Exactly calculable field components of electric dipoles in planar boundary

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The Sommerfeld integrals for the electromagnetic fields in the planar boundary between air and a homogeneous, isotropic medium, due to a horizontal and a vertical electric dipole each lying along the interface, are examined in detail. In the case of the horizontal dipole, the tangential electric field is given in terms of series that involve confluent hypergeometric functions, namely, the Fresnel and exponential integrals. A similar exposition is presented for the magnetic and vertical electric fields of the vertical dipole. When the index of refraction of the adjacent space is of a sufficiently large magnitude, the derived series converge rapidly and uniformly with the distance from the source. Specifically, their rates of convergence are shown to be independent of distance. It is pointed out that the corresponding formulas of King *et al.* are valid down to any distance close to the source, where they smoothly connect to known ''quasi-static'' approximations. [DOI: 10.1063/1.1330731]

I. INTRODUCTION

Almost a century ago, Sommerfeld[1] first formulated the problem of the radiating vertical electric dipole located in the planar boundary between two homogeneous and isotropic half spaces by invoking the Hertz vector $\mathbf{\Pi}$ rather than the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. With the use of the Fourier–Bessel representations in cylindrical coordinates, Sommerfeld proposed approximate formulas for $\mathbf{\Pi}$ for distances of many wavelengths in air away from the source. Soon after, his student Hörschelman[2] applied the same method to the case of the horizontal electric dipole in air. Other authors revisited these problems aiming at alternative representations for $\mathbf{\Pi}$ that could be amenable to asymptotic evaluations for sufficiently large distances. A historical account and extensive list of references can be found in the monograph by Baños.[3]

Serious efforts to derive accurate expressions for the $\mathbf{\Pi}$ of electric or magnetic dipoles were often made under the simplifying yet practically significant assumption that both the source and the observation point lie at the interface.[3–10] Some of the components of $\mathbf{\Pi}$ then involve the Fourier–Bessel integrals,[3]

$$U(\rho)=\int_0^\infty d\lambda\lambda\,\frac{1}{\sqrt{\lambda^2-k_1^2}+\sqrt{\lambda^2-k_2^2}}\,J_0(\lambda\rho), \tag{1.1}$$

$$V(\rho)=\int_0^\infty d\lambda\lambda\,\frac{1}{k_2^2\sqrt{\lambda^2-k_1^2}+k_1^2\sqrt{\lambda^2-k_2^2}}\,J_0(\lambda\rho), \tag{1.2}$$

where k_j $(j=1,2)$ is the complex wave number in medium j, ρ is the polar distance from the source, and J_0 is the Bessel function of order 0. Van der Pol[6] showed that $U(\rho)$ is given in terms of elementary functions, while $V(\rho)$ can be converted to a finite, one-dimensional integral of an elementary function that readily yields Sommerfeld's approximate result. On the basis of Van der

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Pol's formula for $V(\rho)$, Rice[10] derived exact series expansions that, although credited as being ''uniform'' in the distance ρ,[11] become impractical when $|k_j\rho|\gg 1$. He also proposed disparate asymptotic expansions for these distances when the refraction index k_1/k_2 is close to 1, k_2 being the wave number in air. Along the same lines is the exposition by Wise.[7] Noteworthy is Fock's[9] expansion for $V(\rho)$ in terms of products of Bessel functions with half-integer indices, where the expansion parameter $(k_1^2-k_2^2)/(k_1^2+k_2^2)$ is assumed to be of magnitude less than 1. Baños[3] re-derived the Sommerfeld–Van der Pol formula by applying a version of the steepest descent method, where a simple pole is extracted from the vicinity of a saddle point, and neglecting high orders in k_2/k_1. However, the issue of connecting this formula, which is valid in the range $k_2^3\rho/|k_1^2|\geq O(1)$, to the respective approximation for $|k_1\rho|\ll 1$ was not essentially addressed.

In a series of works,[12] Wait gave asymptotic formulas for the Sommerfeld integrals in different ranges of polar distances and source heights. Consider, for example, the ranges $k_2\rho\ll 1$, $k_2\rho=O(1)$, and $k_2\rho\gg 1$ when the dipole and the observation point both lie in the boundary;[13] even in this simplest nontrivial case, Wait's approximations seem to be based on intuitive arguments. In particular, in the ''quasi-static approach,''[13] the fields in air are regarded as solutions of Laplace's or Poisson's equation with no practical restriction on $k_1\rho$, but there is no clear indication, for instance, about the convergence or the magnitude of the remainder of the underlying expansion when $k_1=O(k_2)$ with $k_2<|k_1|$. In the spirit of the quasi-static approach, the computation of the Hertz vector is carried out in Refs. 14–16 for low frequencies via a convenient resummation of the λ-Maclaurin expansion for the radical under the integral sign. The ensuing simple expressions are interpreted as superpositions of primary and reflected fields, where the earth is replaced by a perfectly conducting medium with the boundary being shifted by the distance $1/k_1$.[16] Notably, the electric and magnetic fields are obtained through direct differentiations of the approximate formulas for $\mathbf{\Pi}$.

Recently, integrated formulas were derived by King *et al.*[17] for the electromagnetic field in air over an imperfectly conducting or dielectric earth when the source is a horizontal or vertical electric dipole. Their major simplifying conditions are $k_2^2\ll|k_1^2|$ and $k_2r>O(1)$, r being the radial distance from the source. Some of the novelties of their approach can be outlined as follows. First, these authors deal directly and systematically with the field itself and not the Hertz vector; their set of formulas satisfy Maxwell's equations and the required boundary conditions consistently to the desired order in k_2^2/k_1^2. Second, in their sequence of approximation steps, the direct and the ideal-image fields are singled out, some of the remaining integrals are computed exactly by analytical means, and large-argument approximations for the Bessel functions are only applied to the remainders that involve the Sommerfeld pole. The results advance the works of Baños[3] and others both quantitatively, with the retainment of a larger number of terms, and qualitatively, with the notions of the surface and lateral waves being dissociated in the mathematical treatment from that of a saddle-point in the vicinity of a pole.

In a recent paper,[18] King and Wu make use of the approximate formulas of Ref. 17 for the horizontal dipole to calculate the electromagnetic field in air of infinitely long transmission lines above the earth. However, as pointed out in Ref. 18, the violation of the condition $|k_1r|>1$ at extremely low frequencies introduces an inaccuracy for the axial component of the electric field. A formula for this component that is uniform in distance was later derived in a more elaborate analysis by Margetis.[19] The inaccuracy mentioned above signifies one of the instances where approximate formulas that are known to hold sufficiently far from the source are forced to be extended to distances too close to the source. An interesting question is whether it is possible, and if so in what sense, to connect the lateral-wave formulas of King *et al.*[17] to known near-field expressions, such as those given by Wait for $k_2\rho\ll 1$,[13] so that the final formulas adequately describe the field for all reasonable distances when $k_2^2\ll|k_1^2|$.[20] Various interesting references and formulas for the evaluation of Sommerfeld-type integrals are provided in Ref. 21. Noteworthy among these formulas are the representations in terms of incomplete cylindrical functions.

The purpose of this paper is twofold. The first is to evaluate exactly, in terms of series that are uniform in ρ, those Sommerfeld integrals that are given by integrals of elementary functions, by relaxing the condition $k_2^2\ll|k_1^2|$. This task is carried out in Secs. III and IV for the electromagnetic

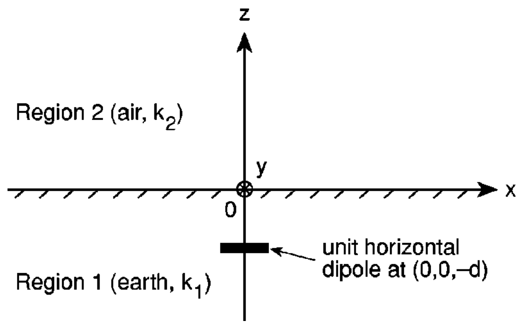

FIG. 1. The geometry and Cartesian coordinate system for a unit horizontal dipole in the earth. The height d is allowed to approach zero ($d\to 0^+$).

field of electric dipoles lying in the planar interface; the use of the Hertz vector is entirely avoided, in the spirit of Ref. 17. The expansion parameter is the inverse of the refraction index, k_2/k_1, which is assumed to be of magnitude less than 1, and the coefficients are known transcendental functions, namely, the exponential and Fresnel integrals. These series are believed to be new. In particular, the rates of convergence of the derived series are shown to depend only on the ratio k_2^2/k_1^2.[11] Emphasis is also placed on obtaining bounds and estimates for the remainders when a finite number of terms are summed. As a consequence, stringent conditions for the validity of simplifications under $k_2^2 \ll |k_1^2|$ can follow. All derivations are subject to routine mathematical rigor, and comparisons with numerical computations are beyond the scope of this paper.[22] A discussion on the merits of the present analysis for numerical evaluations is provided in Sec. VI.

The second purpose is to demonstrate that the corresponding lateral-wave formulas in Ref. 17 may indeed be extended to distances from the source that are short compared to the wavelength in air. In Sec. V we deal precisely with this task via the step-by-step approximations of the exact series. Finally, in Appendix A we calculate analytically a class of integrals involving Bessel functions through a generalized Schwinger–Feynman representation; Van der Pol's formula[6] essentially follows as a special case. The nature of the field asymptotic expansions for $k_2\rho \gg 1$ is analyzed in Appendix B on the basis of the derived series, while in Appendix C we revisit the simplifications of the original integrals in the limiting cases $k_2\rho \ll 1$ and $k_2\rho \gg 1$. The time dependence $e^{-i\omega t}$ is suppressed throughout the analysis.

II. FORMAL REPRESENTATIONS

A. Horizontal electric dipole

The geometry and Cartesian coordinate system are shown in Fig. 1. As the source and the observation point approach the boundary from below ($d\to 0^+$) and from above ($z\to 0^+$), respectively, the Fourier–Bessel representation for the electromagnetic field in the cylindrical coordinates (ρ,ϕ,z) with $x=\rho\cos\phi$ and $y=\rho\sin\phi$ $(0\le\phi<2\pi)$ is[17]

$$E_{2z}=\frac{k_1^2}{k_2^2}E_{1z}=\frac{i\omega\mu_0}{4\pi k_2^2}\int_0^\infty d\lambda\,\lambda^2\frac{k_2^2\sqrt{k_1^2-\lambda^2}-k_1^2\sqrt{k_2^2-\lambda^2}}{k_2^2\sqrt{k_1^2-\lambda^2}+k_1^2\sqrt{k_2^2-\lambda^2}}J_1(\lambda\rho)\cos\phi, \tag{2.1}$$

$$E_{2\phi}=E_{1\phi}=\frac{\omega\mu_0}{4\pi}\int_0^\infty d\lambda\,\lambda\left\{\frac{\sqrt{k_1^2-\lambda^2}\,\sqrt{k_2^2-\lambda^2}}{k_1^2\sqrt{k_2^2-\lambda^2}+k_2^2\sqrt{k_1^2-\lambda^2}}[J_0(\lambda\rho)+J_2(\lambda\rho)]\right.$$
$$\left.+\frac{1}{\sqrt{k_1^2-\lambda^2}+\sqrt{k_2^2-\lambda^2}}[J_0(\lambda\rho)-J_2(\lambda\rho)]\right\}\sin\phi, \tag{2.2}$$

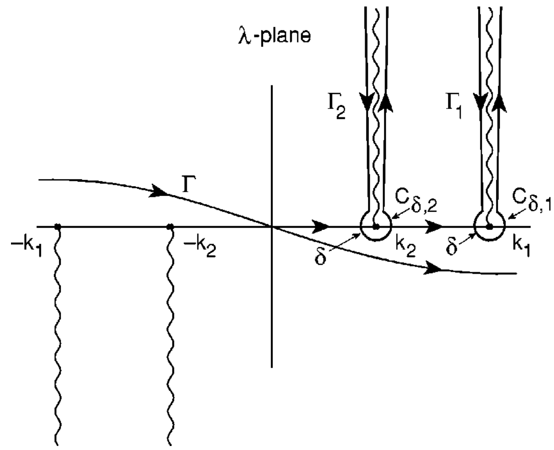

FIG. 2. Branch-cut configuration and integration paths pertaining to the Sommerfeld integrals (2.1)–(2.6) for the horizontal electric dipole and (2.10)–(2.12) for the vertical electric dipole. The original integration path is shown with arrows in the positive real axis. The contours Γ and Γ_j $(j=1,2)$ serve the asymptotic evaluations for $k_2\rho \gg 1$ carried out in Appendix C.

$$E_{2\rho}=E_{1\rho}=-\frac{\omega\mu_0}{4\pi}\int_0^\infty d\lambda\,\lambda\left\{\frac{\sqrt{k_1^2-\lambda^2}\sqrt{k_2^2-\lambda^2}}{k_1^2\sqrt{k_2^2-\lambda^2}+k_2^2\sqrt{k_1^2-\lambda^2}}[J_0(\lambda\rho)-J_2(\lambda\rho)]\right.$$
$$\left.+\frac{1}{\sqrt{k_1^2-\lambda^2}+\sqrt{k_2^2-\lambda^2}}[J_0(\lambda\rho)+J_2(\lambda\rho)]\right\}\cos\phi, \tag{2.3}$$

$$B_{2z}=B_{1z}=i\frac{\mu_0}{2\pi}\int_0^\infty d\lambda\,\lambda^2\frac{1}{\sqrt{k_1^2-\lambda^2}+\sqrt{k_2^2-\lambda^2}}J_1(\lambda\rho)\sin\phi, \tag{2.4}$$

$$B_{2\phi}=B_{1\phi}=-\frac{\mu_0}{8\pi}\int_0^\infty d\lambda\,\lambda\left\{\frac{\sqrt{k_2^2-\lambda^2}-\sqrt{k_1^2-\lambda^2}}{\sqrt{k_2^2-\lambda^2}+\sqrt{k_1^2-\lambda^2}}[J_0(\lambda\rho)+J_2(\lambda\rho)]\right.$$
$$\left.+\frac{k_2^2\sqrt{k_1^2-\lambda^2}-k_1^2\sqrt{k_2^2-\lambda^2}}{k_2^2\sqrt{k_1^2-\lambda^2}+k_1^2\sqrt{k_2^2-\lambda^2}}[J_0(\lambda\rho)-J_2(\lambda\rho)]\right\}\cos\phi, \tag{2.5}$$

$$B_{2\rho}=B_{1\rho}=-\frac{\mu_0}{8\pi}\int_0^\infty d\lambda\,\lambda\left\{\frac{\sqrt{k_2^2-\lambda^2}-\sqrt{k_1^2-\lambda^2}}{\sqrt{k_2^2-\lambda^2}+\sqrt{k_1^2-\lambda^2}}[J_0(\lambda\rho)-J_2(\lambda\rho)]\right.$$
$$\left.+\frac{k_2^2\sqrt{k_1^2-\lambda^2}-k_1^2\sqrt{k_2^2-\lambda^2}}{k_2^2\sqrt{k_1^2-\lambda^2}+k_1^2\sqrt{k_2^2-\lambda^2}}[J_0(\lambda\rho)+J_2(\lambda\rho)]\right\}\sin\phi, \tag{2.6}$$

the first subscript in each component referring to the region (1 for $z<0$ and 2 for $z>0$).

