

Radiation of horizontal electric dipole on large dielectric sphere

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The electromagnetic field in air of a radiating electric dipole located below and tangential to the surface of a homogeneous, isotropic and optically dense sphere is studied anew. The starting point is the eigenfunction expansion for the field in spherical harmonics, which is now converted into series of integrals via the Poisson summation formula. A creeping-wave structure for all six components along the boundary is revealed that consists of waves exponentially decreasing through air and rays bouncing and circulating inside the sphere. The character of individual modes of propagation and the interplay between “electric” and “magnetic” types of polarization are investigated. Connections with and differences from standard ray optics and the cases of the radiating vertical dipole and scalar plane-wave scattering are outlined. © 2002 American Institute of Physics.

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I. INTRODUCTION

The scattering and diffraction of electromagnetic waves have long been understood as boundary-value problems of Maxwell's equations. In principle, the field can be determined everywhere by specifying the source and the boundary conditions. In practice, even when closed-form solutions are then obtained, the chosen representations may not be amenable to quantitative understanding. This difficulty often plagues analyses where current sources lie too close to the boundary separating two media. Such a case arises in connection with the long-distance communication along the sea surface at very low frequencies.^{1,2}

In the present article, a three-dimensional idealized model is studied in which the source is an electric dipole located below and tangential to the surface of a homogeneous and isotropic, electrically large sphere surrounded by air. Of course, none of the field components can be made to vanish identically in this case. A dipole vertical to the spherical boundary,^{3,4} on the other hand, introduces an axis of symmetry, having only three nonzero spherical components; these admit eigenfunction expansions of simpler structure. In both problems, the formal solution is easily attainable in terms of spherical harmonics but is not directly amenable to computations and interpretation. One of the objectives of this work is to unveil the underlying physical picture by asymptotic methods for the case of the horizontal dipole, describing the interplay between two coexisting polarizations. The assumption of an optically dense sphere is thus imposed while attention is restricted to points lying in the spherical boundary. The analysis is also intended to reveal differences from the known case of plane-wave scattering in the context of the scalar wave and Schrödinger's equations. Because the emphasis is on the physical concepts and the tools needed to expose such concepts, actual numerical calculations are beyond the scope of this article.

There is a fairly long sequence of papers in connection with the present problem. Noteworthy is Mie's formal expansion⁵ in partial waves for a plane wave incident on a homogeneous sphere. Another formulation found in a later paper by Debye⁶ is related to his previous studies of high-frequency plane-wave scattering by an infinitely long cylinder.⁷ An exposition and discussion of

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these as well as of other works was given by van de Hulst.⁸ Watson⁹ appears to be the first to investigate systematically the radiation of a point source in the presence of a sphere with radius large compared to the wavelength. In his formulation the source was an electric dipole located above and vertical to the surface of a perfectly conducting sphere; his focus was on scalar potentials that furnish the electromagnetic field via successive differentiations. The merits of Watson’s approach are unquestionable: the slowly converging expansion in partial waves was converted to an integral which in turn generated a rapidly converging series. This method was later invoked by other authors^{10–12} in their efforts to treat the case of a finitely conducting sphere. Among these authors, Gray,¹² for example, identified a “magnetic” type of wave propagating and attenuating through air with an attenuation rate independent of the adjacent medium, when the source is a magnetic dipole vertical to the surface of a lossy sphere.

Being aware of these works, Norton¹³ proposed simplified formulas and graphs for the field intensity of vertical and horizontal dipoles elevated over a spherical earth. Bremmer^{14,15} compared the fields of the two configurations by considering the direct wave and the leading reflected wave in free space; his analysis pointed to a simple picture for wave propagation in air for distances exceeding the free-space wavelength. The radiation of a horizontal dipole above a finitely conducting sphere was investigated by Fock¹⁶ by use of scalar potentials. He approximated the field through air in the “shadow region” in terms of exponentially decreasing waves, and gave the corresponding attenuation rates as solutions to two uncoupled transcendental equations. Fock started with an extension of Watson’s method⁹ by neglecting the field that travels through the sphere and not examining the transition to planar-earth formulas. In the same spirit, the problem was essentially revisited by Wait¹⁷ in the 1950s; he concluded that at “low radio frequencies” the horizontal component of the electric field is negligibly small compared to the vertical one. On the basis of Watson’s method,⁹ geometrical-ray pictures were invoked in that same period of time in the study of elastic waves inside spherical cavities.^{18,19} (See Refs. 20 and 21 for later developments in the theory of elastic-wave propagation.)

In a remarkable paper, Wu²² invoked the concept of the creeping wave in order to study the high-frequency scattering of plane waves by impenetrable cylinders and spheres in the context of Schrödinger’s and Maxwell’s equations. He derived asymptotic expansions for the total scattering cross sections that went well beyond the standard geometrical optics, and pointed out that a mathematical tool leading to the creeping wave in the case of a sphere is the Poisson summation formula. Notably, Wu²² extended the familiar concept of the creeping wave in two space dimensions^{23–26} from high frequencies to all positive frequencies by arguments of algebraic topology.²⁷ The underlying physical idea was soon after generalized to other scatterers by Seshadri.²⁸ A similar analysis based on the Poisson summation formula was later used by Nussenzveig,²⁹ who referred to Ref. 22, for the study of high-frequency, plane-wave scattering by transparent spheres. Key points in his analysis were the imposition of a large index of refraction and the expansion of the total scattering amplitude in a series of the Debye type.⁷ Nussenzveig provided a description in terms of waves that attenuate exponentially along the boundary and rays bouncing and circulating inside the sphere. The relevant Poisson summation formula can be written as³⁰

$$\sum_{l=0}^{\infty} g(l) = \sum_{n=-\infty}^{\infty} e^{-in\pi} \int_0^{\infty} d\nu g\left(\nu - \frac{1}{2}\right) e^{i2\pi n\nu}, \tag{1.1}$$

where the left-hand side is the starting eigenfunction expansion. The right-hand side of this equation was interpreted in terms of “classical paths” by Berry and Mount.³¹ This interpretation stems from noticing that for high frequencies each index n identifies a path that encircles the origin n times.^{31,32} Accordingly, these authors invoked stationary-phase calculations and elaborate uniform approximations.

Recently, Houdzoumis^{3,4,33} applied the Poisson summation formula in order to study the radiation of a vertical electric dipole over a sphere, by imposing the simplifying assumption of a large index of refraction. As mentioned earlier, the number of nonzero field components is reduced

to three in this case, with the corresponding polarization being termed as of the “electric type.” Houdzoumis placed equal emphasis on the wave that attenuates exponentially along the boundary in air and the rays that circulate around the origin inside the sphere. King *et al.*^{1,2} made use of these results for the surface wave in order to calculate the field of an antenna on the sea surface in the range of very low frequencies (VLF).

In the spirit of Houdzoumis’ analysis,^{3,4,33} the present article has a threefold purpose. The first purpose is to give the complete solution for the field of an electric dipole located inside a sphere without recourse to scalar potentials. The dipole is taken to lie in the plane defined by a local tangent and the center of the sphere, and is parallel to the tangent. The second purpose is to evaluate all six components when the dipole approaches the boundary from below by converting the series of partial waves into integrals according to the Poisson summation formula. In this limit all series diverge in the usual sense and some care needs to be exercised. By use of asymptotics, it is found that, apart from waves that reach the observation point with the air and earth phase velocities, significant contributions may arise from rays bouncing and circulating inside the sphere. The third purpose is to compare these findings with other, known cases such as the plane-wave scattering and the radiation of a vertical electric dipole.

The remainder of the article is organized as follows. Section II starts with Maxwell’s equations; the ordinary differential equations in the radial distance for the θ - and ϕ -components of the field in spherical coordinates are solved explicitly. The eigenfunction expansions in spherical harmonics are then converted into series of integrals, and both the source and the observation point are allowed to approach the boundary. In Sec. III, integral expressions are obtained with the inverse electrical radius used as the expansion parameter. The lowest-order terms are known Sommerfeld integrals.³⁴ Corrections to these integrals account for the finite curvature of the boundary, describing waves that travel through air and through the dense medium. Integrated expressions are given for these waves when $k_2^2 \ll |k_1^2|$, $k_2 \rho \gg 1$ and the observation angle θ is sufficiently small, where k_2 is the wave number in air, assumed to be real throughout the paper, k_1 is the complex wave number in the sphere, and ρ is the cylindrical distance from the source. The expressions for the wave through the sphere are new to the author’s knowledge. The accompanying zeroth-order integral terms have been evaluated via rapidly converging series elsewhere.^{34,35} In Sec. IV, the aforementioned asymptotic formulas are modified to account for the transition of the planar-boundary formulas to waves decreasing exponentially through air, with the distinction of two modes of propagation pertaining to polarizations of an “electric” and a “magnetic” type. A description is also provided for the field with the phase velocity of the enclosed medium. The analysis in Sec. V shows that, for $\theta = O(1)$ and $\pi - \theta = O(1)$, the total field consists of waves that attenuate exponentially, and rays that circulate in the sphere and are multiply reflected at the boundary. The contributions of these rays are negligible for a finitely conducting sphere. A few limiting cases of the ray contributions are placed under scrutiny, an example being the case where the antipodal point at $\theta = \pi$ is approached along the boundary. The interplay between contributions of the two polarizations to the ray amplitudes as well as deviations from the ray character are discussed.

In some of the calculations presented in this article, such as those in Appendix A, the wave number in the sphere, k_1 , is treated as real for the sake of simplicity. This poses no restriction on the final asymptotic formulas, however, which are usable for complex values of k_1 with $0 \leq \text{Arg } k_1 \leq \pi/4$, and are therefore applicable to spheres of finite conductivity. The $e^{-i\omega t}$ time dependence is suppressed throughout the analysis.

