilon_C - \epsilon}{\epsilon_C - \epsilon_B} \theta(\epsilon - \epsilon_B) \theta(\epsilon(\epsilon_C - \epsilon)) \right].
\]

(\ref{eq:rho_D_3})

The total density of states is then obtained by the sum

\[
\rho(\epsilon) = \sum_\Delta \rho_\Delta(\epsilon).
\]

The weight function \( f_k \) can now be included by linear interpolation. With

\[
f(\epsilon) = f_B \frac{\epsilon - \epsilon_A}{\epsilon_B - \epsilon_A} \theta(\epsilon - \epsilon_A) \theta(\epsilon_B - \epsilon) \\
+ f_A \frac{\epsilon_B - \epsilon}{\epsilon_B - \epsilon_A} \theta(\epsilon - \epsilon_A) \theta(\epsilon_B - \epsilon) \\
+ f_C \frac{\epsilon_C - \epsilon}{\epsilon_C - \epsilon_B} \theta(\epsilon - \epsilon_B) \theta(\epsilon_C - \epsilon) \\
+ f_B \frac{\epsilon_C - \epsilon}{\epsilon_C - \epsilon_B} \theta(\epsilon - \epsilon_B) \theta(\epsilon_C - \epsilon) ,
\]

(\ref{eq:f_omega})

and reincorporation of the degeneracy factor \( g \), the general density of states is thus approximated by

\[
\rho_\Delta(\epsilon) = \frac{g \tilde{J}}{(2\pi)^2} f(\epsilon) \left[ \frac{\epsilon - \epsilon_A}{\epsilon_B - \epsilon_A} \theta(\epsilon - \epsilon_A) \theta(\epsilon_B - \epsilon) \\
+ \frac{\epsilon_C - \epsilon}{\epsilon_C - \epsilon_B} \theta(\epsilon - \epsilon_B) \theta(\epsilon(\epsilon_C - \epsilon)) \right].
\]

(\ref{eq:rho_D_4})

Apart from increasing the discretization, the numerical results can be further smoothened by explicitly taking advantage of the rotational symmetry, i.e., \( 2 \tilde{j}_\text{tot} \cdot \tilde{j}_\text{tot} = (j_x^1 + j_x^2)^2 + (j_y^1 + j_y^2)^2, \) \( 2 \tilde{j}_\text{mag} \cdot \tilde{j}_\text{mag} = (j_x^1 - j_x^2)^2 + (j_y^1 - j_y^2)^2, \) and \( 2 \tilde{j}_xy \cdot \tilde{j}_xy = j_x^1 j_y^2 - j_x^2 j_y^1. \)

### Appendix B: Real part of interband conductivity: Analytical derivations

In this appendix, we describe analytically the real part of the interband conductivity for \( \mu = 0 \) and \( T = 0 \), by use of the two-band model introduced in Refs. 62 and 63. In this model, the Dirac cone coexists with a parabolic profile in the Hamiltonian. We focus on the limits of the interband conductivity as \( \omega \to 0 \) and \( \omega \to \infty \).

![Fig. 7](image_url)

**Fig. 7.** (A) The rhombic Brillouin zone defined by the reciprocal lattice vectors \( G_1 \) and \( G_2 \). (B) Zoom-In of a small rhombus with side length \( |G_1|/N \). The vertices are labeled by \( i = A, B, C, D \) and characterized by \( k_i, \epsilon_i \), and optionally also by \( f_i \).

1. **Model Hamiltonian**

The reduced, two-band Hamiltonian without a gap reads

\[
H_{\text{red}} = - \begin{pmatrix} 0 & \omega \omega^* + \eta \omega \\ \omega^* & 0 \end{pmatrix},
\]

(\ref{eq:Ham_red})

where \( \omega = k_x \mp ik_y \) in the vicinity of \( K \) (\( K' \)). Here, we have set \( \hbar = 1 \) in \( k_y \) for later algebraic convenience. The parameter \( \eta \) is assumed positive and small \( (0 < \eta \ll 1) \). It expresses the relative strength of the Dirac cone. From now on, we focus on the point \( K \). We will comment on the case with \( \eta < 0 \) below.