These integrals are divergent in the conventional sense. The procedure implied by allowing $d\to 0^+$ and $z\to 0^+$ in Fig. 1 dictates that they be interpreted in the sense of Abel.[23] The first Riemann sheet is such that $(j=1,2)$

$$\text{Im}\,\sqrt{k_j^2-\lambda^2}\geqslant 0,\quad \lambda>0, \tag{2.7}$$

with the branch-cut configuration of Fig. 2 where k_1 is taken to be real and $k_2<k_1$. Note that each $\sqrt{k_j^2-\lambda^2}$ is even in λ and the denominator,

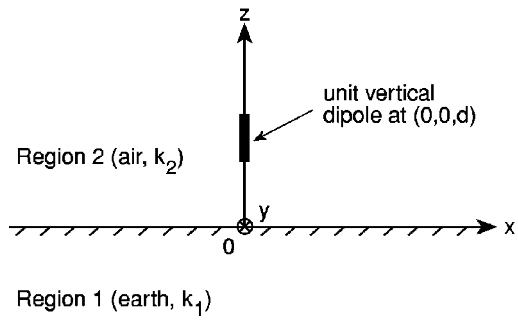

FIG. 3. The geometry and Cartesian coordinate system for a unit vertical dipole in air. The height d is allowed to approach zero ($d \to 0^+$).

$$D(\lambda) = k_1^2 \sqrt{k_2^2 - \lambda^2} + k_2^2 \sqrt{k_1^2 - \lambda^2}, \tag{2.8}$$

has four simple zeros in the Riemann surface. These are located at

$$\lambda = \pm k_S = \pm \frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}}, \tag{2.9}$$

and are not present in the first Riemann sheet.

B. Vertical electric dipole

The $\hat{\mathbf{z}}$-directed unit dipole is immersed in air (region 2, $z>0$), as depicted in Fig. 3. In the limit $d \to 0^+$ and $z \to 0^+$ the field is[17]

$$B_{2\phi} = B_{1\phi} = i \frac{\mu_0 k_1^2}{2\pi} \int_0^\infty d\lambda \lambda^2 \frac{1}{k_2^2 \sqrt{k_1^2 - \lambda^2} + k_1^2 \sqrt{k_2^2 - \lambda^2}} J_1(\lambda\rho), \tag{2.10}$$

$$E_{2z} = \frac{k_1^2}{k_2^2} E_{1z} = -\frac{\omega \mu_0 k_1^2}{2\pi k_2^2} \int_0^\infty d\lambda \lambda^3 \frac{1}{k_2^2 \sqrt{k_1^2 - \lambda^2} + k_1^2 \sqrt{k_2^2 - \lambda^2}} J_0(\lambda\rho) = \frac{i\omega}{k_2^2} \frac{1}{\rho} \frac{d}{d\rho} \rho B_{2\phi}, \tag{2.11}$$

$$E_{2\rho} = E_{1\rho} = -\frac{i\omega\mu_0}{4\pi k_2^2} \int_0^\infty d\lambda \lambda^2 \frac{k_2^2 \sqrt{k_1^2 - \lambda^2} - k_1^2 \sqrt{k_2^2 - \lambda^2}}{k_2^2 \sqrt{k_1^2 - \lambda^2} + k_1^2 \sqrt{k_2^2 - \lambda^2}} J_1(\lambda\rho). \tag{2.12}$$

The first Riemann sheet along with the branch-cut configuration, and the integration path are chosen as described in Sec. II A and shown in Fig. 2. Throughout the following analysis, it is assumed that

$$0 < k_2 < |k_1|, \quad 0 \leqslant \operatorname{Arg} \; k_1 < \pi/4. \tag{2.13}$$

III. EXACT B_{2z}, $E_{2\phi}$, AND $E_{2\rho}$ OF HORIZONTAL ELECTRIC DIPOLE

For mathematical convenience, consider the replacements

$$k_j = i q_j, \quad \sqrt{k_j^2 - \lambda^2} = i \sqrt{\lambda^2 + q_j^2} \quad (j = 1, 2). \tag{3.1}$$

The Sommerfeld pole corresponds to $q_S = -ik_S$. For the purpose of carrying out the requisite integrations, q_1 and q_2 are thought of as positive with $q_2 < q_1$, unless it is stated or implied otherwise. The final formulas are continued analytically to complex $q_j = -ik_j$ $(j=1,2)$ in view of restrictions (2.13).

A. The *z*-component of the magnetic field

It is verified that B_{2z} is expressed in terms of elementary functions.[13] The requisite integral equals

$$B_{2z} = \frac{\mu_0}{2\pi} \frac{1}{q_1^2 - q_2^2} \frac{1}{\rho^4} [\mathcal{I}_m(q_1\rho) - \mathcal{I}_m(q_2\rho)] \sin\phi, \tag{3.2}$$

where

$$\mathcal{I}_m(\alpha) = \int_0^\infty dx\, x^2 \sqrt{x^2+\alpha^2} J_1(x) = \frac{\alpha^{5/2}}{2^{-3/2}\Gamma\left(-\frac{1}{2}\right)} K_{5/2}(\alpha), \tag{3.3}$$

via the analytic continuation to $\mu = -3/2$ of the right-hand side of the equation,[24]

$$\int_0^\infty dx \frac{x^2 J_1(x)}{(x^2+\alpha^2)^{\mu+1}} = \frac{\alpha^{1-\mu}}{2^\mu \Gamma(\mu+1)} K_{1-\mu}(\alpha).$$

$K_\nu(\alpha)$ is the modified Bessel function of the third kind.[25] Hence,

$$\mathcal{I}_m(\alpha) = -\alpha^2 \left(1 + \frac{3}{\alpha} + \frac{3}{\alpha^2}\right) e^{-\alpha}. \tag{3.4}$$

It follows that

$$B_{2z} = -\frac{\mu_0}{2\pi} \frac{1}{(k_1^2 - k_2^2)\rho^2} \left[e^{ik_1\rho} k_1^2 \left(1 + \frac{3i}{k_1\rho} - \frac{3}{k_1^2\rho^2}\right) - e^{ik_2\rho} k_2^2 \left(1 + \frac{3i}{k_2\rho} - \frac{3}{k_2^2\rho^2}\right)\right] \sin\phi. \tag{3.5}$$

This result is also derived by Wait through differentiation of the Hertz vector.[13] Discussions on a similar integral appearing in the problem of the radiating vertical magnetic dipole can be found in the books by Baños[3] and Kong.[26]

B. The ϕ-component of the electric field

With the definition $q_S = -ik_S$, consider the decomposition

$$\frac{\sqrt{\lambda^2+q_1^2}\sqrt{\lambda^2+q_2^2}}{q_1^2\sqrt{\lambda^2+q_2^2} + q_2^2\sqrt{\lambda^2+q_1^2}} = \frac{1}{q_1^4 - q_2^4} [q_1^2 \sqrt{\lambda^2+q_1^2} - q_2^2 \sqrt{\lambda^2+q_2^2}] + \frac{q_1^3 q_2^3}{q_1^4 - q_2^4} \times \frac{1}{q_1^2+q_2^2} \left[\frac{q_2}{q_1} \frac{\sqrt{\lambda^2+q_1^2}}{\lambda^2+q_S^2} - \frac{q_1}{q_2} \frac{\sqrt{\lambda^2+q_2^2}}{\lambda^2+q_S^2} \right]. \tag{3.6}$$

Accordingly, Eq. (2.2) reads as

$$E_{2\phi} = -\frac{i\omega\mu_0}{2\pi}\frac{1}{q_1^2-q_2^2}\left\{\frac{d}{d\rho}\left[\frac{\mathcal{I}_e(q_1\rho)-\mathcal{I}_e(q_2\rho)}{\rho^2}\right]+\frac{1}{q_1^2+q_2^2}\frac{1}{\rho}\left[\frac{q_1^2\mathcal{I}_e(q_1\rho)-q_2^2\mathcal{I}_e(q_2\rho)}{\rho^2}\right]\right.$$
$$\left.+\frac{q_1^3q_2^3}{(q_1^2+q_2^2)^2}\frac{1}{\rho}\mathcal{W}(\rho)\right\}\sin\phi, \tag{3.7}$$

where

$$\mathcal{I}_e(\alpha) = \int_0^\infty dx\,\sqrt{x^2+\alpha^2}\,J_1(x) = \alpha + e^{-\alpha}, \tag{3.8}$$

with recourse to Ref. 24, and

$$\mathcal{W}(\rho) = \int_0^\infty d\lambda\left[\frac{q_2}{q_1}\frac{\sqrt{\lambda^2+q_1^2}}{\lambda^2+q_S^2}-\frac{q_1}{q_2}\frac{\sqrt{\lambda^2+q_2^2}}{\lambda^2+q_S^2}\right]J_1(\lambda\rho). \tag{3.9}$$

The task is to express $\mathcal{W}(\rho)$ in terms of known transcendental functions.

Following Van der Pol,[6] a first step is to convert the representation (3.9) into an integral of elementary functions. The radical in the integrand reads as follows:

$$\frac{q_2}{q_1}\frac{\sqrt{\lambda^2+q_1^2}}{\lambda^2+q_S^2}-\frac{q_1}{q_2}\frac{\sqrt{\lambda^2+q_2^2}}{\lambda^2+q_S^2} = \frac{1}{\lambda^2+q_S^2}\frac{\sqrt{\lambda^2+q_s^2x^2}}{\sqrt{x^2-1}}\Bigg|_{x=q_2/q_S}^{q_1/q_S}$$
$$= \int_{q_2/q_S}^{q_1/q_S} d[(x^2-1)^{-1/2}]\frac{1}{\sqrt{\lambda^2+q_S^2x^2}}. \tag{3.10}$$

The interchange of the order of integration yields

$$\mathcal{W}(\rho) = \int_{q_2/q_S}^{q_1/q_S} d\left(\frac{1}{\sqrt{x^2-1}}\right)\int_0^\infty d\lambda\,\frac{J_1(\lambda\rho)}{\sqrt{\lambda^2+q_S^2x^2}}, \tag{3.11}$$

where, from Ref. 24 or Eq. (A6) of Appendix A,

$$\int_0^\infty d\lambda\,\frac{J_1(\lambda\rho)}{\sqrt{\lambda^2+q_S^2x^2}} = \frac{1}{q_S\rho x}(1-e^{-q_S\rho x}). \tag{3.12}$$

Therefore, through integration by parts,

$$\mathcal{W}(\rho) = -\frac{1}{q_S\rho}\left[\frac{q_1-q_2}{q_S}-\frac{q_1e^{-q_2\rho}-q_2e^{-q_1\rho}}{q_S}+\rho W(\rho)\right], \tag{3.13}$$

where

$$W(\rho)=W(\rho;q_1,q_2,q_S), \quad W(\rho;\xi_1,\xi_2,\xi_3) = \int_{\xi_2}^{\xi_1} dv\,\frac{v}{\sqrt{v^2-\xi_3^2}}e^{-v\rho}, \quad \xi_3\leqslant\xi_2<\xi_1. \tag{3.14}$$

The procedure described hitherto is not different from the one in Ref. 27 for the Hertz vector of a vertical dipole. An alternative derivation of the last equation, that is amenable to generalizations, is provided in Appendix A. It is noted that $W(\rho)$ can be expressed in terms of incomplete cylindrical functions as further discussed in Sec. VI. Despite this fact, it is more advantageous to rewrite $W(\rho)$ as

$$W(\rho)=W(\rho;\infty,q_S,q_S)-W(\rho;\infty,q_1,q_S)-W(\rho;q_2,q_S,q_S). \tag{3.15}$$

The first term is calculated explicitly:[25]

$$W(\rho;\infty,q_S,q_S)=q_S\int_0^\infty dy\cosh y\, e^{-q_S\rho\cosh y}=q_S K_1(q_S\rho). \tag{3.16}$$

1. Integral $W(\rho;\infty,q_1,q_S)$

By invoking the identity

$$(1-u)^{-1/2}=\sum_{m=0}^{M-1}\frac{(\frac{1}{2})_m}{m!}u^m+\frac{(\frac{1}{2})_M}{(M-1)!}u^M\int_0^1 dt(1-t)^{M-1}(1-ut)^{-M-1/2}, \tag{3.17}$$

with $u=q_S^2 v^{-2}$ and a positive integer M, the second term in Eq. (3.15) reads as

$$W(\rho;\infty,q_1,q_S)=\frac{e^{-q_1\rho}}{\rho}\left[\sum_{m=0}^{M-1}U_m(\rho)+R_{1M}(\rho)\right], \tag{3.18}$$

where

$$U_m(\rho)=\frac{(\frac{1}{2})_m}{m!}(q_S\rho)^{2m}g_{2m}(q_1\rho), \tag{3.19}$$

$$g_n(z)=\int_z^\infty d\tau\,\tau^{-n}e^{-\tau+z}, \tag{3.20}$$

$$R_{1M}(\rho)=q_1\rho\frac{(\frac{1}{2})_M}{(M-1)!}\left(\frac{q_S}{q_1}\right)^{2M}\int_1^\infty d\eta\,\eta^{-2M}e^{-q_1\rho(\eta-1)}$$

$$\times\int_0^1 dt(1-t)^{M-1}(1-q_S^2q_1^{-2}\eta^{-2}t)^{-M-1/2} \tag{3.21a}$$

$$=q_1\rho\frac{(\frac{1}{2})_M}{M!}\left(\frac{q_S}{q_1}\right)^{2M}\int_1^\infty d\eta\,\eta^{-2M}{}_2F_1(M+\tfrac{1}{2},1;M+1;q_S^2q_1^{-2}\eta^{-2})e^{-q_1\rho(\eta-1)}. \tag{3.21b}$$

In the above, ${}_2F_1$ is the hypergeometric function[28] and $(a)_m$ is Pochhammer's symbol.[28]

By bearing in mind that $1-t\leqslant|1-wt|$ for $0\leqslant t\leqslant 1$ and $|w|\leqslant 1$, it is inferred that for admissible complex q_1 and q_2 ($\mathrm{Re}\, q_1\geqslant 0$),

$$|R_{1M}(\rho)|<|q_1\rho|(1-|q_S^2|/|q_1^2|)^{-3/2}\frac{(\frac{1}{2})_M}{M!}\left|\frac{q_S^2}{q_1^2}\right|^M,\quad M=1,2,\ldots, \tag{3.22a}$$

which can be used to prove the convergence of the corresponding series as $M\to\infty$. This relation must be supplemented with the formula

$$R_{1M}(\rho)\sim\frac{(\frac{1}{2})_M}{M!}{}_2F_1(M+\tfrac{1}{2},1;M+1;q_S^2/q_1^2)\frac{q_1\rho}{2M+q_1\rho}\left(\frac{q_S}{q_1}\right)^{2M},\quad |q_1\rho|\gg 1, \tag{3.22b}$$

in order to show that $|R_{1M}(\rho)|$ remains bounded as $|q_1\rho|\to\infty$. It is noted in passing that for $m=0,1,2,\ldots,$

$$ {}_2F_1(m+\tfrac{3}{2},1;m+2;z)=(-1)^m \frac{m+1}{(\frac{1}{2})_{m+1}}(1-z)^{-1/2}\frac{d^m}{dz^m}\left[\frac{(1-z)^m}{1+\sqrt{1-z}}\right]. $$