II. FORMULATION

A. Formal representations

The geometry of the problem is depicted in Fig. 1. It consists of an x -directed electric dipole S of unit moment located inside a homogeneous, isotropic and nonmagnetic sphere (region 1, $r < a$) at a distance b from the origin. The sphere is surrounded by air (region 2, $r > a$). Maxwell’s equations in each region j ($j = 1, 2$) read as follows:

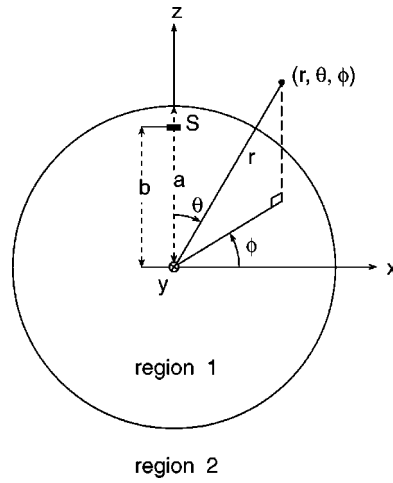


FIG. 1. Spherical coordinates and horizontal electric dipole S inside an isotropic and homogeneous sphere.

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta E_{j\phi}) - \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (E_{j\theta}) = i\omega B_{jr}, \tag{2.1}$$

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (E_{jr}) - \frac{1}{r} \frac{\partial}{\partial r} (rE_{j\phi}) = i\omega B_{j\theta}, \tag{2.2}$$

$$\frac{1}{r} \frac{\partial}{\partial r} (rE_{j\theta}) - \frac{1}{r} \frac{\partial}{\partial \theta} (E_{jr}) = i\omega B_{j\phi}, \tag{2.3}$$

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta B_{j\phi}) - \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (B_{j\theta}) = -i \frac{k_j^2}{\omega} E_{jr}, \tag{2.4}$$

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (B_{jr}) - \frac{1}{r} \frac{\partial}{\partial r} (rB_{j\phi}) = \mu_0 J_{j\theta} - i \frac{k_j^2}{\omega} E_{j\theta}, \tag{2.5}$$

$$\frac{1}{r} \frac{\partial}{\partial r} (rB_{j\theta}) - \frac{1}{r} \frac{\partial}{\partial \theta} (B_{jr}) = \mu_0 J_{j\phi} - i \frac{k_j^2}{\omega} E_{j\phi}, \tag{2.6}$$

where the current density is

$$\mathbf{J}(\mathbf{r}) = \delta(x) \delta(y) \delta(z-b) \hat{\mathbf{x}}. \tag{2.7}$$

The field in region 1 is the superposition of a primary and a secondary field, viz.,

$$\mathbf{F}_1 = \mathbf{F}_1^{(pr)} + \mathbf{F}_1^{(sc)}, \quad \mathbf{F} = \mathbf{E}, \mathbf{B}. \tag{2.8}$$

In order to calculate the primary field, introduce the vector potential $\mathbf{A}_1^{(pr)} = G(\mathbf{r}; b) \hat{\mathbf{x}}$, where

$$G(\mathbf{r}; b) = \frac{\mu_0}{4\pi} \frac{e^{ik_1 R}}{R} = \frac{i\mu_0 k_1}{4\pi} \sum_{l=0}^{\infty} (2l+1) j_l(k_1 r_<) h_l^{(1)}(k_1 r_>) P_l(\cos \theta), \tag{2.9}$$

$R = \sqrt{r^2 + b^2 - 2rb \cos \theta}$, $r_>$ is the larger of r and b , and $r_<$ is the smaller. j_l and $h_l^{(1)}$ denote the spherical Bessel and Hankel functions, and P_l^m is the Legendre function of the first kind.³⁶ Hence,

$$B_{1r}^{(pr)} = b_{1r}^{(pr)}(r, \theta) \sin \phi = -\frac{1}{r} \frac{\partial G}{\partial \theta} \sin \phi = -\frac{i\mu_0 k_1}{4\pi r} \sin \phi \sum_{l=0}^{\infty} (2l+1) j_l(k_1 r_{<}) h_l^{(1)}(k_1 r_{>}) P_l^1(\cos \theta), \tag{2.10}$$

$$B_{1\theta}^{(pr)} = b_{1\theta}^{(pr)}(r, \theta) \sin \phi, \quad B_{1\phi}^{(pr)} = b_{1\phi}^{(pr)}(r, \theta) \cos \phi, \tag{2.11}$$

$$\begin{aligned} E_{1r}^{(pr)} &= e_{1r}^{(pr)}(r, \theta) \cos \phi = -\frac{i\omega}{k_1^2 r} \frac{\partial}{\partial \theta} \left(\frac{\partial G}{\partial b} + \frac{1}{b} G \right) \cos \phi \\ &= \frac{\omega\mu_0}{4\pi b} \frac{1}{k_1 r} \cos \phi \sum_{l=0}^{\infty} (2l+1) \psi_l(k_1 r) [\check{\psi}_l(k_1 b) + k_1 b \check{\psi}'_l(k_1 b)] P_l^1(\cos \theta), \end{aligned} \tag{2.12}$$

$$E_{1\theta}^{(pr)} = e_{1\theta}^{(pr)}(r, \theta) \cos \phi, \quad E_{1\phi}^{(pr)} = e_{1\phi}^{(pr)}(r, \theta) \sin \phi, \tag{2.13}$$

where $(\psi_l, \check{\psi}_l) = (j_l, h_l^{(1)})$ if $r < b$, $(\psi_l, \check{\psi}_l) = (h_l^{(1)}, j_l)$ if $r > b$, and the prime denotes differentiation with respect to the argument. Let

$$B_{jr} = b_{jr}(r, \theta) \sin \phi, \quad B_{j\theta} = b_{j\theta}(r, \theta) \sin \phi, \quad B_{j\phi} = b_{j\phi}(r, \theta) \cos \phi, \tag{2.14}$$

$$E_{jr} = e_{jr}(r, \theta) \cos \phi, \quad E_{j\theta} = e_{j\theta}(r, \theta) \cos \phi, \quad E_{j\phi} = e_{j\phi}(r, \theta) \sin \phi. \tag{2.15}$$

It follows that

$$\frac{\partial^2}{\partial r^2} (rb_{j\phi}) + k_j^2 (rb_{j\phi}) = \frac{1}{\sin \theta} \frac{\partial}{\partial r} (b_{jr}) + \frac{ik_j^2}{\omega} \frac{\partial}{\partial \theta} (e_{jr}), \tag{2.16}$$

$$\frac{\partial^2}{\partial r^2} (re_{j\phi}) + k_j^2 (re_{j\phi}) = -i\omega \frac{\partial}{\partial \theta} (b_{jr}) - \frac{1}{\sin \theta} \frac{\partial}{\partial r} (e_{jr}), \tag{2.17}$$

$$b_{j\theta} = \frac{i}{\omega} \left[\frac{1}{r \sin \theta} e_{jr} + \frac{1}{r} \frac{\partial}{\partial r} (re_{j\phi}) \right], \tag{2.18}$$

$$e_{j\theta} = \frac{i\omega}{k_j^2} \left[\frac{1}{r \sin \theta} b_{jr} - \frac{1}{r} \frac{\partial}{\partial r} (rb_{j\phi}) \right]. \tag{2.19}$$

The total field must be bounded at the origin and satisfy the usual radiation conditions.³⁷ It is natural to set

$$e_{1r}^{(sc)} = -\frac{\mu_0 \omega}{4\pi r} \frac{1}{k_1 b} \sum_{l=0}^{\infty} (2l+1) \tilde{A}_{lj} j_l(k_1 r) [j_l(k_1 b) + k_1 b j'_l(k_1 b)] P_l^1(\cos \theta), \tag{2.20}$$

$$e_{2r} = \frac{\mu_0 \omega}{4\pi r} \frac{1}{k_1 b} \sum_{l=0}^{\infty} (2l+1) \tilde{B}_{lj} h_l^{(1)}(k_2 r) [j_l(k_1 b) + k_1 b j'_l(k_1 b)] P_l^1(\cos \theta), \tag{2.21}$$

$$b_{1r}^{(sc)} = \frac{i\mu_0 k_1}{4\pi r} \sum_{l=0}^{\infty} (2l+1) \tilde{C}_{lj} j_l(k_1 r) j_l(k_1 b) P_l^1(\cos \theta), \tag{2.22}$$

$$b_{2r} = -\frac{i\mu_0 k_1}{4\pi r} \sum_{l=0}^{\infty} (2l+1) \tilde{D}_{lj} h_l^{(1)}(k_2 r) j_l(k_1 b) P_l^1(\cos \theta), \tag{2.23}$$

where $\tilde{A}_l, \tilde{B}_l, \tilde{C}_l,$ and \tilde{D}_l are coefficients yet to be determined.

In region 1, Eqs. (2.16) and (2.17) split into two equations by virtue of Eq. (2.8). In compact notation, these equations are

$$\frac{\partial^2}{\partial r^2}(rw) + k^2(rw) = \frac{1}{\sin \theta} \frac{\partial g}{\partial r}, \tag{2.24a}$$

$$\frac{\partial^2}{\partial r^2}(rw) + k^2(rw) = \frac{\partial g}{\partial \theta}, \tag{2.24b}$$

where $g = g(r, \theta)$ ($g = b_{jr}, e_{jr}$) is treated as known. Specifically,

$$g(r, \theta) = r^{-1} \sum_{l=0}^{\infty} (2l+1) c_l \psi_l(kr) P_l^1(\cos \theta). \tag{2.25}$$

In the above, $\psi_l = j_l$ and $k = k_1$ if $r < a$, $\psi_l = h_l^{(1)}$ and $k = k_2$ if $r > a$.

A solution to Eq. (2.24a) is

$$rw(r, \theta) = \frac{1}{\sin \theta} \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} c_l w_l(r) P_l^1(\cos \theta). \tag{2.26}$$

Each $w_l(r)$ ($l = 0, 1, 2, \dots$) should of course satisfy

$$\left(\frac{d^2}{dr^2} + k^2 \right) w_l(r) = - \frac{l(l+1)}{r^2} [\psi_l(kr) - kr \psi_l'(kr)], \tag{2.27}$$

and therefore equals $w_l(r) = (d/dr)[r \psi_l(kr)]$. With regard to Eq. (2.24b),

$$rw(r, \theta) = \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} c_l w_l(r) \frac{\partial P_l^1}{\partial \theta}, \tag{2.28}$$

where $w_l(r)$ is forced to satisfy

$$\left(\frac{d^2}{dr^2} + k^2 \right) w_l(r) = \frac{l(l+1)}{r} \psi_l(kr), \tag{2.29}$$

with an admissible solution $w_l(r) = r \psi_l(kr)$. Once $b_{j\phi}$ and $e_{j\phi}$ are determined in this fashion, $b_{j\theta}$ and $e_{j\theta}$ follow from Eqs. (2.18) and (2.19).