This Hamiltonian yields the eigenenergies

\[
\epsilon_k, \pm = \pm |F_k| , \quad F_k = \omega^2 + \eta \omega ,
\]

(\ref{eq:ep_k})

and the normalized eigenvectors

\[
|\pm\rangle_k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp e^{i \theta_k} \end{pmatrix} , \quad \theta_k = \text{Arg} F_k .
\]

(\ref{eq:theta_k})

The eigenenergies are expressed explicitly by

\[
\epsilon_{k, \pm} = \pm \sqrt{k_x^2 + 2\eta k_x^3 \cos(3\theta) + \eta^2 k^2} = \pm \epsilon(k; \eta) ,
\]

(\ref{eq:ep_k_2})

in the polar coordinates \( (k, \theta) \) with center at \( K \).

Evidently, the scaling of the momentum with \( \eta \) according to \( k = \eta \tilde{k} \) results in \( \epsilon(k; \eta) = \eta \tilde{\epsilon}(\tilde{k}) \) where

\[
\tilde{\epsilon}(\tilde{k}) = \sqrt{k_x^2 + 2k_x^3 \cos(3\theta) + k^2} = \epsilon(k; 1) .
\]

(\ref{eq:ep_k_3})

It is algebraically convenient to use the scaled momentum and eigenenergy (see, however, Eq. (B7)). For ease of notation, we henceforth drop the tildes from \( \tilde{k} \) and \( \tilde{\epsilon} \).

Next, we describe the local minima of \( \epsilon_{k, +} = \epsilon(k) \). By \( \nabla_k(\epsilon^2) = 0 \) we obtain \( k = k_c (\cos \theta_0, \sin \theta_0) \) where \( k_c = 0 \), or \( k_c = 1 \) with \( \theta_c = \pi - 2\pi n/3, n \in \mathbb{N} \). These points yield zero bandgap. The other critical points of \( \epsilon(k) \)
correspond to saddle points, with nonzero bandgap, and are disregarded. If \( \eta < 0 \), the local minima correspond to \( k_c = 0 \), or \( k_c = 1 \) with \( \theta_c = 2\pi n/3 \) (by \( k_x \to -k_x \)).

We turn our attention to the velocity matrix element needed for the interband conductivity. By setting \( F = F_R + iF_I \) (\( F_R = \Re F \) and \( F_I = \Im F \)), we have

\[
\langle -|\nabla_k H_{red}|^+ \rangle = -i\epsilon(k) \nabla_k \theta_k = \frac{i}{\epsilon(k)} \{ (\nabla_k F_R) F_I - F_R (\nabla_k F_I) \}.
\]

Let’s compute the \( x \)-component, for example. We find

\[
\langle -|\partial_{k_x} H_{red}|^+ \rangle = - 2i \frac{\Lambda_x(k)}{\epsilon(k)}
\]

where

\[
\Lambda_x(k) = k_y \left\{ \left( k_x - \frac{1}{2} \right)^2 + k_y^2 - \frac{3}{4} \right\}. \tag{B6}
\]

For \( K' \), one simply has to replace \( k_x \) by \(-k_x\). Note that \( \Lambda_x(k) = 0 \) at the local minima of \( \epsilon(k) \) determined above.

2. Integral of interband conductivity

The diagonal elements of the interband (regular) conductivity are computed from the formula \( (\alpha = x, y) \)

\[
\sigma_{\alpha\alpha}^R(\omega) = 4ig_s g_v g_t \sigma_G (\omega + i0^+) \int \frac{d^2k}{(2\pi)^2} \frac{1}{\epsilon(k) \epsilon(k)^2 - (\omega + i0^+)^2} \frac{\Lambda_x(k)^2}{\epsilon(k) \epsilon(k)^2 - (\omega + i0^+)^2} \tag{B7}
\]

where \( \epsilon(k) \) is given by Eq. (B5) and

\[
\tilde{\omega} = \frac{\omega}{2\eta^2}.
\]

We will keep the symbol \( \tilde{\omega} \) (with tilde) throughout.