Use of the asymptotic formula ($\eta \gg 1$),

$$ \int_0^1 dt(1-t)^{M-1}(1-q_S^2 q_1^{-2}\eta^{-2}t)^{-M-1/2} \sim \frac{1}{M-1-(M+1/2)q_S^2 q_1^{-2}\eta^{-2}}, \quad M \gg 1, \tag{3.23} $$

in Eq. (3.21a) leads to

$$ R_{1M}(\rho) \sim \frac{(\frac{1}{2})_M}{M!}\left(\frac{q_S^2}{q_1^2}\right)^M (1-q_S^2/q_1^2)^{-1}\frac{q_1\rho}{2M+q_1\rho}, \quad M \gg 1. \tag{3.24} $$

By inspection of Eq. (3.14), the rate of convergence of the series from Eq. (3.18) is essentially independent of ρ. In particular, $U_m(\rho)$ is approximated by

$$ U_m(\rho) \sim \frac{q_1\rho}{2m+q_1\rho}\frac{(\frac{1}{2})_m}{m!}\left(\frac{q_S^2}{q_1^2}\right)^m, \quad m \gg 1. \tag{3.25} $$

This formula also holds when $m=O(1)$ and $|q_1\rho| \gg 1$ with $\mathrm{Re}\, q_1 \geqslant 0$, and becomes exact when $m=0$ for any $q_1\rho$. Hence,

$$ \frac{U_{m+1}(\rho)}{U_m(\rho)} \sim \frac{m+1/2}{m+1}\frac{2m+q_1\rho}{2m+2+q_1\rho}\frac{q_S^2}{q_1^2}, \quad |2m+q_1\rho| \gg 1. \tag{3.26} $$

In the sense of Cauchy for convergence,

$$ W(\rho;\infty,q_1,q_S) = \frac{e^{-q_1\rho}}{\rho}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m}{m!}(q_S\rho)^{2m} g_{2m}(q_1\rho). \tag{3.27} $$

The coefficients $g_n(q_1\rho)$ are partial derivatives in x of the generating function,

$$ \sum_{n=1}^{\infty} g_n(z)x^{n-1} = -e^{z-x}\,\mathrm{Ei}(x-z), \tag{3.28} $$

where $\mathrm{Ei}(-z)$ is the exponential integral.[28] Finally,

$$ g_n(z) = \begin{cases} 1, & n=0, \\ \dfrac{(-1)^n}{(n-1)!}\dfrac{d^{n-1}}{dz^{n-1}}[e^z \mathrm{Ei}(-z)], & n=1,2,\ldots . \end{cases} \tag{3.29} $$

An asymptotic expansion for $W(\rho;\infty,q_1,q_S)$ when $|q_1\rho| \gg 1$ is derived in Appendix B.

2. Integral $W(\rho;q_2,q_S,q_S)$

With the change of variable $\xi = v - q_S$ in the original integral from Eqs. (3.14) and use of the identity

$$(1+2u)(1+u)^{-1/2}=\sum_{m=0}^{M-1}(-1)^{m+1}\frac{(\frac{1}{2})_m}{m!}\frac{m+1/2}{m-1/2}u^m+(-1)^{M-1}\frac{(\frac{1}{2})_{M-1}}{(M-1)!}u^M$$
$$\times\int_0^1 dt(1-t)^{M-1}(M+1/2+ut)(1+ut)^{-M-1/2}, \tag{3.30}$$

where now $u=(2q_S)^{-1}\xi$, it is straightforward to get

$$W(\rho;q_2,q_S,q_S)=\frac{e^{-q_S\rho}}{\sqrt{2q_S}}\int_0^{q_2-q_S}\frac{d\xi}{\sqrt{\xi}}(\xi+q_S)\left(1+\frac{\xi}{2q_S}\right)^{-1/2}e^{-\xi\rho}, \tag{3.31a}$$

$$=-q_Se^{-q_2\rho}\left[\sum_{m=0}^{M-1}V_m(\rho)+R_{2M}(\rho)\right]. \tag{3.31b}$$

In the above,

$$V_m(\rho)=-e^{i\pi/4}i^{m+1}\frac{(\frac{1}{2})_m}{m!}\frac{m+1/2}{m-1/2}(2q_S\rho)^{-m-1/2}f_m(i(q_2-q_S)\rho), \tag{3.32}$$

$$f_m(z)=\int_0^z d\tau\tau^{m-1/2}e^{-i(z-\tau)}, \tag{3.33}$$

$$R_{2M}(\rho)=(-1)^M\frac{(\frac{1}{2})_{M-1}}{(M-1)!}\left(\frac{q_2-q_S}{2q_S}\right)^{M+1/2}\int_0^1 d\eta\,\eta^{M-1/2}e^{(q_2-q_S)\rho(1-\eta)}$$
$$\times\int_0^1 dt(1-t)^{M-1}\left(M+\frac{1}{2}+\frac{q_2-q_S}{2q_S}\eta t\right)\left(1+\frac{q_2-q_S}{2q_S}\eta t\right)^{-M-1/2}. \tag{3.34}$$

Because

$$\mathrm{Re}\,\frac{q_2-q_S}{2q_S}=\mathrm{Re}\,\frac{k_2-k_\mathrm{S}}{2k_\mathrm{S}}>0, \tag{3.35}$$

one may employ the inequality $|1+wt|\geqslant 1$ for $t\geqslant 0$ and Re $w>0$, to show that for complex q_1 and q_2 with Re $(q_2-q_S)\leqslant 0$,

$$|R_{2M}(\rho)|<\frac{(\frac{1}{2})_M}{M!}\frac{M+1}{M^2-1/4}\left|\frac{q_2-q_S}{2q_S}\right|^{M+1/2}, \quad M=1,2,\ldots. \tag{3.36a}$$

The convergence of the right-hand side of Eq. (3.31b) as $M\to\infty$ follows. Furthermore,

$$R_{2M}(\rho)\sim(-1)^M\frac{(\frac{1}{2})_M}{M!}\left[{}_2F_1(M+\tfrac{1}{2},1;M+1;-w)\right.$$
$$\left.+\frac{1}{M-1/2}\,{}_2F_1(M-\tfrac{1}{2},1;M+1;-w)\right]_{w=(q_2-q_S)/(2q_S)}$$
$$\times\frac{1}{M-1/2-(q_2-q_S)\rho}\left(\frac{q_2-q_S}{2q_S}\right)^{M+1/2}, \quad |(q_2-q_S)\rho|\gg 1. \tag{3.36b}$$

M is any positive integer. In the above, the hypergeometric functions reduce to elementary functions. For instance, by setting $M=1$ in the second line,

$$ {}_2F_1(\tfrac{1}{2},1;2;z)=\frac{2}{1+\sqrt{1-z}}. $$

On the other hand, by virtue of formula (3.23),

$$ R_{2M}(\rho)\sim(-1)^M\frac{(\frac{1}{2})_{M-1}}{M!}\left(\frac{q_2-q_S}{2q_S}\right)^{M+1/2}\int_0^1 d\eta\,\frac{M+1/2}{1+\frac{q_2-q_S}{2q_S}\eta}\,e^{(M-1/2)\ln\eta+(q_2-q_S)\rho(1-\eta)} $$
$$ \sim\frac{2q_S}{q_2+q_S}(-1)^M\frac{(\frac{1}{2})_M}{M!}\frac{M+1/2}{M-1/2}\frac{1}{M-1/2-(q_2-q_S)\rho}\left(\frac{q_2-q_S}{2q_S}\right)^{M+1/2},\quad M\gg 1. \tag{3.37} $$

When $|q_2\rho|\gg 1$, the corresponding sum needs to be combined with the asymptotic expansion for the modified Hankel function of Eq. (3.16), as discussed in Appendix B. In some analogy with expressions (3.25) and (3.26),

$$ V_m(\rho)\sim\frac{(-1)^m}{m-1/2-(q_2-q_S)\rho}\frac{(\frac{1}{2})_m}{m!}\frac{m+1/2}{m-1/2}\left(\frac{q_2-q_S}{2q_S}\right)^{m+1/2}, \tag{3.38} $$

which in turn leads to

$$ \frac{V_{m+1}(\rho)}{V_m(\rho)}\sim-\frac{m-\frac{1}{2}}{m+\frac{1}{2}}\frac{m+\frac{3}{2}}{m+1}\frac{m-\frac{1}{2}-(q_2-q_S)\rho}{m+\frac{1}{2}-(q_2-q_S)\rho}\frac{q_2-q_S}{2q_S},\quad |m-(q_2-q_S)\rho|\gg 1, \tag{3.39} $$

provided that m is a positive integer.

The aforementioned considerations indicate some rather attractive convergence properties of the series expansions when $|q_2^2|\ll|q_1^2|$. Their termwise differentiation with respect to ρ is legitimate and preserves the uniform-in-ρ convergence. The series from Eq. (3.31b) is

$$ W(\rho;q_2,q_S,q_S)=e^{i\pi/4}\frac{q_Se^{-q_2\rho}}{\sqrt{2q_S\rho}}\sum_{m=0}^{\infty} i^{m+1}\frac{(\frac{1}{2})_m}{m!}\frac{m+1/2}{m-1/2}(2q_S\rho)^{-m}f_m(i(q_2-q_S)\rho). \tag{3.40} $$

The generating function for $f_m(z)$ is

$$ \sum_{m=0}^{\infty}\frac{f_m(z)}{m!}(ix)^m=\sqrt{2\pi}e^{-iz}(1+x)^{-1/2}F_0((1+x)z), \tag{3.41} $$

where

$$ F_0(z)=\int_0^z dt\,\frac{e^{it}}{\sqrt{2\pi t}}=C(z)+iS(z), \tag{3.42} $$

and $C(z)$ and $S(z)$ are the Fresnel integrals,[28]

$$ C(z)=\int_0^z dx\,\frac{\cos x}{\sqrt{2\pi x}},\qquad S(z)=\int_0^z dx\,\frac{\sin x}{\sqrt{2\pi x}}. \tag{3.43} $$

Hence,

$$f_m(z)=\sqrt{2\pi}(-i)^m e^{-iz} z^{1/2+m} \frac{d^m}{dz^m}[z^{-1/2}F_0(z)]. \tag{3.44}$$

3. Exact formula for $E_{2\phi}$

The combination of Eqs. (3.15), (3.16), (3.27), and (3.40), with Eqs. (3.19) and (3.32), furnishes the desired expression for $W(\rho)$, viz.,

$$W(\rho)=q_S K_1(q_S\rho)+q_S e^{-q_2\rho}\sum_{m=0}^{\infty} V_m(\rho)-\frac{e^{-q_1\rho}}{\rho}\sum_{m=0}^{\infty} U_m(\rho). \tag{3.45}$$

From Eq. (3.13),

$$\mathcal{W}(\rho)=-i\frac{k_1-k_2}{k_S^2\rho}+i\frac{k_1 e^{ik_2\rho}-k_2 e^{ik_1\rho}}{k_S^2\rho}+\frac{\pi}{2}H_1^{(1)}(k_S\rho)-e^{ik_2\rho}\sum_{m=0}^{\infty} V_m(\rho)-\frac{e^{ik_1\rho}}{ik_S\rho}\sum_{m=0}^{\infty} U_m(\rho), \tag{3.46}$$

where

$$U_m(\rho)=\begin{cases} 1, & m=0,\\ \dfrac{(\frac{1}{2})_m}{m!(2m-1)!}(-k_S^2\rho^2)^m \dfrac{d^{2m-1}}{dz^{2m-1}}[e^z\ \mathrm{Ei}(-z)]_{z=-ik_1\rho}, & m=1,2,\ldots, \end{cases} \tag{3.47}$$

$$V_m(\rho)=\sqrt{2\pi}i^m\frac{(\frac{1}{2})_m}{m!}\frac{m+1/2}{m-1/2}\left(\frac{k_2-k_S}{2k_S}\right)^{m+1/2}\left\{e^{-iz}\frac{d^m}{dz^m}[z^{-1/2}F_0(z)]\right\}_{z=(k_2-k_S)\rho}. \tag{3.48}$$

The substitution of Eqs. (3.8) and (3.46) into (3.7) gives

$$\begin{aligned} E_{2\phi}=&-\frac{i\omega\mu_0}{2\pi}\frac{k_2^2}{k_1^4-k_2^4}\left\{-ie^{ik_1\rho}k_1\left[\frac{k_1^2+k_2^2(2-k_S/k_2)}{k_2^2\rho^2}+i\frac{k_1^2+2k_2^2}{k_1k_2^2\rho^3}\right]\right.\\ &+ie^{ik_2\rho}\frac{2k_1^2+k_2^2}{k_2\rho^2}\left(1+\frac{i}{k_2\rho}\right)-i\frac{k_1k_S^2}{k_2\rho}\left[\frac{i\pi}{2}H_1^{(1)}(k_S\rho)-ie^{ik_2\rho}\sum_{m=0}^{\infty}V_m(\rho)\right.\\ &\left.\left.-\frac{e^{ik_1\rho}}{k_S\rho}\sum_{m=0}^{\infty}U_{m+1}(\rho)\right]\right\}\sin\phi. \end{aligned} \tag{3.49}$$

Note that when $k_2^2 \ll |k_1^2|$, the argument of each Fresnel integral becomes

$$(k_2-k_S)\rho=k_2\left[1-\left(1+\frac{k_2^2}{k_1^2}\right)^{-1/2}\right]\rho\sim\frac{k_2^3\rho}{2k_1^2}=\wp, \tag{3.50}$$

where $|(k_2-k_S)\rho|$ is the Sommerfeld "numerical distance."[29]

For $k_2\rho \gg 1$, the Hankel function in Eq. (3.49) is approximated by an expansion with the phase factor $e^{ik_S\rho}$. This expansion exactly cancels terms produced by the Fresnel integrals, so that the final expression describes only waves traveling with the phase velocity of medium 1 or 2 (terms $\propto e^{ik_j\rho}$, $j=1,2$), as shown in Appendix B.

C. The ρ-component of the electric field

The integral for $E_{2\rho}$ is evaluated via the interchange of $1/\rho$ and the operator $(d/d\rho)$ in Eq. (3.7). The series that result through the term-by-term differentiation of expansions (3.27) and (3.40) also exhibit rapid convergence for $k_2^2 \ll |k_1^2|$, with a rate which is essentially independent of the distance ρ. Without further ado,

$$E_{2\rho}=\frac{i\omega\mu_0}{2\pi}\frac{k_2^2}{k_1^4-k_2^4}\left\{e^{ik_1\rho}\left(\frac{k_1^2-k_2^2}{k_2^2\rho^3}-ik_1\frac{k_1^2-k_2^2}{k_2^2\rho^2}+\frac{k_1^2}{\rho}\right)\right.$$
$$\left.+e^{ik_2\rho}\left(\frac{k_1^2-k_2^2}{k_2^2\rho^3}-i\frac{k_1^2-k_2^2}{k_2\rho^2}-\frac{k_1^2}{\rho}\right)-i\frac{k_1}{k_2}k_S W'(\rho)\right\}\cos\phi, \tag{3.51a}$$

$$=\frac{i\omega\mu_0}{2\pi}\frac{k_2^2}{k_1^4-k_2^4}\left\{e^{ik_1\rho}\left(\frac{k_1^2-k_2^2}{k_2^2\rho^3}-ik_1\frac{k_1^2-k_2^2}{k_2^2\rho^2}+\frac{k_1^2}{\rho}\right)\right.$$
$$+e^{ik_2\rho}\left(\frac{k_1^2-k_2^2}{k_2^2\rho^3}-i\frac{k_1^2-k_2^2}{k_2\rho^2}-\frac{k_1^2}{\rho}\right)+\frac{\pi}{2}\frac{k_1}{k_2}k_S^3 H_1^{(1)\prime}(k_S\rho)$$
$$\left.-\frac{k_1}{k_2}k_S^2\frac{d}{d\rho}\left[\frac{e^{ik_1\rho}}{ik_S\rho}\sum_{m=0}^{\infty}U_m(\rho)+e^{ik_2\rho}\sum_{m=0}^{\infty}V_m(\rho)\right]\right\}\cos\phi, \tag{3.51b}$$

where $W(\rho)$ is defined by Eq. (3.14) and the prime here denotes differentiation with respect to the argument.