The coefficients $\tilde{A}_l, \tilde{B}_l, \tilde{C}_l,$ and \tilde{D}_l in Eqs. (2.20)–(2.23) are calculated via imposition of the continuity of E_ϕ and B_θ at $r = a$. These conditions yield two independent systems of linear equations, namely, one for \tilde{A}_l, \tilde{B}_l and one for \tilde{C}_l and \tilde{D}_l . The former set describes a magnetic-type (H -) polarization ($B_r = 0, E_r \neq 0$), while the latter one pertains to an electric-type (E -) polarization ($E_r = 0, B_r \neq 0$), in correspondence to the case with a planar boundary. Explicitly,

$$\tilde{A}_l = \frac{h_l^{(1)}(k_1 a)}{j_l(k_1 a)} \left\{ \frac{1}{k_2 a} + \frac{h_l^{(1)'}(k_2 a)}{h_l^{(1)}(k_2 a)} - \frac{k_2}{k_1} \left[\frac{1}{k_1 a} + \frac{h_l^{(1)'}(k_1 a)}{h_l^{(1)}(k_1 a)} \right] \right\} \frac{1}{K_l}, \tag{2.30}$$

$$\tilde{B}_l = \frac{i}{k_1 k_2 a^2 j_l(k_1 a) h_l^{(1)}(k_2 a) K_l}, \tag{2.31}$$

$$\tilde{C}_l = \frac{h_l^{(1)}(k_1 a)}{j_l(k_1 a)} \left[\frac{h_l^{(1)'}(k_1 a)}{h_l^{(1)}(k_1 a)} - \frac{k_2}{k_1} \frac{h_l^{(1)'}(k_2 a)}{h_l^{(1)}(k_2 a)} \right] \frac{1}{M_l}, \tag{2.32}$$

$$\tilde{D}_l = - \frac{i}{(k_1 a)^2 j_l(k_1 a) h_l^{(1)}(k_2 a) M_l}, \tag{2.33}$$

where the two principal denominators read as

$$K_l = \frac{h_l^{(1)'}(k_2 a)}{h_l^{(1)}(k_2 a)} + \frac{1}{k_2 a} - \frac{k_2}{k_1} \left[\frac{1}{k_1 a} + \frac{j_l'(k_1 a)}{j_l(k_1 a)} \right], \tag{2.34}$$

$$M_l = \frac{j_l'(k_1 a)}{j_l(k_1 a)} - \frac{k_2}{k_1} \frac{h_l^{(1)'}(k_2 a)}{h_l^{(1)}(k_2 a)}. \tag{2.35}$$

The terminology above primarily serves the purpose of distinguishing between contributions from these two denominators.

The field in region 2 is

$$E_{2r} = \frac{i \omega \mu_0}{4 \pi k_1 r k_2 a^2} \cos \phi \sum_{l=0}^{\infty} (2l+1) \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{K_l} \left[\frac{1}{k_1 b} + \frac{j_l'(k_1 b)}{j_l(k_1 b)} \right] P_l^1(\cos \theta), \tag{2.36}$$

$$\begin{aligned} E_{2\theta} &= \frac{i \omega \mu_0}{4 \pi k_1 a^2} \cos \phi \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{K_l} \left[\frac{1}{k_1 b} + \frac{j_l'(k_1 b)}{j_l(k_1 b)} \right] \\ &\quad \times \left[\frac{1}{k_2 r} + \frac{h_l^{(1)'}(k_2 r)}{h_l^{(1)}(k_2 r)} \right] \frac{\partial P_l^1}{\partial \theta} \\ &\quad - \frac{i \omega \mu_0}{4 \pi k_1 a^2 \sin \theta} \cos \phi \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{M_l} P_l^1(\cos \theta), \end{aligned} \tag{2.37}$$

$$\begin{aligned} E_{2\phi} &= - \frac{i \omega \mu_0}{4 \pi k_1 a^2 \sin \theta} \sin \phi \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{K_l} \left[\frac{1}{k_1 b} + \frac{j_l'(k_1 b)}{j_l(k_1 b)} \right] \\ &\quad \times \left[\frac{1}{k_2 r} + \frac{h_l^{(1)'}(k_2 r)}{h_l^{(1)}(k_2 r)} \right] P_l^1(\cos \theta) \\ &\quad + \frac{i \omega \mu_0}{4 \pi k_1 a^2} \sin \phi \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{M_l} \frac{\partial P_l^1}{\partial \theta}, \end{aligned} \tag{2.38}$$

$$B_{2r} = - \frac{\mu_0}{4 \pi k_1 r a^2} \sin \phi \sum_{l=0}^{\infty} (2l+1) \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{M_l} P_l^1(\cos \theta), \tag{2.39}$$

$$\begin{aligned} B_{2\theta} &= - \frac{\mu_0 k_2}{4 \pi k_1 a^2} \sin \phi \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{M_l} \left[\frac{1}{k_2 r} + \frac{h_l^{(1)'}(k_2 r)}{h_l^{(1)}(k_2 r)} \right] \frac{\partial P_l^1}{\partial \theta} \\ &\quad - \frac{\mu_0 k_2}{4 \pi k_1 a^2 \sin \theta} \sin \phi \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{K_l} \left[\frac{1}{k_1 b} + \frac{j_l'(k_1 b)}{j_l(k_1 b)} \right] P_l^1(\cos \theta), \end{aligned} \tag{2.40}$$

$$\begin{aligned}
 B_{2\phi} = & -\frac{\mu_0 k_2}{4\pi k_1 a^2 \sin \theta} \cos \phi \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{M_l} \left[\frac{1}{k_2 r} + \frac{h_l^{(1)'}(k_2 r)}{h_l^{(1)}(k_2 r)} \right] P_l^1(\cos \theta) \\
 & -\frac{\mu_0 k_2}{4\pi k_1 a^2} \cos \phi \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1)} \frac{j_l(k_1 b)}{j_l(k_1 a)} \frac{h_l^{(1)}(k_2 r)}{h_l^{(1)}(k_2 a)} \frac{1}{K_l} \left[\frac{1}{k_1 b} + \frac{j_l'(k_1 b)}{j_l(k_1 b)} \right] \frac{\partial P_l^1}{\partial \theta}. \tag{2.41}
 \end{aligned}$$

Expansions (2.36)–(2.41) converge uniformly in all parameters if $r \neq a$ or $b \neq a$. For large l each series is majorized by a geometric series with expansion parameter $(b/a) \times (\min\{r,a\}/\max\{r,a\})$. For $k_2 a \gg 1$, the summands with $l \leq O(k_1 a)$ oscillate rapidly, hindering physical interpretation and rendering direct computations impractical.

B. Electromagnetic field on the surface

When the dipole and the observation point are allowed to approach the boundary ($b \rightarrow a^-$, $r \rightarrow a^+$), the series expansions for the field diverge.³⁸ Application of the Poisson summation formula (1.1) converts Eqs. (2.36)–(2.41) into the following series:

$$E_{2r} = \frac{i\omega\mu_0}{2\pi k_1 k_2 a^3} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\mathcal{K}(\nu)} \left[\frac{1}{2k_1 a} + \frac{J'_\nu(k_1 a)}{J_\nu(k_1 a)} \right] P_{\nu-1/2}^1(\cos \theta), \tag{2.42}$$

$$\begin{aligned}
 E_{2\theta} = & \frac{i\omega\mu_0}{2\pi k_1 a^2} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{K}(\nu)} \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] \\
 & \times \left[\frac{1}{2k_1 a} + \frac{J'_\nu(k_1 a)}{J_\nu(k_1 a)} \right] \frac{\partial P_{\nu-1/2}^1}{\partial \theta} - \frac{i\omega\mu_0}{2\pi k_1 a^2 \sin \theta} \cos \phi \\
 & \times \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{M}(\nu)} P_{\nu-1/2}^1(\cos \theta), \tag{2.43}
 \end{aligned}$$

$$\begin{aligned}
 E_{2\phi} = & -\frac{i\omega\mu_0}{2\pi k_1 a^2 \sin \theta} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{K}(\nu)} \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] \\
 & \times \left[\frac{1}{2k_1 a} + \frac{J'_\nu(k_1 a)}{J_\nu(k_1 a)} \right] P_{\nu-1/2}^1(\cos \theta) \\
 & + \frac{i\omega\mu_0}{2\pi k_1 a^2} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{M}(\nu)} \frac{\partial P_{\nu-1/2}^1}{\partial \theta}, \tag{2.44}
 \end{aligned}$$

$$B_{2r} = -\frac{\mu_0}{2\pi k_1 a^3} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\mathcal{M}(\nu)} P_{\nu-1/2}^1(\cos \theta), \tag{2.45}$$

$$\begin{aligned}
 B_{2\theta} = & -\frac{\mu_0 k_2}{2\pi k_1 a^2} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n \nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{M}(\nu)} \left[\frac{1}{2k_2 a} + \frac{H_{\nu}^{(1)'}(k_2 a)}{H_{\nu}^{(1)}(k_2 a)} \right] \frac{\partial P_{\nu-1/2}^1}{\partial \theta} \\
 & -\frac{\mu_0 k_2}{2\pi k_1 a^2} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n \nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{K}(\nu)} \\
 & \times \left[\frac{1}{2k_1 a} + \frac{J_{\nu}'(k_1 a)}{J_{\nu}(k_1 a)} \right] P_{\nu-1/2}^1(\cos \theta), \tag{2.46}
 \end{aligned}$$

$$\begin{aligned}
 B_{2\phi} = & -\frac{\mu_0 k_2}{2\pi k_1 a^2} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n \nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{M}(\nu)} \\
 & \times \left[\frac{1}{2k_2 a} + \frac{H_{\nu}^{(1)'}(k_2 a)}{H_{\nu}^{(1)}(k_2 a)} \right] P_{\nu-1/2}^1(\cos \theta) \\
 & -\frac{\mu_0 k_2}{2\pi k_1 a^2} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n \nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{K}(\nu)} \left[\frac{1}{2k_1 a} + \frac{J_{\nu}'(k_1 a)}{J_{\nu}(k_1 a)} \right] \frac{\partial P_{\nu-1/2}^1}{\partial \theta}, \tag{2.47}
 \end{aligned}$$

where

$$\mathcal{K}(\nu) = \frac{H_{\nu}^{(1)'}(k_2 a)}{H_{\nu}^{(1)}(k_2 a)} + \frac{1}{2k_2 a} - \frac{k_2}{k_1} \left[\frac{1}{2k_1 a} + \frac{J_{\nu}'(k_1 a)}{J_{\nu}(k_1 a)} \right], \tag{2.48}$$

$$\mathcal{M}(\nu) = \frac{J_{\nu}'(k_1 a)}{J_{\nu}(k_1 a)} - \frac{k_2}{k_1} \frac{H_{\nu}^{(1)'}(k_2 a)}{H_{\nu}^{(1)}(k_2 a)}. \tag{2.49}$$