Our task is to compute \( \Re \sigma_{xx}^R \) by carrying out the integration in \textit{local} polar coordinates by consideration of points \( \mathbf{k} = \mathbf{k}_c \) such that \( \epsilon(k) = \tilde{\omega} \) (if \( \tilde{\omega} > 0 \)). A difficulty is that these points may locally form non-circular curves.

The integral for \( \Re \sigma_{xx}^R \) has significant contributions from the vicinity of each curve. We study the following limits:

(i) \( \tilde{\omega} \to 0 \), when the curves of interest are formed near local minima of \( \epsilon(k) \); and (ii) \( \tilde{\omega} \to +\infty \), when \( k_c \) is large.

\[ q = |\mathbf{q}| \text{ as a function of } \phi \text{ and } \tilde{\omega}. \]

Let \( q_* (\phi) \) be such a solution. Subsequently, we expand \( \epsilon(k)^2 \) near \( q = q_* \).

First, consider \( k_c = 0 \), which amounts to the center point (\( K \)). After some algebra, we obtain

\[
q_* = \tilde{\omega} \{ 1 - \tilde{\omega} \cos(3\phi) + O(\tilde{\omega}^2) \} \tag{B10}
\]

where \( O(\tilde{\omega}^2) \) denotes a correction of the order of \( \tilde{\omega}^2 \). This formula entails an approximation of the form

\[
\epsilon(k)^2 - \tilde{\omega}^2 \approx Q_1(\phi) (q - q_*) + Q_2(\phi) (q - q_*)^2 \tag{B11}
\]

where \( Q_1(\phi) = 2q_1 [1 + 3q_1 \cos(3\phi)] \) and \( Q_2(\phi) = 1 \).

Second, consider \( k_c = 1 \) with \( \theta_c = \pi \), which amounts to the critical point at \( \mathbf{k} = (-1, 0) \), for \( n = 0 \). We find

\[
q_* \cong \frac{\tilde{\omega}}{\sqrt{1 + 8 \sin^2 \phi}} \left\{ 1 + \tilde{\omega} \frac{\cos \phi (1 + 4 \sin^2 \phi)}{(1 + 8 \sin^2 \phi)^2} \right\}. \tag{B12}
\]

This formula implies expansion (B11) with

\[ Q_1(\phi) = 2q_* [1 + 8 \sin^2 \phi + 3q_* (1 + 4 \sin^2 \phi)] \tag{B13} \]

and \( Q_2(\phi) = 1 + 8 \sin^2 \phi \).

Third, we consider the critical points with \( k_c = 1 \) and \( \theta_c = \pi - 2\pi n/3 \) for \( n = 1, 2 \), i.e., at \( \mathbf{k} = (0, \pm \sqrt{3}/2) \). We thus obtain the following expansions for \( q_* = q_*(\phi) \):

\[
q_* \cong \frac{\tilde{\omega}}{\sqrt{4 \cos^2 \phi + 3 + 2\sqrt{3} \sin(2\phi)}} \left\{ 1 - \tilde{\omega} \frac{2 \cos \phi (1 + 4 \sin^2 \phi)}{[4 \cos^2 \phi + 3 + 2\sqrt{3} \sin(2\phi)]^2} \right\}. \tag{B14}
\]

Each of these formulas implies expansion (B11) with
\[ Q_1(\phi) = 2q_1 \left\{ 4 \cos^2 \phi + 3 = 2\sqrt{3} \sin(2\phi) + 3q_1 \left[ 2 \cos \phi \cos(2\phi) + \sqrt{3} \sin \phi \right] + 2q_2 \right\} \]  

(B15)

and \( Q_2(\phi) = 4 \cos^2 \phi + 3 = 2\sqrt{3} \sin(2\phi) \).

In all of the above cases, we have \( Q_1(\phi) \neq 0 \) for every \( \phi \). The expansions for \( \epsilon(k) \) near local minima are uniform in \( \phi \); and capture the zero bandgap with a negligible correction of the order of \( \tilde{\omega}^{1/2} \) or smaller. This property can be used to show (as a self-consistency check) that our leading-order result for \( \Re \sigma_{xx}^R \), given below, has a negligible correction if \( \tilde{\omega} \ll 1 \). We omit details on this here.