It is desirable to further manipulate this formula. Decomposition (3.15) entails

$$W'(\rho)=ik_S^2\frac{\pi}{2}H_1^{(1)\prime}(k_S\rho)+\tilde{W}(\rho;\infty,-ik_1,-ik_S)+\tilde{W}(\rho;-ik_2,-ik_S,-ik_S), \tag{3.52a}$$

where

$$\tilde{W}(\rho;\xi_1,\xi_2,\xi_3)=\int_{\xi_2}^{\xi_1}dv\frac{v^2}{\sqrt{v^2-\xi_3^2}}e^{-v\rho}, \quad \xi_3\leqslant\xi_2<\xi_1. \tag{3.52b}$$

1. Integral $\tilde{W}(\rho;\infty,q_1,q_S)$

With the steps of Sec. III B and for $M=1,2,\ldots,$

$$\tilde{W}(\rho;\infty,q_1,q_S)=\frac{e^{-q_1\rho}}{\rho^2}(1+q_1\rho)-q_S^2e^{-q_1\rho}\left[\sum_{m=0}^{M-1}\tilde{U}_m(\rho)+\tilde{R}_{1M}(\rho)\right], \tag{3.53}$$

where

$$\tilde{U}_m(\rho)=-\frac{(\frac{1}{2})_{m+1}}{(m+1)!}(q_S\rho)^{2m}g_{2m+1}(q_1\rho), \quad m=0,1,2,\ldots, \tag{3.54}$$

with $g_n(z)$ defined by Eq. (3.20), and

$$\tilde{R}_{1M}(\rho)=-\frac{(\frac{1}{2})_{M+1}}{M!}\left(\frac{q_S}{q_1}\right)^{2M}\int_1^{\infty}d\eta\,\eta^{-2M-1}e^{-q_1\rho(\eta-1)}$$
$$\times\int_0^1 dt(1-t)^M(1-q_S^2q_1^{-2}\eta^{-2}t)^{-M-3/2}. \tag{3.55}$$

It is easily verified that

$$|\tilde{R}_{1M}(\rho)|<(1-|k_S^2|/|k_1^2|)^{-3/2}\frac{(\frac{1}{2})_{M+1}}{(M+1)!}\left|\frac{k_S^2}{k_1^2}\right|^M, \quad M=1,2,\ldots, \tag{3.56a}$$

$$\tilde{R}_{1M}(\rho)\sim-\frac{(\frac{1}{2})_{M+1}}{(M+1)!}\,{}_2F_1(M+\tfrac{3}{2},1;M+2;k_S^2/k_1^2)\,\frac{1}{2M+1-ik_1\rho}\left(\frac{k_S^2}{k_1^2}\right)^M, \quad |k_1\rho|\gg 1, \tag{3.56b}$$

$$\tilde{R}_{1M}(\rho)\sim-(1-k_S^2/k_1^2)^{-1}\frac{(\frac{1}{2})_{M+1}}{(M+1)!}\,\frac{1}{2M+1-ik_1\rho}\left(\frac{k_S^2}{k_1^2}\right)^M, \quad M\gg 1, \tag{3.56c}$$

while, for $|2m+1-ik_1\rho|\gg 1$,

$$\tilde{U}_m(\rho)\sim-\frac{(\frac{1}{2})_{m+1}}{(m+1)!}\,\frac{1}{2m+1-ik_1\rho}\left(\frac{k_S^2}{k_1^2}\right)^m, \tag{3.57a}$$

$$\frac{\tilde{U}_{m+1}(\rho)}{\tilde{U}_m(\rho)}\sim\frac{m+3/2}{m+2}\,\frac{2m+1-ik_1\rho}{2m+3-ik_1\rho}\,\frac{k_S^2}{k_1^2}. \tag{3.57b}$$

2. Integral $\tilde{W}(\rho;q_2,q_S,q_S)$

Likewise,

$$\tilde{W}(\rho;q_2,q_S,q_S)=q_S^2e^{-q_2\rho}\left[\sum_{m=0}^{M-1}\tilde{V}_m(\rho)+\tilde{R}_{2M}(\rho)\right], \tag{3.58}$$

where, for $m=0,1,2,\ldots$,

$$\tilde{V}_m(\rho)=-e^{i\pi/4}i^{m+1}\frac{(\frac{1}{2})_m}{m!}\,\frac{m^2+3/4}{(m-1/2)(m-3/2)}\,(2q_S\rho)^{-m-1/2}f_m(i(q_2-q_S)\rho), \tag{3.59}$$

with $f_m(z)$ defined by Eq. (3.44), and

$$\begin{aligned}\tilde{R}_{2M}(\rho)=&(-1)^M\frac{(\frac{1}{2})_{M-1}}{(M-1)!}\,\frac{1}{M-3/2}\left(\frac{q_2-q_S}{2q_S}\right)^{M+1/2}\int_0^1 d\eta\,\eta^{M-1/2}e^{(q_2-q_S)\rho(1-\eta)}\\&\times\int_0^1 dt(1-t)^{M-1}\left[M^2+\frac{3}{4}+\left(2M+3+3\,\frac{q_2-q_S}{2q_S}\,\eta t\right)\frac{q_2-q_S}{2q_S}\,\eta t\right]\\&\times\left(1+\frac{q_2-q_S}{2q_S}\,\eta t\right)^{-M-1/2}.\end{aligned} \tag{3.60}$$

It is of interest to note the relations

$$|\tilde{R}_{2M}(\rho)|<\frac{(\frac{1}{2})_M}{M!}\,\frac{M^2+M+3}{|M-3/2|(M^2-1/4)}\left|\frac{k_2-k_S}{2k_S}\right|^{M+1/2}, \quad M=1,2,\ldots, \tag{3.61a}$$

$$\tilde{R}_{2M}(\rho)\sim(-1)^M\frac{(\frac{1}{2})_M}{M!}\bigg[{}_2F_1(M+\tfrac{1}{2},1;M+1;-w)+\frac{2}{M-1/2}\,{}_2F_1(M-\tfrac{1}{2},1;M+1;-w)$$
$$+\frac{3}{(M-1/2)(M-3/2)}\,{}_2F_1(M-\tfrac{3}{2},1;M+1;-w)\bigg]_{w=(k_2-k_S)/(2k_S)}$$
$$\times\frac{1}{M-1/2+i(k_2-k_S)\rho}\left(\frac{k_2-k_S}{2k_S}\right)^{M+1/2},\quad |(k_2-k_S)\rho|\gg 1, \tag{3.61b}$$

$$\tilde{R}_{2M}(\rho)\sim\frac{2k_S}{k_2+k_S}(-1)^M\frac{(\frac{1}{2})_M}{M!}\,\frac{M^2+3/4}{(M-1/2)(M-3/2)}\,\frac{1}{M-1/2+i(k_2-k_S)\rho}$$
$$\times\left(\frac{k_2-k_S}{2k_S}\right)^{M+1/2},\quad M\gg 1. \tag{3.61c}$$

Again, the hypergeometric functions here are calculable in terms of elementary functions. For $m\gg 1$ and $|m+i(k_2-k_S)\rho|\gg 1$,

$$\tilde{V}_m(\rho)\sim(-1)^m\frac{(\frac{1}{2})_m}{m!}\,\frac{m^2+3/4}{(m-1/2)(m-3/2)}\,\frac{1}{m-1/2+i(k_2-k_S)\rho}\left(\frac{k_2-k_S}{2k_S}\right)^{m+1/2}, \tag{3.62a}$$

$$\frac{\tilde{V}_{m+1}(\rho)}{\tilde{V}_m(\rho)}\sim-\frac{m+1/2}{m+1}\,\frac{m-3/2}{m+1/2}\,\frac{(m+1)^2+3/4}{m^2+3/4}\,\frac{m-1/2+i(k_2-k_S)\rho}{m+1/2+i(k_2-k_S)\rho}\,\frac{k_2-k_S}{2k_S}. \tag{3.62b}$$

3. Exact formula for $E_{2\rho}$

It follows that in the limit $M\to\infty$ all series converge uniformly in ρ. $W'(\rho)$ from Eq. (3.52) reads as

$$W'(\rho)=e^{ik_1\rho}\left(\frac{1}{\rho^2}-\frac{ik_1}{\rho}\right)+k_S^2\left[\frac{i\pi}{2}H_1^{(1)'}(k_S\rho)-e^{ik_2\rho}\sum_{m=0}^{\infty}\tilde{V}_m(\rho)+e^{ik_1\rho}\sum_{m=0}^{\infty}\tilde{U}_m(\rho)\right]. \tag{3.63}$$

Finally, substituting $W'(\rho)$ in Eq. (3.51a) yields

$$E_{2\rho}=\frac{i\omega\mu_0}{2\pi}\,\frac{k_2^2}{k_1^4-k_2^4}\Bigg\{e^{ik_1\rho}\left[\frac{k_1^2-k_2^2}{k_2^2\rho^3}-ik_1\frac{k_1^2-k_2^2(1-k_S/k_2)}{k_2^2\rho^2}+\frac{k_1^2(1-k_S/k_2)}{\rho}\right]$$
$$+e^{ik_2\rho}\left(\frac{k_1^2-k_2^2}{k_2^2\rho^3}-i\,\frac{k_1^2-k_2^2}{k_2\rho^2}-\frac{k_1^2}{\rho}\right)$$
$$-i\frac{k_1}{k_2}k_S^3\left[\frac{i\pi}{2}H_1^{(1)'}(k_S\rho)-e^{ik_2\rho}\sum_{m=0}^{\infty}\tilde{V}_m(\rho)+e^{ik_1\rho}\sum_{m=0}^{\infty}\tilde{U}_m(\rho)\right]\Bigg\}\cos\phi, \tag{3.64}$$

where $\tilde{U}_m(\rho)$ and $\tilde{V}_m(\rho)$ are given by Eqs. (3.54) and (3.59). An asymptotic formula for $k_2\rho\gg 1$ can be derived along the lines of Appendix B.

IV. EXACT $B_{2\phi}$ AND E_{2z} OF VERTICAL ELECTRIC DIPOLE

A. Magnetic field

In consideration of Eq. (3.1) with $q_S=-ik_S$ and the decomposition,

$$\frac{1}{q_1^2\sqrt{\lambda^2+q_2^2}+q_2^2\sqrt{\lambda^2+q_1^2}}=\frac{q_1q_2}{q_1^4-q_2^4}\left[\frac{q_1}{q_2}\,\frac{\sqrt{\lambda^2+q_2^2}}{\lambda^2+q_S^2}-\frac{q_2}{q_1}\,\frac{\sqrt{\lambda^2+q_1^2}}{\lambda^2+q_S^2}\right], \tag{4.1}$$

Eq. (2.10) becomes

$$B_{2\phi}=-\frac{\mu_0}{2\pi}\,\frac{q_1^3q_2}{q_1^4-q_2^4}\int_0^\infty d\lambda\,\lambda^2\left[\frac{q_2}{q_1}\,\frac{\sqrt{\lambda^2+q_1^2}}{\lambda^2+q_S^2}-\frac{q_1}{q_2}\,\frac{\sqrt{\lambda^2+q_2^2}}{\lambda^2+q_S^2}\right]J_1(\lambda\rho). \tag{4.2}$$

From Eq. (3.10), one gets

$$B_{2\phi}=-\frac{\mu_0}{2\pi}\,\frac{q_1^3q_2}{q_1^4-q_2^4}\int_{q_2/q_S}^{q_1/q_S}d[(x^2-1)^{-1/2}]\int_0^\infty d\lambda\,\lambda^2\frac{J_1(\lambda\rho)}{\sqrt{\lambda^2+q_S^2x^2}}, \tag{4.3}$$

where

$$\int_0^\infty d\lambda\,\lambda^2\frac{J_1(\lambda\rho)}{\sqrt{\lambda^2+q_S^2x^2}}=\sqrt{\frac{2}{\pi\rho}}\,(q_Sx)^{3/2}K_{3/2}(q_S\rho x)=\frac{1}{\rho^2}(1+q_S\rho x)e^{-q_S\rho x}, \tag{4.4}$$

via the analytic continuation to $\nu=1$ of the right-hand side of the formula[24]

$$\int_0^\infty dx\,\frac{x^{\nu+1}J_\nu(ax)}{\sqrt{x^2+k^2}}=\frac{a^{-1/2}k^{\nu+1/2}}{2^{-1/2}\sqrt{\pi}}K_{\nu+1/2}(ka).$$

Alternatively,

$$B_{2\phi}=-\frac{\mu_0}{2\pi}\,\frac{q_1^3q_2}{q_1^4-q_2^4}\left\{\frac{1}{\rho^2}\left[\frac{q_2}{q_1}\mathcal{I}_e(q_1\rho)-\frac{q_1}{q_2}\mathcal{I}_e(q_2\rho)\right]-q_S^2\mathcal{W}(\rho)\right\}, \tag{4.5}$$

where $\mathcal{I}_e(\alpha)$ is given by Eq. (3.8) and $\mathcal{W}(\rho)$ is defined by Eq. (3.9). $B_{2\phi}$ can be expressed in terms of incomplete cylindrical functions.[27]

With the $W(\rho)$ introduced in Eq. (3.13), the exact $B_{2\phi}$ from Eq. (4.3) reads as

$$\begin{aligned}B_{2\phi}&=-\frac{\mu_0}{2\pi}\,\frac{q_1^3q_2}{q_1^4-q_2^4}\left\{\frac{1}{\rho^2}\left[e^{-q_1\rho}\frac{q_2}{q_1}(1+q_1\rho)-e^{-q_2\rho}\frac{q_1}{q_2}(1+q_2\rho)\right]+q_SW(\rho)\right\}\\&=-\frac{\mu_0}{2\pi}\,\frac{k_1^3k_2}{k_1^4-k_2^4}\left\{e^{ik_1\rho}\frac{k_2}{k_1}\left[-\frac{ik_1(1-k_S/k_2)}{\rho}+\frac{1}{\rho^2}\right]-e^{ik_2\rho}\frac{k_1}{k_2}\left(-\frac{ik_2}{\rho}+\frac{1}{\rho^2}\right)\right.\\&\quad\left.-ik_S^2\left[\frac{i\pi}{2}H_1^{(1)}(k_S\rho)-ie^{ik_2\rho}\sum_{m=0}^\infty V_m(\rho)-\frac{e^{ik_1\rho}}{k_S\rho}\sum_{m=0}^\infty U_{m+1}(\rho)\right]\right\}.\end{aligned} \tag{4.6}$$

U_m and V_m $(m=0,1,\ 2,\dots)$ are given by Eqs. (3.47) and (3.48). For obtaining an asymptotic formula for $B_{2\phi}$ when $k_2\rho\gg1$, one may follow the steps of Appendix B.