The integrands in Eqs. (2.42)–(2.47) are meromorphic functions of ν . The zeros of $\mathcal{K}(\nu)$ and $\mathcal{M}(\nu)$ are sometimes called Regge poles in the literature (for example, see Ref. 29). The study of the possible resonances associated with these poles, which are often believed to give rise to various effects of absorption and scattering from spheres and other scatterers, lies beyond the scope of this analysis. A discussion on the location of the zeros of $\mathcal{K}(\nu)$ and $\mathcal{M}(\nu)$ is provided in Appendix A. Each integral diverges in the usual sense, but is interpreted unambiguously as³⁸

$$\int_0^{\infty} d\nu (\dots) = \lim_{\nu \rightarrow 0^+} \int_0^{\infty} d\nu (\dots) e^{-\nu \nu}. \tag{2.50}$$

III. SOMMERFELD INTEGRALS AND LOWEST-ORDER CORRECTIONS

A. Approximate integral formulas

As $\theta \rightarrow 0^+$, Eqs. (2.42)–(2.47) should reduce to known integral formulas.³⁴ The following steps are taken when $\theta \ll 1$: (i) Only the $n=0$ terms are retained, since the integrands with $n \neq 0$ are highly oscillatory. (ii) The Bessel functions are replaced by asymptotic formulas that are valid outside the transitional regions.³⁹ (iii) The Legendre functions are replaced by MacDonald’s formulas³⁶ that involve Bessel functions. (iv) The integration path is properly deformed in the fourth quadrant of the ν -plane, as suggested by the analysis in Appendix A.

Accordingly, $\mathcal{K}^{-1}(\nu)$ and $\mathcal{M}^{-1}(\nu)$ from Eqs. (2.48) and (2.49) are approximated as follows:

$$\frac{1}{\mathcal{K}(\nu)} \sim -i \left[\sqrt{1 - \left(\frac{\nu}{k_2 a}\right)^2} + \frac{k_2}{k_1} \sqrt{1 - \left(\frac{\nu}{k_1 a}\right)^2} \right]^{-1} - \frac{1}{2k_2 a} \frac{(\nu/k_2 a)^2 [1 - (\nu/k_2 a)^2]^{-1} - (k_2^2/k_1^2) (\nu/k_1 a)^2 [1 - (\nu/k_1 a)^2]^{-1}}{[\sqrt{1 - (\nu/k_2 a)^2} + (k_2/k_1) \sqrt{1 - (\nu/k_1 a)^2}]^2}, \quad (3.1)$$

$$\frac{1}{\mathcal{M}(\nu)} \sim i \left[\sqrt{1 - \left(\frac{\nu}{k_1 a}\right)^2} + \frac{k_2}{k_1} \sqrt{1 - \left(\frac{\nu}{k_2 a}\right)^2} \right]^{-1} - \frac{1}{2k_1 a} \frac{[1 - (\nu/k_1 a)^2]^{-1} - [1 - (\nu/k_2 a)^2]^{-1}}{\left[\sqrt{1 - (\nu/k_1 a)^2} + \frac{k_2}{k_1} \sqrt{1 - (\nu/k_2 a)^2}\right]^2}. \quad (3.2)$$

Note that the simplified $\mathcal{K}^{-1}(\nu = \lambda a)$ exhibits a pair of poles at

$$\lambda = \pm k_S = \pm \frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}} \sim \pm \left(k_2 - \frac{k_2^3}{2k_1^2} \right), \quad k_2^2 \ll |k_1^2|. \quad (3.3)$$

No poles exist in the approximation for $\mathcal{M}^{-1}(\lambda a)$.

With $\lambda = \nu/a$ and $\rho = a\theta$, the field components reduce to

$$E_{2r}^{n=0} \sim i \frac{\omega \mu_0 k_2^2}{2\pi k_1} (I_{ez} - I_{ez}^c) \cos \phi, \quad (3.4)$$

$$E_{2\theta}^{n=0} \sim - \frac{\omega \mu_0 k_2^2}{4\pi k_1} (I_{e\rho} + I_{e\rho}^c) \cos \phi, \quad (3.5)$$

$$E_{2\phi}^{n=0} \sim \frac{\omega \mu_0 k_2^2}{4\pi k_1} (I_{e\phi} + I_{e\phi}^c) \sin \phi, \quad (3.6)$$

$$B_{2r}^{n=0} \sim i \frac{\mu_0 k_2^2}{2\pi} (I_{bz} + I_{bz}^c) \sin \phi, \quad (3.7)$$

$$B_{2\theta}^{n=0} \sim - \frac{\mu_0 k_2^3}{4\pi k_1} (I_{b\rho} + I_{b\rho}^c) \sin \phi, \quad (3.8)$$

$$B_{2\phi}^{n=0} \sim - \frac{\mu_0 k_2^3}{4\pi k_1} (I_{b\phi} + I_{b\phi}^c) \cos \phi. \quad (3.9)$$

In the above, $I_{f\kappa}$ ($f = e, b$; $\kappa = \rho, \phi, z$) denote the Sommerfeld integrals,³⁴ viz.,

$$I_{ez} = k_2^{-3} \int_0^\infty d\lambda \frac{\sqrt{1 - (\lambda/k_1)^2}}{\sqrt{1 - (\lambda/k_2)^2} + (k_2/k_1) \sqrt{1 - (\lambda/k_1)^2}} \lambda^2 J_1(\lambda\rho), \quad (3.10)$$

$$I_{e\rho} = k_2^{-2} \int_0^\infty d\lambda \lambda \left\{ \frac{\sqrt{1 - (\lambda/k_1)^2} \sqrt{1 - (\lambda/k_2)^2}}{\sqrt{1 - (\lambda/k_2)^2} + (k_2/k_1) \sqrt{1 - (\lambda/k_1)^2}} [J_0(\lambda\rho) - J_2(\lambda\rho)] + \frac{1}{\sqrt{1 - (\lambda/k_1)^2} + (k_2/k_1) \sqrt{1 - (\lambda/k_2)^2}} [J_0(\lambda\rho) + J_2(\lambda\rho)] \right\}, \quad (3.11)$$

$$I_{e\phi} = k_2^{-2} \int_0^\infty d\lambda \lambda \left\{ \frac{\sqrt{1-(\lambda/k_1)^2} \sqrt{1-(\lambda/k_2)^2}}{\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}} [J_0(\lambda\rho) + J_2(\lambda\rho)] \right. \\ \left. + \frac{1}{\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}} [J_0(\lambda\rho) - J_2(\lambda\rho)] \right\}, \tag{3.12}$$

$$I_{bz} = k_1^{-1} k_2^{-2} \int_0^\infty d\lambda \frac{1}{\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}} \lambda^2 J_1(\lambda\rho), \tag{3.13}$$

$$I_{b\rho} = k_2^{-2} \int_0^\infty d\lambda \lambda \left\{ \frac{\sqrt{1-(\lambda/k_2)^2}}{\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}} [J_0(\lambda\rho) - J_2(\lambda\rho)] \right. \\ \left. + \frac{\sqrt{1-(\lambda/k_1)^2}}{\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}} [J_0(\lambda\rho) + J_2(\lambda\rho)] \right\}, \tag{3.14}$$

$$I_{b\phi} = k_2^{-2} \int_0^\infty d\lambda \lambda \left\{ \frac{\sqrt{1-(\lambda/k_2)^2}}{\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}} [J_0(\lambda\rho) + J_2(\lambda\rho)] \right. \\ \left. + \frac{\sqrt{1-(\lambda/k_1)^2}}{\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}} [J_0(\lambda\rho) - J_2(\lambda\rho)] \right\}. \tag{3.15}$$

The lowest-order corrections $I_{f\kappa}^c$ read as

$$I_{ez}^c = \frac{i}{2k_2^4 a} \int_0^\infty d\lambda \left\{ \frac{k_2}{k_1} \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2} \frac{1}{\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}} \right. \\ \left. + \sqrt{1-(\lambda/k_1)^2} \frac{\frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} - (k_2^2/k_1^2) \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2}}{[\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}]^2} \right\} \lambda^2 J_1(\lambda\rho), \tag{3.16}$$

$$I_{e\rho}^c = \frac{i}{2k_2^3 a} \int_0^\infty d\lambda \left\{ \frac{\sqrt{1-(\lambda/k_1)^2} \frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} - (k_2/k_1) \sqrt{1-(\lambda/k_2)^2} \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2}}{\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}} \right. \\ \left. - \sqrt{1-(\lambda/k_2)^2} \sqrt{1-(\lambda/k_1)^2} \frac{\frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} - (k_2^2/k_1^2) \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2}}{[\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}]^2} \right\} \lambda [J_0(\lambda\rho) \\ - J_2(\lambda\rho)] + \frac{i}{2k_1 k_2^2 a} \int_0^\infty d\lambda \frac{[1-(\lambda/k_1)^2]^{-1} - [1-(\lambda/k_2)^2]^{-1}}{[\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}]^2} \lambda [J_0(\lambda\rho) + J_2(\lambda\rho)], \tag{3.17}$$

$$\begin{aligned}
 I_{e\phi}^c = & \frac{i}{2k_2^3 a} \int_0^\infty d\lambda \left\{ \frac{\sqrt{1-(\lambda/k_1)^2} \frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} - (k_2/k_1) \sqrt{1-(\lambda/k_2)^2} \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2}}{\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}} \right. \\
 & \left. - \frac{\sqrt{1-(\lambda/k_2)^2} \sqrt{1-(\lambda/k_1)^2} \frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} - (k_2^2/k_1^2) \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2}}{[\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}]^2} \right\} \lambda [J_0(\lambda\rho) + J_2(\lambda\rho)] \\
 & + \frac{i}{2k_1 k_2^2 a} \int_0^\infty d\lambda \frac{[1-(\lambda/k_1)^2]^{-1} - [1-(\lambda/k_2)^2]^{-1}}{[\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}]^2} \lambda [J_0(\lambda\rho) - J_2(\lambda\rho)], \quad (3.18)
 \end{aligned}$$