Next, by Eq. (B8), we split the integral for \( \sigma_{xx}^R(\omega) \) into four contributions, one for each local minimum of \( \epsilon(k) \). Using the local polar coordinates \((q, \phi)\), we first carry out the integration in \( q \) by employing the formula

\[ \frac{1}{\epsilon(k)^2 - (\omega + i0^+)^2} = i\pi Q_1(\phi)^{-1} \delta(q - q_0(\phi)) + \mathcal{P} \left( \frac{1}{q - q_0} \right) \]

for each contribution. In the above, \( \mathcal{P}(\cdot) \) indicates the principal-value integral. The two-dimensional integral for \( \Re \sigma_{xx}^R \) immediately reduces to an integral with respect to the polar angle \( \phi \), from the delta function term.

Accordingly, we perform the remaining integration, with respect to \( \phi \). For \( \tilde{\omega} \ll 1 \), we write

\[ \Re \sigma_{xx}^R(\omega) \approx \frac{1}{2} g_t g_s g_c \sigma_G \left\{ I^{(c)} + 9 \sum_{n=0}^{2} I^{(n)} \right\} , \]  

(B16)

where \( I^{(c)} \) and \( I^{(n)} \) correspond to the center point \((k_c = 0)\) and the points \( k_c = 1 \) and \( \theta_c = \pi - 2\pi n / 3 \), respectively. We define and compute the following requisite integrals:

\[ I^{(c)} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \sin^2 \phi = \frac{1}{2} \]

and

\[ I^{(0)} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{\sin^2 \phi}{(1 + 8 \sin^2 \phi)^2} = \frac{1}{54} \].

Hence, we finally obtain

\[ \Re \sigma_{xx}^R(\omega) = \frac{3}{2} g_t g_s g_c \sigma_G = 12 \sigma_G \quad \text{as} \quad \tilde{\omega} \to 0 , \]

(B17)

for the TBG system. The anticipated correction to this result for small nonzero \( \tilde{\omega} \) is of the order of \( \tilde{\omega}^2 \).

**b. Limit \( \tilde{\omega} \to \infty \)**

In this case, we apply a procedure similar to the above. In particular, we solve the equation \( \epsilon(k) = \tilde{\omega} \) for large \( \tilde{\omega} \), to find \( k = k_\omega(\phi) \gg 1 \). Then we expand the difference \( \epsilon(k)^2 - \tilde{\omega}^2 \) in powers of \( k - k_\omega \). The quantity \( \Re \sigma_{xx}^R \) is determined by integration near the curve \( k = k_\omega(\phi) \).

In detail, by perturbations for \( \tilde{\omega} \gg 1 \), we obtain

\[ k_\omega = \sqrt{\tilde{\omega}} \left\{ 1 + \frac{1}{2} \tilde{\omega}^{-1/2} \cos(3\theta) + O(\tilde{\omega}^{-1}) \right\} . \]  

(B18)

Here, we use the polar angle \( \theta \), where \( k = k(\cos \theta, \sin \theta) \). The above formula implies expansion (B11) with \( q = k \) and \( \phi = \theta \) (since \( k = q \) here), while

\[ Q_1(\theta) = 2k_\omega \left[ 2k_\omega^2 - 3k_\omega \cos(3\theta) + 1 \right] \]  

(B19)

and \( Q_2(\theta) = 6k_\omega^2 - 6k_\omega \cos(3\theta) + 1 \).

These considerations lead to the simplified integral

\[ \Re \sigma_{xx}^R(\omega) \approx g_t g_s g_c \sigma_G \tilde{\omega}^{-2} \frac{1}{2\pi} \int_0^{2\pi} d\phi \ k_\omega(\theta)^4 \sin^2 \theta \]

\[ \approx \frac{1}{2} g_t g_s g_c \sigma_G = 4 \sigma_G \quad \text{as} \quad \tilde{\omega} \to +\infty , \]

(B20)

for the TBG system.

---

This is only obtained for twist angles which are already sufficiently close to the magic angle, i.e., $\theta = 1.05^\circ$ shows an avoided crossing, but still does not reach this plateau, yet.