B. The z-component of the electric field

By use of Eqs. (4.1) and (3.10), Eq. (2.11) becomes

$$E_{2z}=\frac{i\omega\mu_0 q_1^3}{2\pi q_2}\frac{1}{q_1^4-q_2^4}\int_{q_2/q_S}^{q_1/q_S} d[(x^2-1)^{-1/2}]$$

$$\times\left[\int_0^\infty d\lambda\lambda\sqrt{\lambda^2+q_S^2x^2}J_0(\lambda\rho)-q_S^2x^2\int_0^\infty d\lambda\frac{\lambda J_0(\lambda\rho)}{\sqrt{\lambda^2+q_S^2x^2}}\right]. \tag{4.7}$$

After some straightforward algebra,[24]

$$E_{2z}=-\frac{i\omega\mu_0 k_1^3}{2\pi k_2}\frac{1}{k_1^4-k_2^4}\left\{e^{ik_1\rho}k_2\left[\frac{k_1(1-k_S^2/k_2^2)}{\rho}+\frac{i}{\rho^2}-\frac{1}{k_1\rho^3}\right]\right.$$

$$\left.-e^{ik_2\rho}k_1\left[\frac{k_2(1-k_S^2/k_1^2)}{\rho}+\frac{i}{\rho^2}-\frac{1}{k_2\rho^3}\right]-ik_S^3\check{W}(\rho)\right\}, \tag{4.8}$$

where

$$\check{W}(\rho)=\check{W}(\rho;-ik_1,-ik_2,-ik_S),\quad \check{W}(\rho;\xi_1,\xi_2,\xi_3)=\int_{\xi_2}^{\xi_1}\frac{d\upsilon}{\sqrt{\upsilon^2-\xi_3^2}}e^{-\upsilon\rho},\quad \xi_3\leqslant\xi_2<\xi_1. \tag{4.9}$$

Similar to Eq. (3.15),

$$\check{W}(\rho;q_1,q_2,q_S)=\check{W}(\rho;\infty,q_S,q_S)-\check{W}(\rho;\infty,q_1,q_S)-\check{W}(\rho;q_2,q_S,q_S), \tag{4.10}$$

the first term of which is calculated as[25]

$$\check{W}(\rho;\infty,q_S,q_S)=\int_0^\infty dt e^{-q_S\rho\cosh t}=K_0(q_S\rho). \tag{4.11}$$

1. Integral $\check{W}(\rho;\infty,q_1,q_S)$

On the basis of a term-by-term integration of Eq. (3.17),

$$\check{W}(\rho;\infty,q_1,q_S)=e^{-q_1\rho}\left[\sum_{m=0}^{M-1}\check{U}_m(\rho)+\check{R}_{1M}(\rho)\right],\quad M=1,2,\ldots, \tag{4.12}$$

where

$$\check{U}_m(\rho)=\frac{(\frac{1}{2})_m}{m!}(q_S\rho)^{2m}g_{2m+1}(q_1\rho)=-\frac{(\frac{1}{2})_m}{m!(2m)!}(-k_S^2\rho^2)^m\frac{d^{2m}}{dz^{2m}}[e^z\,\mathrm{Ei}(-z)]_{z=-ik_1\rho}, \tag{4.13}$$

$$\check{R}_{1M}(\rho)=\frac{(\frac{1}{2})_M}{(M-1)!}\left(\frac{q_S}{q_1}\right)^{2M}\int_1^\infty d\eta\,\eta^{-2M-1}e^{-q_1\rho(\eta-1)}$$

$$\times\int_0^1 dt(1-t)^{M-1}(1-q_S^2q_1^{-2}\eta^{-2}t)^{-M-1/2}$$

$$=\frac{(\frac{1}{2})_M}{M!}\left(\frac{q_S}{q_1}\right)^{2M}\int_1^\infty d\eta\,\eta^{-2M-1}{}_2F_1(M+\tfrac{1}{2},1;M+1;q_S^2q_1^{-2}\eta^{-2})e^{-q_1\rho(\eta-1)}. \tag{4.14}$$

Appealing properties of the series expansion ensue from the relations

$$|\check{R}_{1M}(\rho)|<\frac{1}{2}(1-|k_S^2|/|k_1^2|)^{-3/2}\frac{(\frac{1}{2})_M}{M!}\left|\frac{k_S^2}{k_1^2}\right|^M, \quad M=1,2,\ldots, \tag{4.15a}$$

$$\check{R}_{1M}(\rho)\sim\frac{(\frac{1}{2})_M}{M!}\,{}_2F_1(M+\tfrac{1}{2},1;M+1;k_S^2/k_1^2)\,\frac{1}{2M+1-ik_1\rho}\left(\frac{k_S^2}{k_1^2}\right)^M, \quad |k_1\rho|\gg 1, \tag{4.15b}$$

$$\check{R}_{1M}(\rho)\sim(1-k_S^2/k_1^2)^{-1}\frac{(\frac{1}{2})_M}{M!}\,\frac{1}{2M+1-ik_1\rho}\left(\frac{k_S^2}{k_1^2}\right)^M, \quad M\gg 1, \tag{4.15c}$$

and, with $|2m+1-ik_1\rho|\gg 1$,

$$\check{U}_m(\rho)\sim\frac{(\frac{1}{2})_m}{m!}\,\frac{1}{2m+1-ik_1\rho}\left(\frac{k_S^2}{k_1^2}\right)^m, \tag{4.16a}$$

$$\frac{\check{U}_{m+1}(\rho)}{\check{U}_m(\rho)}\sim\frac{m+1/2}{m+1}\,\frac{2m+1-ik_1\rho}{2m+3-ik_1\rho}\,\frac{k_S^2}{k_1^2}. \tag{4.16b}$$

As $M\to\infty$, the remainder $\check{R}_{1M}(\rho)$ approaches zero while being bounded uniformly in distance. The rate of convergence of the exact series is independent of ρ.

2. Integral $\check{W}(\rho;q_2,q_S,q_S)$

By use of Eq. (3.17),

$$\check{W}(\rho;q_2,q_S,q_S)=e^{-q_2\rho}\left[\sum_{m=0}^{M-1}\check{V}_m(\rho)+\check{R}_{2M}(\rho)\right], \tag{4.17}$$

where

$$\begin{aligned}\check{V}_m(\rho)&=-e^{i\pi/4}i^{m+1}\frac{(\frac{1}{2})_m}{m!}(2q_S\rho)^{-m-1/2}f_m(i(q_2-q_S)\rho)\\ &=\sqrt{2\pi}i^m\frac{(\frac{1}{2})_m}{m!}\left(\frac{k_2-k_S}{2k_S}\right)^{m+1/2}\left\{e^{-iz}\frac{d^m}{dz^m}[z^{-1/2}F_0(z)]\right\}_{z=(k_2-k_S)\rho},\end{aligned} \tag{4.18}$$

$$\begin{aligned}\check{R}_{2M}(\rho)&=(-1)^M\frac{(\frac{1}{2})_M}{(M-1)!}\left(\frac{q_2-q_S}{2q_S}\right)^{M+1/2}\int_0^1 d\eta\,\eta^{M-1/2}e^{(q_2-q_S)\rho(1-\eta)}\\ &\quad\times\int_0^1 dt(1-t)^{M-1}\left(1+\frac{q_2-q_S}{2q_S}\,\eta t\right)^{-M-1/2}\\ &=(-1)^M\frac{(\frac{1}{2})_M}{M!}\left(\frac{q_2-q_S}{2q_S}\right)^{M+1/2}\\ &\quad\times\int_1^\infty d\eta\,\eta^{M-1/2}{}_2F_1\left(M+\tfrac{1}{2},1;M+1;-\frac{q_2-q_S}{2q_S}\,\eta\right)e^{(q_2-q_S)\rho(1-\eta)}.\end{aligned} \tag{4.19}$$

$F_0(z)$ is defined by Eq. (3.42).

In analogy with expressions (4.15) and (4.16),

$$|\check{R}_{2M}(\rho)|<\frac{(\frac{1}{2})_M}{M!}\frac{1}{M+1/2}\left|\frac{k_2-k_S}{2k_S}\right|^{M+1/2}, \quad M=1,2,\ldots, \tag{4.20a}$$

$$\check{R}_{2M}(\rho)\sim(-1)^M\frac{(\frac{1}{2})_M}{M!}\,{}_2F_1\left(M+\tfrac{1}{2},1;M+1;-\frac{k_2-k_S}{2k_S}\right)$$
$$\times\frac{1}{M-1/2+i(k_2-k_S)\rho}\left(\frac{k_2-k_S}{2k_S}\right)^{M+1/2}, \quad |(k_2-k_S)\rho|\gg 1, \tag{4.20b}$$

$$\check{R}_{2M}(\rho)\sim\frac{2k_S}{k_2+k_S}(-1)^M\frac{(\frac{1}{2})_M}{M!}\frac{1}{M-1/2+i(k_2-k_S)\rho}\left(\frac{k_2-k_S}{2k_S}\right)^{M+1/2}, \quad M\gg 1, \tag{4.20c}$$

while for $|m+i(k_2-k_S)\rho|\gg 1$ and $m=1,2,\ldots,$

$$\check{V}_m(\rho)\sim(-1)^m\frac{(\frac{1}{2})_m}{m!}\frac{1}{m-1/2+i(k_2-k_S)\rho}\left(\frac{k_2-k_S}{2k_S}\right)^{m+1/2}, \tag{4.21a}$$

$$\frac{\check{V}_{m+1}(\rho)}{\check{V}_m(\rho)}\sim-\frac{m+1/2}{m+1}\frac{m-1/2+i(k_2-k_S)\rho}{m+1/2+i(k_2-k_S)\rho}\frac{k_2-k_S}{2k_S}. \tag{4.21b}$$

3. Exact formula for E_{2z}

By virtue of Eqs. (4.8)–(4.10),

$$E_{2z}=-\frac{i\omega\mu_0k_1^3}{2\pi k_2}\frac{1}{k_1^4-k_2^4}\left\{e^{ik_1\rho}k_2\left[\frac{k_1(1-k_S^2/k_2^2)}{\rho}+\frac{i}{\rho^2}-\frac{1}{k_1\rho^3}\right]\right.$$
$$-e^{ik_2\rho}k_1\left[\frac{k_2(1-k_S^2/k_1^2)}{\rho}+\frac{i}{\rho^2}-\frac{1}{k_2\rho^3}\right]$$
$$\left.-ik_S^3\left[\frac{i\pi}{2}H_0^{(1)}(k_S\rho)-e^{ik_2\rho}\sum_{m=0}^{\infty}\check{V}_m(\rho)-e^{ik_1\rho}\sum_{m=0}^{\infty}\check{U}_m(\rho)\right]\right\}. \tag{4.22}$$

$\check{U}_m(\rho)$ and $\check{V}_m(\rho)$ are defined by Eqs. (4.13) and (4.18).

V. SIMPLIFIED FORMULAS FOR $k_2\ll|k_1|$

The exact results of Secs. III and IV are simplified considerably under the condition

$$k_2\ll|k_1|, \tag{5.1}$$

which holds in many cases of practical interest. In this section, connection formulas for the approximations of Appendix C are recovered to the leading order in k_2/k_1.

A. Horizontal electric dipole

1. The z-component of the magnetic field

Equation (3.5) for B_{2z} becomes

$$B_{2z}\sim\frac{\mu_0}{2\pi k_1^2}\left[e^{ik_2\rho}\left(\frac{k_2^2}{\rho^2}+\frac{3ik_2}{\rho^3}-\frac{3}{\rho^4}\right)-e^{ik_1\rho}\left(\frac{k_1^2}{\rho^2}+\frac{3ik_1}{\rho^3}-\frac{3}{\rho^4}\right)\right]\sin\phi, \tag{5.2}$$

which is identical to the result given in Ref. 17 and agrees with formulas (C4), (C20), and (C21) of Appendix C. Note that condition (5.1) is redundant for establishing a smooth connection to formula (C4).

2. Tangential electric field

With

$$U_1(\rho)=-\frac{k_S^2\rho^2}{2}\left[e^{-ik_1\rho}\,\mathrm{Ei}(ik_1\rho)-\frac{1}{ik_1\rho}\right], \tag{5.3}$$

$$V_0(\rho)=-e^{i\pi/4}e^{-i(k_2-k_S)\rho}\sqrt{\frac{\pi}{2k_S\rho}}+\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho), \tag{5.4}$$

$E_{2\phi}$ of Eq. (3.49) becomes

$$E_{2\phi}\sim-\frac{i\omega\mu_0}{2\pi k_1^2}\left\{e^{ik_2\rho}k_2\left[\frac{2i}{\rho^2}\left(1+\frac{k_2^2}{2k_1^2}\right)-\frac{2}{k_2\rho^3}\left(1+\frac{k_2^2}{2k_1^2}\right)-\frac{k_S^2}{k_1\rho}\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho)\right.\right.$$
$$\left.+\frac{\pi k_S^2}{2k_1\rho}e^{-i(k_2-k_S)\rho}\left(e^{-ik_S\rho}H_1^{(1)}(k_S\rho)+e^{i\pi/4}\sqrt{\frac{2}{\pi k_S\rho}}\right)\right]-e^{ik_1\rho}k_1\left[\frac{i}{\rho^2}\left(1+\frac{k_2^2}{k_1^2}\right)\right.$$
$$\left.\left.-\frac{1}{k_1\rho^3}\left(1+\frac{2k_2^2}{k_1^2}\right)+\frac{ik_2^4}{2k_1^2}\left(e^{-ik_1\rho}\,\mathrm{Ei}(ik_1\rho)-\frac{1}{ik_1\rho}\right)\right]\right\}\sin\phi. \tag{5.5}$$

A close inspection of the terms inside the parentheses containing the exponential integral Ei shows that these contribute to higher orders in k_2^2/k_1^2. On the other hand, the Hankel function and its accompanying term are negligible for $k_2\rho\leqslant O(1)$ under condition (5.1), while they are cancelled by the V_m's when $k_2\rho\gg 1$, as outlined in Appendix B. A moment's reflection leads to the uniform formula

$$E_{2\phi}\sim-\frac{i\omega\mu_0}{2\pi k_1^2}\left\{e^{ik_2\rho}\left[\frac{2ik_2}{\rho^2}-\frac{2}{\rho^3}-\frac{k_S^2k_2}{k_1\rho}\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho)\right]-e^{ik_1\rho}\left(\frac{ik_1}{\rho^2}-\frac{1}{\rho^3}\right)\right\}\sin\phi, \tag{5.6}$$

valid for all distances that are consistent with the planar-earth model. This formula yields approximation (C6) as well as (C24) and (C25) of Appendix C when $k_2\rho\ll 1$ and $k_2\rho\gg 1$, respectively.