$$I_{bz}^c = \frac{i}{2k_1^2 k_2^2 a} \int_0^\infty d\lambda \frac{[1-(\lambda/k_1)^2]^{-1} - [1-(\lambda/k_2)^2]^{-1}}{[\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}]^2} \lambda^2 J_1(\lambda\rho), \quad (3.19)$$

$$\begin{aligned}
 I_{b\rho}^c = & \frac{i}{2k_2^3 a} \int_0^\infty d\lambda \left\{ \frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} \frac{1}{\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}} \right. \\
 & \left. + \frac{k_2}{k_1} \sqrt{1-(\lambda/k_2)^2} \frac{[1-(\lambda/k_1)^2]^{-1} - [1-(\lambda/k_2)^2]^{-1}}{[\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}]^2} \right\} \lambda [J_0(\lambda\rho) - J_2(\lambda\rho)] \\
 & - \frac{i}{2k_2^3 a} \int_0^\infty d\lambda \left\{ \frac{k_2}{k_1} \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2} \frac{1}{\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}} \right. \\
 & \left. + \sqrt{1-(\lambda/k_1)^2} \frac{\frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} - (k_2^2/k_1^2) \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2}}{[\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}]^2} \right\} \lambda [J_0(\lambda\rho) + J_2(\lambda\rho)], \quad (3.20)
 \end{aligned}$$

$$\begin{aligned}
 I_{b\phi}^c = & \frac{i}{2k_2^3 a} \int_0^\infty d\lambda \left\{ \frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} \frac{1}{\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}} \right. \\
 & \left. + \frac{k_2}{k_1} \sqrt{1-(\lambda/k_2)^2} \frac{[1-(\lambda/k_1)^2]^{-1} - [1-(\lambda/k_2)^2]^{-1}}{[\sqrt{1-(\lambda/k_1)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_2)^2}]^2} \right\} \lambda [J_0(\lambda\rho) + J_2(\lambda\rho)] \\
 & - \frac{i}{2k_2^3 a} \int_0^\infty d\lambda \left\{ \frac{k_2}{k_1} \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2} \frac{1}{\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}} \right. \\
 & \left. + \sqrt{1-(\lambda/k_1)^2} \frac{\frac{(\lambda/k_2)^2}{1-(\lambda/k_2)^2} - (k_2^2/k_1^2) \frac{(\lambda/k_1)^2}{1-(\lambda/k_1)^2}}{[\sqrt{1-(\lambda/k_2)^2} + (k_2/k_1) \sqrt{1-(\lambda/k_1)^2}]^2} \right\} \lambda [J_0(\lambda\rho) - J_2(\lambda\rho)]. \quad (3.21)
 \end{aligned}$$

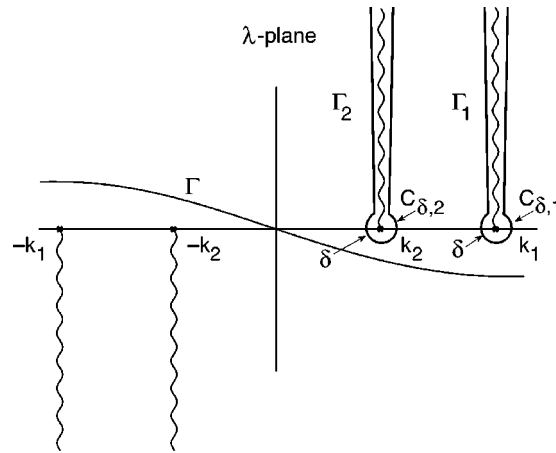


FIG. 2. Branch-cut configuration and integration paths Γ and Γ_j , $j=1, 2$, pertaining to integrals (3.10)–(3.21); k_1 is taken to be real. The final formulas can be extended to complex k_1 .

The first Riemann sheet is chosen so that all square roots are positive for $0 < \lambda < k_2$, if k_1 is real, with the branch-cut configuration of Fig. 2; evidently, no pole lies in this sheet.

B. Integrated formulas, $k_2\rho \gg 1$, $k_2^2 \ll |k_1^2|$

When $k_2\rho \gg 1$, the major contributions to integration in Eqs. (3.10)–(3.21) arise from the vicinities of branch points at $\lambda = k_j$ ($j=1, 2$). Explicit expressions for the integrals $I_{f\kappa}$ are given elsewhere.^{34,35} It is noted in passing that $I_{e\rho}$, $I_{e\phi}$ and I_{bz} are evaluated exactly in terms of well-converging series that involve Fresnel and exponential integrals.³⁵

Attention is now turned to $I_{f\kappa}^c$. By following the procedure in Appendix B of Ref. 35, let $I_{f\kappa,j}^c$ denote the contour integral over the path Γ_j of Fig. 2. Clearly, $I_{f,\kappa}^c = I_{f\kappa,2}^c + I_{f\kappa,1}^c$, since each $I_{f\kappa,j}^c$ follows from $I_{f\kappa}^c$ under

$$\int_0^\infty d\lambda (\dots) J_s(\lambda\rho) \rightarrow \frac{1}{2} \int_{\Gamma_j} d\lambda (\dots) H_s^{(1)}(\lambda\rho). \tag{3.22}$$

With $\lambda = k_j(1 + it)$ in each side of the branch cuts in Fig. 2, it follows that

$$\sqrt{1 - (\lambda/k_j)^2} = \pm e^{-i\pi/4} \sqrt{2t} \sqrt{1 + it/2} \sim \pm e^{-i\pi/4} \sqrt{2t}, \quad t \rightarrow 0^+, \quad \sqrt{t} \geq 0, \tag{3.23}$$

where the upper sign holds along the left-hand side and the lower sign along the right-hand side of each branch cut. Due to the factors $[1 - (\lambda/k_j)^2]^{-1}$, the indentations $C_{\delta,j}$ contribute to the value of $I_{f\kappa,j}^c$ as δ approaches 0. For example, $I_{e_{z,2}}^c$ requires the limit

$$\begin{aligned} & \lim_{\delta \rightarrow 0^+} \int_{C_{\delta,2}} d\lambda \left\{ \frac{(k_2/k_1)(\lambda/k_1)^2 [1 - (\lambda/k_1)^2]^{-1}}{\sqrt{1 - (\lambda/k_2)^2} + (k_2/k_1)\sqrt{1 - (\lambda/k_1)^2}} \right. \\ & \quad \left. + \frac{\sqrt{1 - (\lambda/k_1)^2} \{ (\lambda/k_2)^2 [1 - (\lambda/k_2)^2]^{-1} - (k_2^2/k_1^2)(\lambda/k_1)^2 [1 - (\lambda/k_1)^2]^{-1} \}}{[\sqrt{1 - (\lambda/k_2)^2} + (k_2/k_1)\sqrt{1 - (\lambda/k_1)^2}]^2} \right\} \\ & \quad \times \lambda^2 H_1^{(1)}(\lambda\rho) \\ & = 2\pi i \left(\frac{-k_2}{2} \right) \sqrt{\frac{1 - (k_2/k_1)^2}{[(k_2/k_1)\sqrt{1 - (k_2/k_1)^2}]^2}} k_2^2 H_1^{(1)}(k_2\rho) \sim -e^{i(k_2\rho - \pi/4)} k_1^2 k_2 \sqrt{\frac{2\pi}{k_2\rho}}. \end{aligned} \tag{3.24}$$

Define

$$\varphi = \frac{k_2^3 \rho}{2k_1^2} = \frac{k_2^2}{2k_1^2} (k_2 a) \theta, \tag{3.25}$$

$$F(\varphi) = e^{-i\varphi} \int_{\varphi}^{\infty} dx \frac{e^{ix}}{\sqrt{2\pi x}} = e^{-i\varphi} \left[\frac{1}{2} (1+i) - C(\varphi) - iS(\varphi) \right], \tag{3.26}$$

where $C(\varphi)$ and $S(\varphi)$ are the Fresnel integrals defined as³⁶

$$C(\varphi) = \int_0^{\varphi} dx \frac{\cos x}{\sqrt{2\pi x}}, \quad S(\varphi) = \int_0^{\varphi} dx \frac{\sin x}{\sqrt{2\pi x}}. \tag{3.27}$$

The relevant calculations are illustrated by

$$\begin{aligned} I_{ez,2}^c &\sim -\frac{1}{4} e^{i(k_2 \rho - 3\pi/4)} \frac{1}{k_2 a} \sqrt{\frac{2}{\pi k_2 \rho}} \int_0^{\infty} dt \left\{ \frac{k_2^3}{k_1^3} \left[-\frac{1}{e^{-i\pi/4} \sqrt{2t + k_2/k_1}} + \frac{1}{-e^{-i\pi/4} \sqrt{2t + k_2/k_1}} \right] \right. \\ &\quad \left. + \left(\frac{1}{-2it} - \frac{k_2^4}{k_1^4} \right) \left[-\frac{1}{(e^{-i\pi/4} \sqrt{2t + k_2/k_1})^2} + \frac{1}{(-e^{-i\pi/4} \sqrt{2t + k_2/k_1})^2} \right] \right\} e^{-k_2 \rho t} \\ &\quad + \frac{1}{2} e^{i(k_2 \rho - 3\pi/4)} \frac{k_1^2}{k_2^3 a} \sqrt{\frac{\pi}{2k_2 \rho}} \\ &\sim \frac{i}{2} e^{ik_2 \rho} \frac{1}{k_2 a} \sqrt{\pi k_2 \rho} \left\{ F(\varphi) - i(2\pi\varphi)^{-1/2} + (2i\varphi)^{-1} \left[F(\varphi) - \frac{1}{2}(1+i) \right] \right\} \end{aligned} \tag{3.28a}$$

$$\sim \begin{cases} \frac{1+i}{4} e^{ik_2 \rho} \sqrt{\frac{\pi}{k_2 \rho}} \frac{k_1^2}{k_2^3 a}, & |\varphi| \gg 1, \\ \frac{1-i}{4} e^{ik_2 \rho} \sqrt{\frac{\pi}{k_2 \rho}} \frac{\rho}{a}, & |\varphi| \ll 1, \end{cases} \tag{3.28b}$$

$$I_{ez,1}^c \sim -\frac{1}{2} e^{i(k_1 \rho - \pi/4)} \frac{k_1}{k_2^2 a} \sqrt{\frac{\pi}{2k_1 \rho}}. \tag{3.29}$$