Similar steps can be taken for $E_{2\rho}$ of Eq. (3.64), to obtain

$$E_{2\rho}\sim-\frac{i\omega\mu_0}{2\pi}\frac{k_2^2}{k_1^2}\left\{e^{ik_2\rho}\left[-\frac{1-k_2^2/k_1^2}{k_2^2\rho^3}+i\frac{1-k_2^2/k_1^2}{k_2\rho^2}+\frac{1}{\rho}+\frac{ik_S^3}{k_1k_2}\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho)\right.\right.$$
$$\left.-\frac{\pi k_S^3}{2k_1k_2}e^{-i(k_2-k_S)\rho}\left(e^{-ik_S\rho}H_1^{(1)\prime}(k_S\rho)-e^{-i\pi/4}\sqrt{\frac{2}{\pi k_S\rho}}\right)\right]$$
$$\left.-e^{ik_1\rho}\left[\frac{1-k_2^2/k_1^2}{k_2^2\rho^3}-\frac{ik_1}{k_2^2\rho^2}-\frac{ik_S^3}{2k_1k_2}\left(e^{-ik_1\rho}\,\mathrm{Ei}(ik_1\rho)-\frac{k_2^3}{k_S^3}\frac{1}{ik_1\rho}\right)\right]\right\}\cos\phi. \tag{5.7}$$

In consideration of the asymptotic expansion (B18) of Appendix B, it is inferred that

$$E_{2\rho}\sim-\frac{i\omega\mu_0}{2\pi k_1^2}\left\{e^{ik_2\rho}k_2\left[\frac{k_2}{\rho}+\frac{i}{\rho^2}-\frac{1}{k_2\rho^3}+\frac{ik_S^3}{k_1k_2}\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho)\right]-e^{ik_1\rho}k_1\left(\frac{1}{k_1\rho^3}-\frac{i}{\rho^2}\right)\right\}\cos\phi. \tag{5.8}$$

This formula is useful for all reasonable purposes yet it assumes that

$$k_2\rho\ll|k_1^5/k_2^5|, \tag{5.9}$$

which poses no practical restriction. The formula agrees with approximations (C5), (C22), and (C23) of Appendix C.

B. Vertical electric dipole

1. Magnetic field

Equation (4.6) for $B_{2\phi}$ furnishes

$$B_{2\phi}\sim-\frac{\mu_0}{2\pi}\left\{e^{ik_2\rho}\left[\frac{ik_2}{\rho}-\frac{1}{\rho^2}-\frac{k_2k_S^2}{k_1}\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho)+\frac{\pi k_S^2k_2}{2k_1}e^{-i(k_2-k_S)\rho}\left(e^{-ik_S\rho}H_1^{(1)}(k_S\rho)+e^{i\pi/4}\sqrt{\frac{2}{\pi k_S\rho}}\right)\right]+e^{ik_1\rho}\frac{k_2^2}{k_1^2}\times\left[-\frac{ik_2^2}{2k_1\rho}\left(1-\frac{3}{4}\frac{k_2^2}{k_1^2}\right)+\frac{1}{\rho^2}-\frac{ik_1k_S^3\rho}{2k_2}\left(e^{-ik_1\rho}\,\mathrm{Ei}(ik_1\rho)-\frac{1}{ik_1\rho}\right)\right]\right\}. \tag{5.10}$$

Under the sensible condition

$$k_2\rho\ll|k_1^3/k_2^3|, \tag{5.11}$$

the exponential integral and its accompanying term can be neglected. Consequently,

$$B_{2\phi}\sim-\frac{\mu_0}{2\pi}\left\{e^{ik_2\rho}\left[\frac{ik_2}{\rho}-\frac{1}{\rho^2}-\frac{k_2k_S^2}{k_1}\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho)\right]+e^{ik_1\rho}\frac{k_2^2}{k_1^2}\frac{1}{\rho^2}\right\}, \tag{5.12}$$

which connects smoothly to expressions (C10), and (C32) and (C33) of Appendix C.

2. The *z*-component of the electric field

The retainment of the first term in each series of Eq. (4.22) for E_{2z} yields

$$E_{2z}\sim-\frac{i\omega\mu_0}{2\pi k_2}\left\{-e^{ik_2\rho}\left[\frac{k_2(1-k_2^2/k_1^2)}{\rho}+\frac{i}{\rho^2}-\frac{1}{k_2\rho^3}+\frac{ik_S^3}{k_1}\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho)-\frac{\pi k_S^3}{2k_1}e^{-i(k_2-k_S)\rho}\left(e^{-ik_S\rho}H_0^{(1)}(k_S\rho)-e^{-i\pi/4}\sqrt{\frac{2}{\pi k_S\rho}}\right)\right]+e^{ik_1\rho}\frac{k_2}{k_1}\left[-\frac{k_1}{\rho}\frac{k_2^4}{k_1^4}+\frac{i}{\rho^2}-\frac{1}{k_1\rho^3}-\frac{ik_S^3}{k_2}\left(e^{-ik_1\rho}\,\mathrm{Ei}(ik_1\rho)-\frac{k_2^3}{k_S^3}\frac{1}{ik_1\rho}\right)\right]\right\}. \tag{5.13}$$

With condition (5.11), the preceding expression becomes

$$E_{2z} \sim -\frac{i\omega\mu_0}{2\pi k_2}\left\{ -e^{ik_2\rho}\left[\frac{k_2}{\rho} + \frac{i}{\rho^2} - \frac{1}{k_2\rho^3} + \frac{ik_S^3}{k_1}\sqrt{\frac{\pi}{k_S\rho}}F((k_2-k_S)\rho)\right] + e^{ik_1\rho}\frac{k_2}{k_1}\left(\frac{i}{\rho^2} - \frac{1}{k_1\rho^3}\right)\right\}, \tag{5.14}$$

in agreement with approximations (C11), and (C34) and (C35) of Appendix C.

Approximations (5.6), (5.8), (5.12), and (5.14) are in full agreement with the formulas of King *et al.*,[17] provided that the replacement of $(k_2-k_S)\rho$ by $\wp$ is made according to (3.50).

VI. CONCLUSIONS AND DISCUSSION

We start this paper with the Fourier–Bessel integral representations for the fields in the planar boundary between air and a homogeneous half space of infinitesimal electric dipoles lying in the interface. The focus is on the components E_ρ and E_ϕ of the horizontal dipole and B_ϕ and E_z of the vertical dipole in the cylindrical coordinates of Figs. 1 and 3. These components can be given by one-dimensional integrals of elementary functions, as is known from previous works on the Hertz vector.[4–10] The present analysis is believed to go a step further by relaxing the condition $k_2^2 \ll |k_1^2|$ and replacing the integrals by simple, exact integrated series which are usable for any distance from the source. It is verified that the B_z component of a horizontal dipole is described by simple elementary functions.

The exposition bears two appealing features. The first feature is that the ratio of any successive terms in each series is shown to be proportional to k_2^2/k_1^2, i.e., the inverse of the diffraction index squared, while it remains bounded uniformly in ρ. The relative errors due to the retainment of a finite number of terms in the series, say M, are essentially of the order of $(k_2^2/k_1^2)^M$ regardless of $k_2\rho$ and $k_1\rho$. In most cases of practical interest where $|k_1| \geqslant 3k_2$, at most three or four terms of each expansion suffice for reasonable accuracy.

The second feature is that the summands are expressed in simple closed form as the well-known exponential and Fresnel integrals. These functions explicitly reveal the dependence on the physical parameters such as the $k_1\rho$ and the Sommerfeld numerical distance. They also provide a natural connection to the recently obtained, approximate formulas of King *et al.* that distinguish between the direct and ideal-image fields and the lateral-wave or surface-wave contributions when $k_2^2 \ll |k_1^2|$.[17] The present treatment not only verifies the results of these authors by different, exact means, but also extends their validity to distances close to the source.

The two features mentioned above illustrate the advantages of the proposed formulas over representations of the incomplete Hankel function used for the same purpose.[27] The price that one seems to pay for this simplicity, however, is the limitation in the choice of possible configurations or field components that can be treated exactly in a similar fashion.[21] Obtaining integrated series of analogous properties for the remaining components of Eqs. (2.1)–(2.6) and (2.10)–(2.12) is an open problem for future work.

Higher-order terms of the derived series may become of importance for radiowave propagation over a very dry earth; another example of applications could perhaps be related to the so-called ''low-k'' dielectric insulators.[30] As in Ref. 19, the present model is restricted in its applicability due to the assumption of a planar boundary. Lowest-order correction formulas to take into account the effect of a finite yet sufficiently large radius of curvature are given elsewhere.[31–34]

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APPENDIX A: GENERALIZED FORMULA FOR INTEGRAL $\mathcal{W}(\rho)$

In this appendix, a generalized integration procedure is described which leads to Eq. (3.13) as a special case. More precisely, with the decomposition

$$\mathcal{W}(\rho) = \frac{q_2}{q_1} I(q_1\rho, q_S\rho; -1/2) - \frac{q_1}{q_2} I(q_2\rho, q_S\rho; -1/2), \tag{A1}$$

attention is focused on the integral

$$I(\alpha,\beta;\zeta) = \int_0^\infty dx \frac{(x^2+\alpha^2)^{-\zeta}}{x^2+\beta^2} J_1(x), \quad \alpha \geqslant \beta > 0. \tag{A2}$$

Of course, I is understood as Abel summable in $x \to \infty$. Without loss of generality, α and β are assumed to be positive. The inequality $\alpha \geqslant \beta$ is imposed for definiteness.

1. Case $\alpha=\beta$

With $\alpha=\beta$,

$$I(\alpha,\alpha;\zeta) = \int_0^\infty dx (x^2+\alpha^2)^{-\zeta-1} J_1(x). \tag{A3}$$

The starting point is the known formula[24]

$$\int_0^\infty dx\, x^{\nu+1} \frac{J_\nu(x)}{(x^2+\alpha^2)^{1+\zeta}} = \frac{2^{-\zeta}\alpha^{\nu-\zeta}}{\Gamma(1+\zeta)} K_{\nu-\zeta}(\alpha), \quad -1 < \mathrm{Re}\,\nu < 2\,\mathrm{Re}\,\zeta + 3/2, \tag{A4}$$

where $K_{\nu-\zeta}$ is the modified Bessel function of the third kind. Note that one may not set $\nu=-1$ on both sides of this equation simultaneously. Caution needs to be exercised because allowing $\nu \to -1^+$ in

$$x^{\nu+1} J_\nu(x) = O(x^{2\nu+1}) \text{ as } x \to 0^+,$$

results in a nonintegrable singularity at $x=0$ with a vanishing numerical coefficient.

A remedy is to employ the integral

$$\int_0^\infty dx\, x^{\nu+1} J_\nu(x) = 0, \ \mathrm{Re}\,\nu > -1, \tag{A5}$$

and rewrite $I(\alpha,\alpha;\zeta)$ as

$$\begin{aligned} I(\alpha,\alpha;\zeta) &= -\lim_{\nu\to -1^+} \left\{ \int_0^\infty dx\, x^{\nu+1} [(x^2+\alpha^2)^{-1-\zeta} - \alpha^{-2(1+\zeta)}] J_\nu(x) \right\} \\ &\quad + \alpha^{-2(1+\zeta)} \int_0^\infty dx\, J_1(x) \\ &= -\frac{2^{-\zeta}\,\alpha^{-(1+\zeta)}}{\Gamma(1+\zeta)} K_{1+\zeta}(\alpha) + \alpha^{-2(1+\zeta)}. \end{aligned} \tag{A6}$$

Because $K_{n+1/2}(\alpha)$ is elementary if n is any integer, the conclusion is reached that $I(\alpha,\alpha;\zeta)$ is an elementary function if $\zeta = n-1/2$. Explicitly, one gets[25]

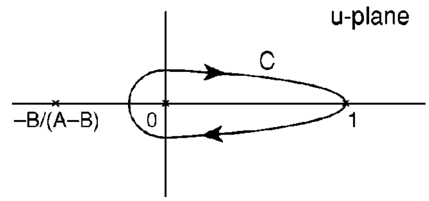

FIG. 4. Contour of integration C for the integral of Eq. (A11).

$$I(\alpha,\alpha;n-1/2)=\begin{cases}\alpha^{-2n-1}+\dfrac{(-1)^{n+1}}{(2n-1)!!}\left(\dfrac{d}{\alpha\, d\alpha}\right)^{n}\dfrac{e^{-\alpha}}{\alpha}, & n=1,2,\ldots,\\ \alpha^{-1}(1-e^{-\alpha}), & n=0,\\ \alpha^{-2n-1}\left[1+\{(2|n|-1)!!\}\left(\dfrac{d}{\alpha\, d\alpha}\right)^{|n|-1}\dfrac{e^{-\alpha}}{\alpha}\right], & -n=1,2,\ldots,\end{cases} \tag{A7}$$

where, in the usual notation, $(2m-1)!!=1\cdot 3\cdots(2m-1)$ for positive integer m.

2. Case $\alpha>\beta$

Let

$$A=x^2+\alpha^2,\quad B=x^2+\beta^2, \tag{A8}$$

and consider the integral representations

$$A^{-\zeta}=\frac{1}{\Gamma(\zeta)}\int_0^\infty d\xi\,\xi^{-1+\zeta}e^{-A\xi},\quad \mathrm{Re}\,\zeta>0,$$

$$B^{-1}=\int_0^\infty d\eta\, e^{-B\eta}. \tag{A9}$$

The radical in Eq. (A2) is recast in the form

$$A^{-\zeta}B^{-1}=\frac{1}{\Gamma(\zeta)}\int_0^\infty\int_0^\infty d\xi\, d\eta\,\xi^{-1+\zeta}e^{-(A\xi+B\eta)}\quad [\xi=uv,\eta=(1-u)v]$$

$$=\zeta\int_0^1 du\, u^{-1+\zeta}[Au+B(1-u)]^{-1-\zeta},\quad \mathrm{Re}\,\zeta>0. \tag{A10}$$

Analytic continuation to complex ζ with $\mathrm{Re}\,\zeta<0$ is brought about via the integral

$$A^{-\zeta}B^{-1}=\frac{\zeta}{1-e^{i2\pi\zeta}}\int_C du\, u^{-1+\zeta}[Au+B(1-u)]^{-1-\zeta}, \tag{A11}$$

where C is a closed contour in the u-plane. C originates from $u=1$ in the first Riemann sheet and encircles the origin in the clockwise sense, as shown in Fig. 4. The first Riemann sheet is defined so that

$$u^{-1+\zeta}>0 \quad\text{and}\quad [Au+B(1-u)]^{-1-\zeta}>0,\ 0<u<1, \tag{A12}$$

with the associated branch cuts lying along the positive and negative real axis. Note that in addition to the branch point at $u=0$ another branch point exists in the negative axis at $u=-B/(A-B)$.

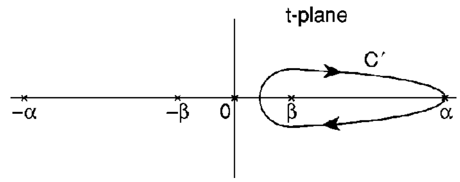

FIG. 5. Map of C of Fig. 4 in the t-plane via $t^2=\alpha^2 u+\beta^2(1-u)$.

With Eq. (A6), it follows that

$$I(\alpha,\beta;\zeta)=\frac{\zeta}{1-e^{i2\pi\zeta}}\int_C du\, u^{-1+\zeta}\int_0^\infty dx\{x^2+[\alpha^2 u+\beta^2(1-u)]\}^{-1-\zeta}J_1(x)$$
$$=-\frac{\zeta e^{-i\pi\zeta}}{2i\sin\pi\zeta}\int_C du\, u^{-1+\zeta}[\alpha^2 u+\beta^2(1-u)]^{-(1+\zeta)}$$
$$\times\left\{1-\frac{2^{-\zeta}}{\Gamma(1+\zeta)}[\alpha^2 u+\beta^2(1-u)]^{(1+\zeta)/2}K_{1+\zeta}(\sqrt{\alpha^2 u+\beta^2(1-u)})\right\}. \quad \text{(A13)}$$

Hence, $I(\alpha,\beta;\zeta)$ is an integral of an elementary function if $\zeta=n-1/2$, n: integer.

Let

$$t=t(u)=\sqrt{\alpha^2 u+\beta^2(1-u)},\quad t(1)=\alpha,\ \ t(0)=\beta. \quad \text{(A14)}$$

This transformation maps C onto C' (Fig. 5):

$$I(\alpha,\beta;\zeta)=-\frac{\zeta e^{-i\pi\zeta}}{i\sin\pi\zeta}\frac{1}{(\alpha^2-\beta^2)^\zeta}\int_{C'}dt\, t^{-1-2\zeta}(t^2-\beta^2)^{-1+\zeta}$$
$$\times\left\{1-\frac{2^{-\zeta}}{\Gamma(1+\zeta)}t^{1+\zeta}K_{1+\zeta}(t)\right\}. \quad \text{(A15)}$$

By virtue of the identity

$$\frac{d}{dt}[t^{-2\zeta}(t^2-\beta^2)^\zeta]=2\beta^2\zeta t^{-1-2\zeta}(t^2-\beta^2)^{-1+\zeta}, \quad \text{(A16)}$$

application of integration by parts to Eq. (A15) furnishes

$$I(\alpha,\beta;\zeta)=\alpha^{-2\zeta}\beta^{-2}\left[1-\frac{2^{-\zeta}\alpha^{1+\zeta}}{\Gamma(1+\zeta)}K_{1+\zeta}(\alpha)\right]$$
$$-\frac{e^{-i\pi\zeta}}{2i\sin\pi\zeta}\frac{2^{-\zeta}}{\Gamma(1+\zeta)}\frac{1}{\beta^2(\alpha^2-\beta^2)^\zeta}\int_{C'}dt\, t^{-2\zeta}(t^2-\beta^2)^\zeta\frac{d}{dt}[t^{1+\zeta}K_{1+\zeta}(t)]. \quad \text{(A17)}$$

It is of some interest to write down recursive formulas that are particularly useful for $\zeta=n-1/2$, where n is any integer. Clearly, from Eq. (A2),

$$I(\alpha,\beta;\zeta+1)=-\frac{1}{2\alpha}\frac{1}{\zeta}\frac{\partial}{\partial\alpha}I(\alpha,\beta;\zeta), \quad \text{(A18)}$$

while

$$I(\alpha,\beta;\zeta-1)=(\alpha^2-\beta^2)I(\alpha,\beta;\zeta)+I(\alpha,\alpha;\zeta-1)$$
$$=(\alpha^2-\beta^2)I(\alpha,\beta;\zeta)+\alpha^{-2\zeta}-\frac{2^{-\zeta+1}\alpha^{-\zeta}}{\Gamma(\zeta)}K_{-\zeta}(\alpha). \tag{A19}$$

Therefore, it suffices to evaluate $I(\alpha,\beta;\zeta)$ for $\zeta=-1/2$. Equation (A18) can be employed for $\zeta=-1/2, 1/2, 3/2,\ldots,$ and Eq. (A19) is adequate for $\zeta=-1/2,-3/2,-5/2,\ldots$.

The substitution $\zeta=-1/2$ in Eq. (A17) gives

$$I(\alpha,\beta;-1/2)=\frac{\alpha}{\beta^2}(1-e^{-\alpha})-\frac{\sqrt{\alpha^2-\beta^2}}{2\beta^2}\int_{C'}dt\,\frac{t}{\sqrt{t^2-\beta^2}}e^{-t}. \tag{A20}$$

Since the integrand has now an integrable singularity at $t=\beta$, the path can be indented back to the positive real axis:

$$I(\alpha,\beta;-1/2)=\frac{\alpha}{\beta^2}(1-e^{-\alpha})-\frac{\sqrt{\alpha^2-\beta^2}}{\beta^2}\int_{\beta}^{\alpha}dt\,\frac{t}{\sqrt{t^2-\beta^2}}e^{-t}. \tag{A21}$$

This result agrees with Eqs. (3.13) and (3.14). Evidently, applying Eq. (A18) does not produce any new integrals of elementary functions.

APPENDIX B: AN ASYMPTOTIC EXPANSION FOR $E_{2\phi}$

In this appendix, an asymptotic formula is derived for the $E_{2\phi}$ of Sec. III when $k_2\rho\gg 1$ and $k_2<|k_1|$ on the basis of its exact series expansion. Consider the $W(\rho)$ of Eq. (3.14). The combination of the first and third terms in Eq. (3.15) yields

$$W(\rho)=W(\rho;\infty,-ik_2,-ik_S)-W(\rho;\infty,-ik_1,-ik_S), \tag{B1}$$

where the second term here is given by Eq. (3.27) and

$$W(\rho;\infty,q_2,q_S)=e^{-q_S\rho}q_S\int_{(q_2-q_S)/q_S}^{\infty}\frac{d\eta}{\sqrt{\eta}}(1+\eta)(2+\eta)^{-1/2}e^{-q_S\rho\eta}. \tag{B2}$$

1. Wave through region 2

When $|q_S\rho|\gg 1$, the major contribution to integral (B2) arises from the vicinity of the lower endpoint of width $O[(q_S\rho)^{-1}]$. If in addition $|(q_2-q_S)\rho|\leqslant O(1)$, $\eta=0$ falls inside the critical region and the radical can be replaced by a Maclaurin expansion. Accordingly,

$$W(\rho;\infty,q_2,q_S)=-e^{-q_2\rho}q_S\left[\sum_{m=0}^{M-1}\mathcal{V}_m(\rho)-e^{(q_2-q_S)\rho}\Upsilon_M(\rho)-R_{2M}(\rho)\right],\quad M\geqslant 1, \tag{B3}$$

where

$$\mathcal{V}_m(\rho)=-e^{i\pi/4}i^{m+1}\frac{(\frac{1}{2})_m}{m!}\frac{m+1/2}{m-1/2}(2q_S\rho)^{-m-1/2}\varphi_m(i(q_2-q_S)\rho), \tag{B4}$$

$$\varphi_m(z)=\sqrt{2\pi}(-i)^m e^{-iz}z^{1/2+m}\frac{d^m}{dz^m}[z^{-1/2}e^{iz}F(z)], \tag{B5}$$

$$F(z)=e^{-iz}\int_z^{\infty}dx\,\frac{e^{ix}}{\sqrt{2\pi x}}=e^{-iz}\left[\frac{1}{2}(1+i)-F_0(z)\right]. \tag{B6}$$

$F_0(z)$ is introduced in Eqs. (3.41) and (3.42), $R_{2M}(\rho)$ is given by Eq. (3.34), and $\Upsilon_M(\rho)$ is the remainder in the sense of Poincaré[35] of an asymptotic expansion for $e^{q_S\rho}K_1(q_S\rho)$ when M terms are summed,[25] i.e., $\Upsilon_M(\rho)=O[(k_S\rho)^{-M-1/2}]$ as $|k_S\rho|\to\infty$ for $M=O(1)$, uniformly in $\mathrm{Arg}(k_S\rho)$. Compare with Eqs. (3.31).

For $1\leqslant M=O(1)$ and $|(k_2-k_S)/(2k_S)|\ll 1$,

$$R_{2M}(\rho)=O\left[\left(\frac{k_2-k_S}{2k_S}\right)^{M+1/2}\right],\quad |(k_2-k_S)\rho|\leqslant O(1), \tag{B7}$$

leading to

$$\frac{e^{-i(k_2-k_S)\rho}\Upsilon_M(\rho)+R_{2M}(\rho)}{\mathcal{V}_{M-1}(\rho)}=O\left(\frac{1}{k_2\rho}\right). \tag{B8}$$

When $|(k_2-k_S)\rho|\gg 1$, the remainder $R_{2M}(\rho)$ dominates over the $\Upsilon_M(\rho)$ in Eq. (B3). It follows that

$$R_{2M}(\rho)=O\left[\left(\frac{k_2-k_S}{2k_S}\right)^{M+1/2}\frac{1}{(k_2-k_S)\rho}\right],\quad |(k_2-k_S)\rho|\gg 1,\ \ M=O(1), \tag{B9}$$

by inspection of Eq. (3.36b), and

$$\frac{e^{-i(k_2-k_S)\rho}\Upsilon_M(\rho)+R_{2M}(\rho)}{-\mathcal{V}_{M-1}(\rho)}=O\left(\frac{k_2-k_S}{2k_S}\right). \tag{B10}$$

Of course, in the limit $M\to\infty$ the remainder $\Upsilon_M(\rho)$ is unbounded and the series from Eq. (B3) diverges. In the sense implied by Eq. (B3),

$$W(\rho;\infty,q_2,q_S)\sim -e^{-q_2\rho}q_S\sum_{m=0}^{\infty}\mathcal{V}_m(\rho). \tag{B11}$$

This asymptotic expansion can be attained somewhat heuristically from the exact series (3.40) combined with the asymptotic expansion for $H_1^{(1)}(k_S\rho)$. Notice that

$$\frac{d^m}{dz^m}[z^{-1/2}F_0(z)]=\frac{e^{i\pi/4}}{\sqrt{2}}(-1)^m\left(\frac{1}{2}\right)_m z^{-1/2-m}-\frac{d^m}{dz^m}[z^{-1/2}e^{iz}F(z)]. \tag{B12}$$

2. Wave through region 1

From Eq. (B1), consider the integral

$$W(\rho;\infty,q_1,q_S)=q_1\int_1^{\infty}d\eta\,\frac{\eta}{\sqrt{\eta^2-\bar{q}}}e^{-q_1\rho\eta},\quad \bar{q}=q_S^2/q_1^2,\ \ |\bar{q}|<1. \tag{B13}$$

When $|q_1\rho|\gg 1$, the principal contribution to integration comes from the vicinity of $\eta=1$. Accordingly, expand the radical as

$$\frac{\eta}{\sqrt{\eta^2-\bar{q}}}-1=\sum_{n=0}^{N-1}\frac{(-1)^n}{n!}\mathcal{A}_n(\bar{q})(\eta-1)^n+O[(\eta-1)^N],\quad N\geqslant 1, \tag{B14}$$

where

$$\mathcal{A}_0(\overline{q}) = \frac{\overline{q}}{\sqrt{1-\overline{q}}(1+\sqrt{1-\overline{q}})}, \tag{B15}$$

and, for $n=1,2,\ldots,$[28]

$$\begin{aligned}\mathcal{A}_n(\overline{q}) &= (-1)^n \frac{d^n}{d\eta^n} \frac{\eta}{\sqrt{\eta^2-\overline{q}}}\bigg|_{\eta=1} \\ &= \overline{q}\frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(1+\frac{n}{2}\right) {}_2F_1\left(\frac{n+3}{2},\frac{n}{2}+1;2;\overline{q}\right) \\ &= 2^{[(n-1)/2]}\{(2[n/2]+1)!!\}\frac{d^{[(n-1)/2]}}{d\overline{q}^{[(n-1)/2]}}\{\overline{q}^{[(n-1)/2]+1}(1-\overline{q})^{-(2[n/2]+3)/2}\}.\end{aligned} \tag{B16}$$

$[x]$ denotes the integral part of x. Note that the second line of Eq. (B16) holds even for $n=0$.

It follows that

$$W(\rho;\infty,q_1,q_S) = \frac{e^{-q_1\rho}}{\rho}\left\{1+\sum_{n=0}^{N-1}\frac{\mathcal{A}_n(\overline{q})}{(-q_1\rho)^n}+O[(q_1\rho)^{-N}]\right\}, \quad N\geqslant 1. \tag{B17}$$

The first term inside the braces is the $U_0(\rho)$ from Eq. (3.19). This expansion can be verified directly from Eq. (3.18) by invoking the formula[28]

$$e^z\mathrm{Ei}(-z) = \sum_{n=0}^{N-1}(-1)^{n+1}n!\,z^{-1-n}+O(|z|^{-N-1}) \ \text{ as } \ z\to\infty, \quad \begin{cases}N=1,2,\ldots,\\ |\mathrm{Arg}\ \ z|<3\pi/2,\end{cases} \tag{B18}$$

which holds uniformly in Arg z, interchanging the order of summation and subsequently allowing $M\to\infty$.

3. Asymptotic formula for $E_{2\phi}$

The resulting asymptotic expansion for $E_{2\phi}$ reads as

$$\begin{aligned}E_{2\phi} \sim -\frac{i\omega\mu_0}{2\pi}\frac{k_2^2}{k_1^4-k_2^4}\Bigg\{&-ie^{ik_1\rho}k_1\left[\frac{k_1^2+k_2^2(2-k_S/k_2)}{k_2^2\rho^2}+i\frac{k_1^2+2k_2^2}{k_1k_2^2\rho^3}\right] \\ &+ie^{ik_2\rho}\frac{2k_1^2+k_2^2}{k_2\rho^2}\left(1+\frac{i}{k_2\rho}\right)-\frac{k_1k_S^2}{k_2\rho}\left[\frac{e^{ik_1\rho}}{ik_S\rho}\sum_{n=0}^{\infty}\frac{\mathcal{A}_n(k_S^2/k_1^2)}{(ik_1\rho)^n}-e^{ik_2\rho}\sum_{m=0}^{\infty}\mathcal{V}_m(\rho)\right]\Bigg\}\sin\phi.\end{aligned} \tag{B19}$$

Similar expressions can be written down by inspection for the other components of Secs. III and IV.

APPENDIX C: ASYMPTOTIC FORMULAS FOR $k_2\rho\ll 1$ AND $k_2\rho\gg 1$

In this appendix, simple approximations are applied directly to the integrals (2.1)–(2.6) and (2.10)–(2.12) when $k_2^2\ll|k_1^2|$ under the conditions $k_1\rho=O(1)$ and $k_2\rho\gg 1$.