From formulas (3.4)–(3.9), with $k_2 a \gg 1$, $k_2^2 \ll |k_1^2|$, $k_2 a \theta \gg 1$, and θ sufficiently small,

$$\begin{aligned} E_{2r}^{n=0} &\sim i \frac{\omega \mu_0 k_2^2}{2\pi k_1} \cos \phi \left\{ -i e^{ik_2 \rho} \frac{k_2}{k_1} \sqrt{\frac{\pi}{k_2 \rho}} \left[F(\varphi) - i(2\pi\varphi)^{-1/2} + \frac{k_1 \rho}{2k_2 a} \left\{ F(\varphi) - i(2\pi\varphi)^{-1/2} \right. \right. \right. \\ &\quad \left. \left. + (2i\varphi)^{-1} \left[F(\varphi) - \frac{1}{2}(1+i) \right] \right\} \right\} + \frac{e^{ik_1 \rho}}{k_2^2 \rho^2} \left(1 + \frac{1-i}{4} \sqrt{\pi k_1 \rho} \frac{\rho}{a} \right), \end{aligned} \tag{3.30}$$

$$\begin{aligned} E_{2\theta}^{n=0} &\sim \frac{\omega \mu_0 k_2^2}{2\pi k_1} \cos \phi \left\{ e^{ik_2 \rho} \frac{k_2^2}{k_1^2} \sqrt{\frac{\pi}{k_2 \rho}} \left[F(\varphi) - i(2\pi\varphi)^{-1/2} + \frac{k_1 \rho}{2k_2 a} \left\{ F(\varphi) - i(2\pi\varphi)^{-1/2} \right. \right. \right. \\ &\quad \left. \left. + (2i\varphi)^{-1} \left[F(\varphi) - \frac{1}{2}(1+i) \right] \right\} \right\} + \frac{e^{ik_1 \rho}}{k_2^2 \rho^2} \left(1 + \frac{1-i}{4} \sqrt{\pi k_1 \rho} \frac{\rho}{a} \right), \end{aligned} \tag{3.31}$$

$$E_{2\phi}^{n=0} \sim i \frac{\omega \mu_0 k_2^2}{2\pi k_1} \sin \phi \left\{ e^{ik_2\rho} \frac{k_2}{k_1^2 \rho} \sqrt{\frac{\pi}{k_2 \rho}} \left[F(\varphi) - 2i(2\pi\varphi)^{-1/2} + \frac{k_1 \rho}{2k_2 a} \left\{ F(\varphi) - i(2\pi\varphi)^{-1/2} \right. \right. \right. \\ \left. \left. \left. + (2i\varphi)^{-1} \left[F(\varphi) - \frac{1}{2}(1+i)(1-2i\varphi) \right] \right\} \right] + i \frac{e^{ik_1\rho}}{k_2^2 \rho^2} \left(1 + \frac{1-i}{4} \sqrt{\pi k_1 \rho} \frac{\rho}{a} \right) \right\}, \quad (3.32)$$

$$B_{2r}^{n=0} \sim \frac{\mu_0 k_2^2}{2\pi} \sin \phi \left\{ \frac{e^{ik_2\rho}}{k_1^2 \rho^2} \left(1 - \frac{1-i}{4} \sqrt{\pi k_2 \rho} \frac{\rho}{a} \right) - \frac{e^{ik_1\rho}}{k_2^2 \rho^2} \left(1 + \frac{1-i}{4} \sqrt{\pi k_1 \rho} \frac{\rho}{a} \right) \right\}, \quad (3.33)$$

$$B_{2\theta}^{n=0} \sim i \frac{\mu_0 k_2^3}{2\pi k_1} \sin \phi \left\{ \frac{e^{ik_2\rho}}{k_1 \rho} \sqrt{\frac{\pi}{k_2 \rho}} \left[F(\varphi) - 2i(2\pi\varphi)^{-1/2} + \frac{k_1 \rho}{2k_2 a} \left\{ F(\varphi) - i(2\pi\varphi)^{-1/2} \right. \right. \right. \\ \left. \left. \left. + (2i\varphi)^{-1} \left[F(\varphi) - \frac{1}{2}(1+i)(1-2i\varphi) \right] \right\} \right] + e^{ik_1\rho} \frac{k_1}{k_2^3 \rho^2} \left(1 + \frac{1-i}{4} \sqrt{\pi k_1 \rho} \frac{\rho}{a} \right) \right\}, \quad (3.34)$$

$$B_{2\phi}^{n=0} \sim -\frac{\mu_0 k_2^3}{2\pi k_1} \cos \phi \left\{ e^{ik_2\rho} \frac{k_2}{k_1} \sqrt{\frac{\pi}{k_2 \rho}} \left[F(\varphi) - i(2\pi\varphi)^{-1/2} + \frac{k_1 \rho}{2k_2 a} \left\{ F(\varphi) - i(2\pi\varphi)^{-1/2} \right. \right. \right. \\ \left. \left. \left. + (2i\varphi)^{-1} \left[F(\varphi) - \frac{1}{2}(1+i) \right] \right\} \right] + \frac{e^{ik_1\rho}}{k_2^2 \rho^2} \left(i \frac{k_2}{k_1} - \frac{1}{k_2 \rho} \right) \left(1 + \frac{1-i}{4} \sqrt{\pi k_1 \rho} \frac{\rho}{a} \right) \right\}. \quad (3.35)$$

Consequently, $|I_{f\kappa,j}| \sim |I_{f\kappa,j}^c|$ provided that, for real k_1 ,

$$\rho \sim a(k_j a/2)^{-1/3} = \rho_{cr,j} = a\theta_{cr,j}, \quad j=1, 2, \quad (3.36)$$

where $\rho_{cr,2}$ is essentially Fock's "reduced distance."¹⁶ Both $\rho_{cr,j}$, $j=1, 2$, enter as parameters in the analysis for a vertical electric dipole.^{3,4,33} Evidently, expressions (3.30)–(3.35) imply that

$$\theta \ll \theta_{cr,j}. \quad (3.37)$$

Of course, if k_1 is complex, one of these inequalities is replaced by $\theta \ll |\theta_{cr,1}|$.

When $1 \ll k_2 \rho \ll |k_1 \rho| \ll |k_1 \rho_{cr,2}|$, the maximum magnitudes in ϕ of the field components traveling with the air phase velocity satisfy

$$|E_{2r,2}^{n=0}|_m : |E_{2\theta,2}^{n=0}|_m : |E_{2\phi,2}^{n=0}|_m = O(1) : O(k_2/k_1) : O[(k_1 \rho)^{-1}], \quad |\varphi| \leq O(1), \\ = O(1) : O(k_2/k_1) : O(k_2^3/k_1^3), \quad |\varphi| \gg 1, \quad (3.38a)$$

$$|B_{2r,2}^{n=0}|_m : |B_{2\theta,2}^{n=0}|_m : |B_{2\phi,2}^{n=0}|_m = O[(k_1 \rho)^{-1}] : O[(k_2 \rho)^{-1}] : O(1), \quad |\varphi| \leq 1, \\ = O[(k_2 \rho)^{-3/2}] : O[(k_2 \rho)^{-1}] : O(1), \quad \varphi = O(1), \\ = O(k_2^3/k_1^3) : O(k_2^2/k_1^2) : O(1), \quad |\varphi| \gg 1. \quad (3.38b)$$

IV. WAVES IN THE CRITICAL RANGES, $\theta = O(\theta_{cr,j})$

When θ becomes of the order of $\theta_{cr,1}$ or $\theta_{cr,2}$ introduced in Eq. (3.36), approximations (3.4)–(3.9) break down. The approximation of Bessel functions by Airy integrals gives^{39,40}

$$\mathcal{K}^{-1}(v) \sim -\frac{k_2}{k_1} \left[1 + \frac{k_2^2}{k_1^2} \left(\frac{2}{k_1 a} \right)^{1/3} \mathcal{H}(\xi_1; 0) \right], \quad \xi_1 = (k_1 a/2)^{-1/3} (v - k_1 a) = O(1), \quad (4.1a)$$

$$\mathcal{K}^{-1}(\nu) \sim -\left(\frac{k_2 a}{2}\right)^{1/3} [\mathcal{H}(\xi_2; 2\pi/3) - i\alpha]^{-1}, \quad \xi_2 = (k_2 a/2)^{-1/3} (\nu - k_2 a) = O(1), \quad (4.1b)$$

while

$$\mathcal{M}^{-1}(\nu) \sim 1 + \left(\frac{2}{k_1 a}\right)^{1/3} \mathcal{H}(\xi_1; 0), \quad \xi_1 = O(1), \quad (4.2a)$$

$$\mathcal{M}^{-1}(\nu) \sim i + \frac{k_2}{k_1} \left(\frac{2}{k_2 a}\right)^{1/3} \mathcal{H}(\xi_2; 2\pi/3), \quad \xi_2 = O(1). \quad (4.2b)$$

In the above,

$$\mathcal{H}(\xi; \psi) = e^{i\psi} \frac{\text{Ai}'(e^{i\psi}\xi)}{\text{Ai}(e^{i\psi}\xi)}, \quad (4.3)$$

$$\alpha = \frac{k_2}{k_1} \left(\frac{k_2 a}{2}\right)^{1/3}. \quad (4.4)$$

Notice the appearance of the Airy function $\text{Ai}(z)$ and its derivative.⁴⁰ The Legendre functions are replaced by Bessel functions^{36,39} of argument $\nu\theta$ where $\nu\theta \gg 1$.