1. Case $k_2^2\ll|k_1^2|$, $k_1\rho=O(1)$

In terms of the q_j $(j=1,2)$ of Eqs. (3.1), $|q_2\rho|\ll 1$ while $q_1\rho$ is kept fixed and $|q_2^2|\ll|q_1^2|$. Then the principal contribution to integration arises from a range where $\lambda=O(1/\rho)\gg|q_2|$. Accordingly, the following approximation becomes effective:

$$\sqrt{k_2^2-\lambda^2}\sim i\lambda$$

A. Horizontal electric dipole

With these approximations, B_{2z} becomes

$$B_{2z} \sim \frac{\mu_0}{2\pi}\int_0^\infty d\lambda\, \lambda^2 \frac{J_1(\lambda\rho)}{\sqrt{\lambda^2+q_1^2}+\lambda}\sin\phi = \frac{\mu_0}{2\pi q_1^2}\left\{\frac{1}{\rho^4}\mathcal{I}_m(q_1\rho) - \int_0^\infty d\lambda\,\lambda^3 J_1(\lambda\rho)\right\}\sin\phi, \tag{C2}$$

where

$$\int_0^\infty d\lambda\,\lambda^3 J_1(\lambda\rho) = -\frac{3}{\rho^4}, \tag{C3}$$

while the integral $\mathcal{I}_m$ is given by Eq. (3.4). Hence,

$$B_{2z} \sim \frac{\mu_0 k_1^2}{2\pi}\left[e^{ik_1\rho}\left(\frac{3}{k_1^4\rho^4} - \frac{3i}{k_1^3\rho^3} - \frac{1}{k_1^2\rho^2}\right) - \frac{3}{k_1^4\rho^4}\right]\sin\phi. \tag{C4}$$

In the same vein, by virtue of Eq. (3.8) for $\mathcal{I}_e$,

$$\begin{aligned} E_{2\rho} &\sim \frac{i\omega\mu_0}{2\pi q_1^2}\left\{\left(\frac{1}{\rho}+\frac{d}{d\rho}\right)\left[\frac{1}{\rho^2}\mathcal{I}_e(q_1\rho)\right] - \frac{1}{\rho}\int_0^\infty d\lambda\,\lambda J_1(\lambda\rho)\right\}\cos\phi \\ &= \frac{i\omega\mu_0 k_1}{2\pi}\left\{e^{ik_1\rho}\left(\frac{1}{k_1^3\rho^3} - \frac{i}{k_1^2\rho^2}\right) + \frac{1}{k_1^3\rho^3}\right\}\cos\phi, \end{aligned} \tag{C5}$$

$$\begin{aligned} E_{2\phi} &\sim -\frac{i\omega\mu_0}{2\pi q_1^2}\left\{\left(\frac{1}{\rho}+\frac{d}{d\rho}\right)\left[\frac{1}{\rho^2}\mathcal{I}_e(q_1\rho)\right] - \frac{d}{d\rho}\int_0^\infty d\lambda\,\lambda J_1(\lambda\rho)\right\}\sin\phi \\ &= \frac{i\omega\mu_0 k_1}{2\pi}\left\{e^{ik_1\rho}\left(-\frac{1}{k_1^3\rho^3} + \frac{i}{k_1^2\rho^2}\right) + \frac{2}{k_1^3\rho^3}\right\}\sin\phi. \end{aligned} \tag{C6}$$

The computation of the limiting forms of E_{2z}, $B_{2\rho}$ and $B_{2\phi}$ is more involved:

$$\begin{aligned} E_{2z} &\sim \frac{i\omega\mu_0}{2\pi q_1^2}\lim_{\epsilon\to 0^+}\frac{\partial}{\partial\rho}\int_0^\infty d\lambda\,\sqrt{\lambda^2+q_1^2}\,e^{-\epsilon\sqrt{\lambda^2+q_1^2}}J_0(\lambda\rho)\cos\phi \\ &= \frac{i\omega\mu_0}{2\pi q_1^2}\lim_{\epsilon\to 0^+}\frac{\partial}{\partial\rho}\frac{\partial^2}{\partial\epsilon^2}\left\{I_0\left[\frac{q_1}{2}(\sqrt{\epsilon^2+\rho^2}-\epsilon)\right]K_0\left[\frac{q_1}{2}(\sqrt{\epsilon^2+\rho^2}+\epsilon)\right]\right\}\cos\phi \\ &= \frac{\omega\mu_0}{4\rho}J_1(k_1\rho/2)H_1^{(1)}(k_1\rho/2)\cos\phi, \end{aligned} \tag{C7}$$

$$B_{2\rho} \sim -\frac{i\mu_0}{4}\frac{d}{d\rho}\left[\frac{1}{\rho}J_1(k_1\rho/2)H_1^{(1)}(k_1\rho/2)\right]\sin\phi, \tag{C8}$$

where use is made of Ref. 36, and

$$B_{2\phi} \sim -\frac{i\mu_0}{4\rho^2}\frac{k_1}{k_2}J_1(k_1\rho/2)H_1^{(1)}(k_1\rho/2)\cos\phi. \tag{C9}$$

Compare with the exposition in Ref. 13. In principle, the behavior of the z-component of the electric field depends on the limit path. For instance, if the observation point is forced to approach the source along a straight line from region 1 ($z<0$), this component behaves as $\sim 1/\rho^3$. [The divergence can ensue from taking $\epsilon=\rho$ in the second line of Eq. (C7).]

B. Vertical electric dipole

From the Fourier–Bessel integral (2.10),

$$B_{2\phi}\sim \frac{\mu_0}{2\pi q_1^2}\left[q_1^2\int_0^\infty d\lambda\,\lambda J_1(\lambda\rho)-\frac{q_2^2}{\rho^2}\mathcal{I}_{\mathrm{e}}(q_1\rho)\right]$$
$$=-\frac{\mu_0}{2\pi\rho^2}\left[e^{ik_1\rho}\frac{k_2^2}{k_1^2}-\left(1+\frac{k_2}{k_1}ik_2\rho\right)\right]\sim-\frac{\mu_0}{2\pi\rho^2}\left(e^{ik_1\rho}\frac{k_2^2}{k_1^2}-1\right). \quad \text{(C10)}$$

Likewise, starting with Eq. (2.11) gives

$$E_{2z}\sim\frac{i\omega\mu_0}{2\pi\rho}\left[e^{ik_1\rho}\left(\frac{1}{k_1^2\rho^2}-\frac{i}{k_1\rho}\right)-\frac{1}{k_2^2\rho^2}\right]. \quad \text{(C11)}$$

The formula for $E_{2\rho}$ is more involved, in analogy with Eqs. (C7)–(C9). It is

$$E_{2\rho}\sim\frac{i\omega\mu_0}{2\pi\rho}\left[\frac{i\pi}{2}J_1(k_1\rho/2)H_1^{(1)}(k_1\rho/2)-\frac{k_2^2}{k_1^2}\right]. \quad \text{(C12)}$$

2. Case $k_2^2\ll|k_1^2|$, $k_2\rho\gg1$

When $|k_j\rho|\geq 1$ $(j=1,2)$, the oscillations of the Bessel functions in Eqs. (2.1)–(2.6) and (2.10)–(2.12) force the major contributions to integration to arise from the vicinities of the branch points at $\lambda=k_j$ with widths $O(1/\rho)$. The condition $k_2^2\ll|k_1^2|$ permits considerable simplification because, heuristically speaking, the two contributing regions separate.

Following Sommerfeld,[29] one may replace each J_n $(n=0,1,2)$ by $(1/2)[H_n^{(1)}+H_n^{(2)}]$. The contour Γ can be chosen symmetric under inversion through the origin, as shown in Fig. 2. Use is made of the analytic continuation formula

$$H_n^{(2)}(ze^{-i\pi})=(-1)^{n+1}H_n^{(1)}(z),\quad n:\ \text{integer}. \quad \text{(C13)}$$

For notational convenience, let $\mathcal{F}_{2\kappa,j}$ ($\mathcal{F}=E$, B; $\kappa=\rho$, ϕ, z; $j=1,2$) denote the part of the field component that corresponds to the contour integral along Γ_j. Each $\mathcal{F}_{2\kappa,j}$ follows upon the replacements

$$\mathcal{F}_{2\kappa}\rightarrow\mathcal{F}_{2\kappa,j}\ \text{ under }\ \int_0^\infty d\lambda(\cdots)J_n(\lambda\rho)\rightarrow\frac{1}{2}\int_{\Gamma_j}d\lambda(\cdots)H_n^{(1)}(\lambda\rho). \quad \text{(C14)}$$

Each $\mathcal{F}_{2\kappa,j}$ amounts to a wave traveling with the phase velocity of medium j.

With

$$H_0^{(1)}(\lambda\rho)+H_2^{(1)}(\lambda\rho)=\frac{2}{\lambda\rho}H_1^{(1)}(\lambda\rho), \quad \text{(C15)}$$

the Hankel functions in each $\mathcal{F}_{2\kappa,j}$ are approximated according to

$$H_n^{(1)}(k_j\rho(1+it))\sim e^{ik_j\rho-in\pi/2-i\pi/4}\sqrt{\frac{2}{\pi k_j\rho}}e^{-k_j\rho t},\quad |k_j\rho|\gg1,\ t\rightarrow0^+. \quad \text{(C16)}$$

Due to the rapid exponential decrease here, the major contribution to integration on each side of the branch cut (positive t-axis) originates from the corresponding branch point ($t=0$).[37]

Upon the change of variable $\lambda=k_j(1+it)$, it is recognized that for $\lambda\sim k_j$,

$$\sqrt{1-(\lambda/k_j)^2}=\pm e^{-i\pi/4}\sqrt{2t}\sqrt{1+it/2}\sim\pm e^{-i\pi/4}\sqrt{2t},\quad t\to0^+,\quad \sqrt{t}\geqslant0, \tag{C17}$$

where the upper sign holds along the left-hand side and the lower sign along the right-hand side of each branch cut. For instance, $D(\lambda)$ of Eq. (2.8) becomes

$$D(\lambda)\sim\pm e^{-i\pi/4}\sqrt{2t}+k_2/k_1,\quad \lambda\sim k_2, \tag{C18}$$

$$D(\lambda)\sim ik_1/k_2\pm(k_2/k_1)e^{-i\pi/4}\sqrt{2t}\sim ik_1/k_2,\quad \lambda\sim k_1. \tag{C19}$$

In each $\mathcal{F}_{2\kappa,j}$, the contribution from the circle $C_{\delta,j}$ of radius δ vanishes in the limit $\delta\to0^+$.

A. Horizontal electric dipole

The integrals for the z-component of the magnetic field are

$$B_{2z,2}\sim e^{ik_2\rho}\frac{\mu_0}{2\pi}\frac{k_2^4}{k_1^2}\sqrt{\frac{2}{\pi k_2\rho}}\int_0^\infty dt\sqrt{2t}e^{-k_2\rho t}\sin\phi=e^{ik_2\rho}\frac{\mu_0}{2\pi}\frac{k_2^2}{k_1^2\rho^2}\sin\phi, \tag{C20}$$

$$B_{2z,1}\sim -e^{ik_1\rho}\frac{\mu_0}{2\pi}\frac{1}{\rho^2}\sin\phi. \tag{C21}$$

The integrals pertaining to the ρ-component of the electric field become

$$\begin{aligned}E_{2\rho,2}&\sim\frac{\omega\mu_0k_2^3}{2\pi k_1^2}\frac{e^{ik_2\rho}}{\sqrt{\pi k_2\rho}}\left[\int_0^\infty dt\frac{\sqrt{t}}{it+(1/2)k_2^2/k_1^2}e^{-k_2\rho t}+\frac{2i}{k_2\rho}\int_0^\infty dt\sqrt{t}e^{-k_2\rho t}\right]\cos\phi\\&\sim e^{ik_2\rho}\frac{\omega\mu_0k_2^4}{2\pi k_1^3}\sqrt{\frac{\pi}{k_2\rho}}[F(\wp)-i(2\pi\wp)^{-1/2}]\cos\phi,\end{aligned} \tag{C22}$$

$$E_{2\rho,1}\sim e^{ik_1\rho}\frac{\omega\mu_0}{2\pi k_1\rho^2}\cos\phi. \tag{C23}$$

$\wp$ and $F(z)$ are defined by Eqs. (3.50) and (B6), respectively.

By a comparison of Eqs. (2.2) and (2.3), no further calculations need to be done for $E_{2\phi,j}$.

$$\begin{aligned}E_{2\phi,2}&\sim-\frac{\omega\mu_0k_2^3}{2\pi k_1^2}\frac{e^{ik_2\rho}}{\sqrt{\pi k_2\rho}}\left\{\frac{1}{ik_2\rho}\left(\frac{\pi k_2}{k_1}[F(\wp)-i(2\pi\wp)^{-1/2}]\right)+ik_2\rho\left(\frac{i}{k_2^2\rho^2}\sqrt{\frac{\pi}{k_2\rho}}\right)\right\}\sin\phi\\&=e^{ik_2\rho}\frac{i\omega\mu_0k_2^3}{2\pi k_1^3\rho}\sqrt{\frac{\pi}{k_2\rho}}[F(\wp)-2i(2\pi\wp)^{-1/2}]\sin\phi,\end{aligned} \tag{C24}$$

$$E_{2\phi,1}\sim -e^{ik_1\rho}\frac{\omega\mu_0}{2\pi k_1\rho^2}\sin\phi. \tag{C25}$$

The rest of the components are calculated under similar approximations as follows:

$$E_{2z,2} \sim e^{ik_2\rho} \frac{\omega\mu_0 k_2^2}{2\pi k_1} \frac{1}{\sqrt{\pi k_2 \rho}} \int_0^\infty dt \frac{\sqrt{t}}{it+(1/2)k_2^2/k_1^2} e^{-k_2\rho t} \cos\phi$$
$$= e^{ik_2\rho} \frac{\omega\mu_0 k_2^3}{2\pi k_1^2} \sqrt{\frac{\pi}{k_2\rho}} [F(\wp) - i(2\pi\wp)^{-1/2}] \cos\phi, \tag{C26}$$

$$E_{2z,1} \sim e^{ik_1\rho} \frac{i\omega\mu_0}{2\pi k_1 \rho^2} \cos\phi, \tag{C27}$$

$$B_{2\rho,2} \sim e^{ik_2\rho} \frac{i\mu_0 k_2^3}{2\pi k_1^2 \rho} \sqrt{\frac{\pi}{k_2\rho}} [F(\wp) - 2i(2\pi\wp)^{-1/2}] \sin\phi, \tag{C28}$$

$$B_{2\rho,1} \sim e^{ik_1\rho} \frac{i\mu_0}{2\pi\rho^2} \sin\phi, \tag{C29}$$

and, in analogy with formulas (C24) and (C25),

$$B_{2\phi,2} \sim -e^{ik_2\rho} \frac{\mu_0 k_2^4}{2\pi k_1^2} \sqrt{\frac{\pi}{k_2\rho}} [F(\wp) - i(2\pi\wp)^{-1/2}] \cos\phi, \tag{C30}$$

$$B_{2\phi,1} \sim -e^{ik_1\rho} \frac{\mu_0 k_2}{2\pi k_1 \rho^2} \left(i\frac{k_2}{k_1} - \frac{1}{k_2\rho} \right) \cos\phi. \tag{C31}$$

Compare with Ref. 17.

B. Vertical electric dipole

In the same vein, the field of a vertical electric dipole can be calculated with recourse to Eqs. (2.10)–(2.12). See also Ref. 32. Without further ado,

$$B_{2\phi,2} \sim e^{ik_2\rho} \frac{\mu_0 k_2^3}{2\pi k_1} \sqrt{\frac{\pi}{k_2\rho}} [F(\wp) - i(2\pi\wp)^{-1/2}], \tag{C32}$$

$$B_{2\phi,1} \sim -e^{ik_1\rho} \frac{\mu_0}{2\pi} \frac{k_2^2}{k_1^2 \rho^2}, \tag{C33}$$

$$E_{2z,2} \sim -e^{ik_2\rho} \frac{\omega\mu_0 k_2^2}{2\pi k_1} \sqrt{\frac{\pi}{k_2\rho}} [F(\wp) - i(2\pi\wp)^{-1/2}], \tag{C34}$$

$$E_{2z,1} \sim e^{ik_1\rho} \frac{\omega\mu_0}{2\pi k_1 \rho^2}, \tag{C35}$$

$$E_{2\rho,2} \sim -e^{ik_2\rho} \frac{\omega\mu_0 k_2^3}{2\pi k_1^2} \sqrt{\frac{\pi}{k_2\rho}} [F(\wp) - i(2\pi\wp)^{-1/2}], \tag{C36}$$

$$E_{2\rho,1} \sim -e^{ik_1\rho} \frac{i\omega\mu_0}{2\pi k_1 \rho^2}. \tag{C37}$$

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