By using the subscript j to denote the contribution from $\nu = k_j a$ ($j = 1, 2$),

$$E_{2r,2}^{n=0} \sim -\frac{\omega\mu_0}{4\pi a} e^{i(k_2 a\theta + \pi/4)} \frac{k_2}{k_1} \left(\frac{k_2 a}{2}\right)^{2/3} \sqrt{\frac{2}{\pi k_2 a\theta}} \mathcal{I} \cos \phi, \quad (4.5a)$$

$$E_{2r,1}^{n=0} \sim -\frac{\omega\mu_0}{4\pi a} e^{i(k_1 a\theta - \pi/4)} \sqrt{\frac{2}{\pi k_1 a\theta}} \mathcal{I}_1 \cos \phi, \quad (4.5b)$$

$$E_{2\theta,2}^{n=0} \sim -\frac{\omega\mu_0}{4\pi a} e^{i(k_2 a\theta + \pi/4)} \frac{k_2^2}{k_1^2} \left(\frac{k_2 a}{2}\right)^{2/3} \sqrt{\frac{2}{\pi k_2 a\theta}} \left[\mathcal{I} + \frac{i}{k_2 a\theta} \left(\frac{k_2 a}{2}\right)^{-2/3} \mathcal{I}_2 \right] \cos \phi, \quad (4.6a)$$

$$E_{2\theta,1}^{n=0} \sim \frac{\omega\mu_0}{4\pi a} e^{i(k_1 a\theta + \pi/4)} \sqrt{\frac{2}{\pi k_1 a\theta}} \mathcal{I}_1 \cos \phi, \quad (4.6b)$$

$$E_{2\phi,2}^{n=0} \sim -\frac{\omega\mu_0}{4\pi a} e^{i(k_2 a\theta + \pi/4)} \frac{k_2^2}{k_1^2} \sqrt{\frac{2}{\pi k_2 a\theta}} \left[\mathcal{I}_2 + i \frac{\theta_{\text{cr},2}}{2\theta} \mathcal{I} \right] \sin \phi, \quad (4.7a)$$

$$E_{2\phi,1}^{n=0} \sim -\frac{\omega\mu_0}{4\pi a} e^{i(k_1 a\theta + \pi/4)} \sqrt{\frac{2}{\pi k_1 a\theta}} \mathcal{I}_1 \sin \phi, \quad (4.7b)$$

$$B_{2r,2}^{n=0} \sim -\frac{\mu_0 k_2}{4\pi a} e^{i(k_2 a\theta + \pi/4)} \frac{k_2^2}{k_1^2} \sqrt{\frac{2}{\pi k_2 a\theta}} \mathcal{I}_2 \sin \phi, \quad (4.8a)$$

$$B_{2r,1}^{n=0} \sim -\frac{\mu_0 k_1}{4\pi a} e^{i(k_1 a\theta + \pi/4)} \sqrt{\frac{2}{\pi k_1 a\theta}} \mathcal{I}_1 \sin \phi, \quad (4.8b)$$

$$B_{2\theta,2}^{n=0} \sim -\frac{\mu_0 k_1}{4\pi a} e^{i(k_2 a\theta + \pi/4)} \frac{k_2^2}{k_1^2} \sqrt{\frac{2}{\pi k_2 a\theta}} \left[\mathcal{I}_2 + i \frac{\theta_{\text{cr},2}}{2\theta} \mathcal{I} \right] \sin \phi, \quad (4.9a)$$

$$B_{2\theta,1}^{n=0} \sim -\frac{\mu_0 k_1}{4\pi a} e^{i(k_1 a \theta - \pi/4)} \sqrt{\frac{2}{\pi k_1 a \theta}} \mathcal{I}_1 \sin \phi, \tag{4.9b}$$

$$B_{2\phi,2}^{n=0} \sim \frac{\mu_0 k_2}{4\pi a} e^{i(k_2 a \theta + \pi/4)} \frac{k_2}{k_1} \left(\frac{k_2 a}{2}\right)^{2/3} \sqrt{\frac{2}{\pi k_2 a \theta}} \left[\mathcal{I}_+ + \frac{i}{k_2 a \theta} \left(\frac{k_2 a}{2}\right)^{-2/3} \mathcal{I}_2 \right] \cos \phi, \tag{4.10a}$$

$$B_{2\phi,1}^{n=0} \sim \frac{\mu_0 k_2}{4\pi a} e^{i(k_1 a \theta + \pi/4)} \left(\frac{1}{k_2 a \theta} - i \frac{k_2}{k_1}\right) \sqrt{\frac{2}{\pi k_1 a \theta}} \mathcal{I}_1 \cos \phi, \tag{4.10b}$$

where

$$\mathcal{I}_1 = \mathcal{I}_1(\theta) = \int_{-\infty - i\varsigma}^{\infty - i\varsigma} d\xi e^{i(\theta/\theta_{cr,1})\xi} \mathcal{H}(\xi; 0), \tag{4.11}$$

$$\mathcal{I}_2 = \mathcal{I}_2(\theta) = \int_{-\infty - i\varsigma}^{\infty - i\varsigma} d\xi e^{i(\theta/\theta_{cr,2})\xi} \mathcal{H}(\xi; 2\pi/3), \tag{4.12}$$

$$\mathcal{I} = \mathcal{I}(\theta; \alpha) = \int_{-\infty - i\varsigma}^{\infty - i\varsigma} d\xi \frac{e^{i(\theta/\theta_{cr,2})\xi}}{\mathcal{H}(\xi; 2\pi/3) - i\alpha}, \quad \varsigma > 0. \tag{4.13}$$

Because the sole singularities of the integrands are poles in the upper ξ -plane, including the real axis, terms with factors $e^{-i(\theta/\theta_{cr,j})\xi}$ are integrated out to zero. \mathcal{I} and \mathcal{I}_2 describe propagation through region 2.

For $\theta \ll |\theta_{cr,1}|$, the leading contributions to integration in Eqs. (4.11)–(4.13) are determined by the large- ξ behavior of $\mathcal{H}(\xi; \psi)$. Accordingly,⁴⁰

$$\mathcal{I}_1 \sim - \int_{-\infty - i\varsigma}^{\infty - i\varsigma} d\xi e^{i(\theta/\theta_{cr,1})\xi} \left(\sqrt{\xi} + \frac{1}{4\xi} \right) = e^{-i\pi/4} \sqrt{\frac{2\pi}{k_1 a}} \left(\frac{\rho}{a}\right)^{-3/2} - \frac{i\pi}{2}, \tag{4.14}$$

while, for $\theta \ll \theta_{cr,2}$,

$$\mathcal{I}_2 \sim \int_{-\infty - i\varsigma}^{\infty - i\varsigma} d\xi e^{i(\theta/\theta_{cr,2})\xi} \left(\sqrt{\xi} - \frac{1}{4\xi} \right) = -e^{-i\pi/4} \sqrt{\frac{2\pi}{k_2 a}} \left(\frac{\rho}{a}\right)^{-3/2} - \frac{i\pi}{2}, \tag{4.15}$$

$$\begin{aligned} \mathcal{I} \sim \int_{-\infty - i\varsigma}^{\infty - i\varsigma} d\xi \frac{e^{i(\theta/\theta_{cr,2})\xi}}{\sqrt{\xi} - 1/4\xi - i\alpha} \sim i\pi e^{i\pi/4} 2^{3/2} \frac{k_2}{k_1} \left(\frac{k_2 a}{2}\right)^{1/3} & \left\{ F(\varphi) - i(2\pi\varphi)^{-1/2} + \frac{k_1 \rho}{2k_2 a} \left[F(\varphi) \right. \right. \\ & \left. \left. - i(2\pi\varphi)^{-1/2} + (2i\varphi)^{-1} \left\{ F(\varphi) - \frac{1}{2}(1+i) \right\} \right] \right\}, \end{aligned} \tag{4.16}$$

in agreement with formulas (3.30)–(3.35).

A. Propagation through air

The integral \mathcal{I}_2 of Eq. (4.12) is expressed as a general Dirichlet series⁴¹ over the residues associated with poles of $\mathcal{H}(\xi; 2\pi/3)$. (See Appendix A for $\kappa = k_2/k_1$.) By closing the contour in the upper ξ -plane,

$$\mathcal{I}_2 = 2\pi i \sum_{s=1}^{\infty} e^{i(\theta/\theta_{cr,2})|a_s|} e^{i\pi/3}, \tag{4.17}$$

where a_s are the zeros of $\text{Ai}(z)$ numbered in order of ascending magnitude.⁴⁰ Series (4.17) is approximated by its first term if $\theta \gg \theta_{\text{cr},2}$.

The poles associated with \mathcal{I} are obtained by solving

$$e^{i2\pi/3} \frac{\text{Ai}'(\xi e^{i2\pi/3})}{\text{Ai}(\xi e^{i2\pi/3})} = i\alpha, \quad \alpha = \frac{k_2}{k_1} \left(\frac{k_2 a}{2} \right)^{1/3}. \tag{4.18}$$

Let $\{\xi_s = \xi_s(\alpha)\}_{s=1,2,\dots}$ be the sequence of these roots; $\xi_s(0)$ are numbered in order of increasing imaginary part. The integral \mathcal{I} equals

$$\mathcal{I} = 2\pi i \sum_{s=1}^{\infty} \frac{e^{i(\theta/\theta_{\text{cr},2})\xi_s}}{\xi_s + \alpha^2}. \tag{4.19}$$

Because $\{\xi_s\}$ do not have any finite limit, they should approach the Stokes line $\text{Arg } \xi = \pi/3$ as $s \rightarrow \infty$. Each $\xi_s(z)$ satisfies

$$\frac{d\xi_s}{dz} = \frac{i}{\xi_s + z^2}, \quad \xi_s(0) = |\check{a}_s| e^{i\pi/3}, \tag{4.20}$$

via differentiation of both sides of Eq. (4.18) in $\alpha \equiv z$. \check{a}_s denote the zeros of $\text{Ai}'(z)$.⁴⁰ Equation (4.20) was given by Fock¹⁶ and has been studied numerically in the literature.⁴² By integrating (4.20) along a path where $|\xi_s(z)| \gg |z^2|$,

$$\xi_s(\alpha) \sim \xi_s(0) + \frac{i\alpha}{\xi_s(0)}. \tag{4.21}$$

Clearly,

$$\mathcal{I} \sim 2\pi i \frac{e^{i(\theta/\theta_{\text{cr},2})\bar{\xi}}}{\bar{\xi} + \alpha^2}, \quad \theta \gg \theta_{\text{cr},2}, \tag{4.22}$$

where $\bar{\xi} = \bar{\xi}(\alpha)$ is the root of Eq. (4.18) with the smallest imaginary part. It is of interest to compare $\min_s \text{Im } \xi_s(\alpha)$ with its limiting value for $\alpha \rightarrow \infty$. With $\alpha = t e^{-i\vartheta}$ ($0 \leq t < \infty$, $0 \leq \vartheta \leq \pi/4$) and fixed ϑ , the trajectory of each $\xi_j(\alpha) = \beta_j(t) + i\gamma_j(t)$ can be described by the coupled equations

$$\frac{d\beta_j}{dt} = \frac{\gamma_j \cos \vartheta + (\beta_j - t^2) \sin \vartheta}{(\beta_j + t^2 \cos 2\vartheta)^2 + (\gamma_j - t^2 \sin 2\vartheta)^2}, \tag{4.23a}$$

$$\frac{d\gamma_j}{dt} = \frac{(\beta_j + t^2) \cos \vartheta - \gamma_j \sin \vartheta}{(\beta_j + t^2 \cos 2\vartheta)^2 + (\gamma_j - t^2 \sin 2\vartheta)^2}, \tag{4.23b}$$

where $\beta_j(0) = |\check{a}_j|/2$, $\gamma_j(0) = \sqrt{3}|\check{a}_j|/2$, and $|\check{a}_j| < |a_j| < |\check{a}_{j+1}|$. Of course, $\lim_{t \rightarrow \infty} \xi_j(\alpha(t)) = |a_j| e^{i\pi/3}$ uniformly in ϑ . For definiteness, consider $j=1$. If $\vartheta=0$, $\beta_1(t)$ and $\gamma_1(t)$ are monotonically increasing in t , and the slope of $\xi_1(\alpha(t))$ equals $\pi/6$ for $t=0$ and approaches $\pi/2$ as $t \rightarrow \infty$. A close inspection of Eqs. (4.23) shows that $\gamma_1(t)$ remains monotonically increasing for fixed $\vartheta \in (0, \pi/6]$, while $\beta_1(t)$ reaches a maximum. For fixed $\vartheta \in (\pi/6, \pi/4]$, $\gamma_1(t)$ is monotonically decreasing and reaches a minimum, and then progresses monotonically to its limiting value. The lowest minimum of $\gamma_1(t)$ is reached for $\vartheta = \pi/4$, when the slope of $\xi_1(\alpha(t))$ is $-\pi/12$ for $t=0$ and $3\pi/4$ as $t \rightarrow \infty$. By relaxing routine rigor, the assumed analyticity of $\xi_1(\alpha)$ in $\Delta_R = \{\alpha: 0 < |\alpha| < R, -\pi/4 < \text{Arg } \alpha < 0\}$, where R is positive and arbitrarily large, entails that $\text{Im } \xi_1(\alpha)$ is harmonic in Δ_R and hence cannot attain any maximum or minimum there. It follows from Fig. 3 that the maximum occurs along the boundary $\{\alpha: |\alpha|=R, \text{Arg } \alpha = \pi/4\}$.

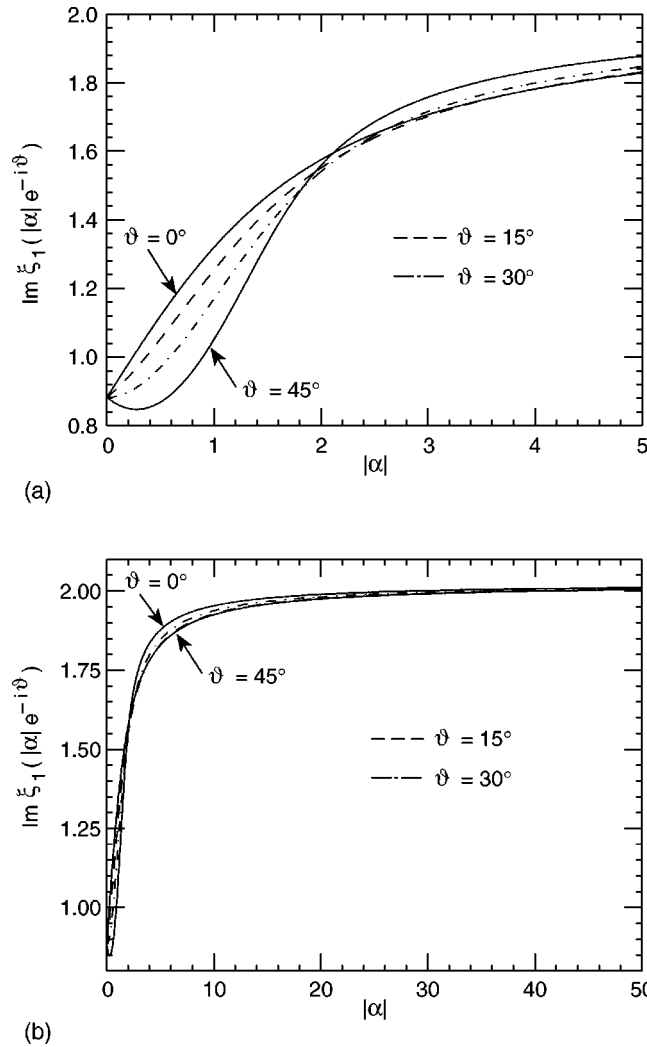


FIG. 3. Imaginary part of the root $\xi_1(\alpha)$ of (4.18) for fixed values of the phase $-\vartheta$ and varying magnitude $|\alpha|$, where $\alpha = |\alpha|e^{-i\vartheta} = (k_2/k_1)(k_2a/2)^{1/3}$, for (a) $0 \leq |\alpha| \leq 5$, and (b) $0 \leq |\alpha| \leq 50$.

It is therefore implied that, for $\alpha \leq O(1)$,

$$\mathcal{I}_2/\mathcal{I} = O[e^{-(\theta/\theta_{cr,2})\bar{h}}], \quad \bar{h} \equiv \frac{1}{2}\sqrt{3}|a_1| - \min_s \text{Im } \xi_s(\alpha), \quad \theta \gg \theta_{cr,2}, \quad (4.24a)$$

while for $\alpha \rightarrow \infty$,

$$\mathcal{I}_2/\mathcal{I} = O(\alpha^2) \quad \text{as } \alpha \rightarrow \infty. \quad (4.24b)$$

Formulas (4.6a) and (4.10a) are further simplified:

$$E_{2\theta,2}^{n=0} \sim -\frac{\omega\mu_0}{4\pi a} e^{i(k_2a\theta + \pi/4)} \frac{k_2^2}{k_1^2} \left(\frac{k_2a}{2}\right)^{2/3} \sqrt{\frac{2}{\pi k_2a\theta}} \mathcal{I} \cos \phi \sim \frac{k_2}{k_1} E_{2r,2}^{n=0}, \quad (4.25)$$

$$B_{2\phi,2}^{n=0} \sim \frac{\mu_0 k_2}{4\pi a} e^{i(k_2a\theta + \pi/4)} \frac{k_2}{k_1} \left(\frac{k_2a}{2}\right)^{2/3} \sqrt{\frac{2}{\pi k_2a\theta}} \mathcal{I} \cos \phi \sim -\frac{1}{c} E_{2r,2}^{n=0}, \quad (4.26)$$

where c is the velocity of light in air.

From expressions (4.7a) and (4.9a),

$$E_{2\phi,2}^{n=0} \sim -i \frac{\omega \mu_0}{8 \pi a} e^{i(k_2 a \theta + \pi/4)} \frac{k_2^2}{k_1^2} \sqrt{\frac{2}{\pi k_2 a \theta}} \frac{\theta_{cr,2}}{\theta} \mathcal{I} \sin \phi, \tag{4.27}$$

$$B_{2\theta,2}^{n=0} \sim -i \frac{\mu_0 k_1}{8 \pi a} e^{i(k_2 a \theta + \pi/4)} \frac{k_2^2}{k_1^2} \sqrt{\frac{2}{\pi k_2 a \theta}} \frac{\theta_{cr,2}}{\theta} \mathcal{I} \sin \phi, \quad \theta \gg \theta_{cr,2}, \quad |\alpha| \leq O(1). \tag{4.28}$$

It is inferred that when $O(\theta_{cr,2}) \leq \theta \leq 1, |\alpha| \leq O(1)$,

$$|E_{2r,2}^{n=0}|_m : |E_{2\theta,2}^{n=0}|_m : |E_{2\phi,2}^{n=0}|_m = O(1) : O(k_2/k_1) : O[(k_1 a \theta)^{-1}], \tag{4.29a}$$

$$|B_{2r,2}^{n=0}|_m : |B_{2\theta,2}^{n=0}|_m : |B_{2\phi,2}^{n=0}|_m = O\left[\frac{\mathcal{I}_2}{\mathcal{I}} \frac{k_2}{k_1} \left(\frac{k_2 a}{2}\right)^{-2/3}\right] : O[(k_2 a \theta)^{-1}] : O(1). \tag{4.29b}$$

B. Propagation through region 1

Difficulties in the evaluation of \mathcal{I}_1 arise because of the presence of poles in the negative real axis. These poles stem from zeros of $\mathcal{K}(\nu)$ or $\mathcal{M}(\nu)$, as outlined in Appendix A. By use of the Wronskian of $\text{Ai}(\zeta e^{-i\pi/3})$ and $\text{Ai}(\zeta e^{i\pi/3})$,

$$\mathcal{I}_1 = \int_{\mathcal{L}} d\zeta [\mathcal{H}(-\zeta; 0) - \mathcal{H}(-\zeta; -2\pi/3)] e^{-i(\theta/\theta_{cr,1})\zeta} = -\frac{e^{i\pi/6}}{2\pi} \int_{\mathcal{L}} d\zeta \frac{e^{-i(\theta/\theta_{cr,1})\zeta}}{\text{Ai}(\zeta e^{i\pi/3}) \text{Ai}(-\zeta)}, \tag{4.30}$$

where \mathcal{L} is a path that extends along the negative real axis, passes through zero and then extends slightly above the positive real axis. This integral can be cast in a form that is amenable to numerical computation.³³ Alternatively, rewrite \mathcal{I}_1 as

$$\begin{aligned} \mathcal{I}_1 &= -\frac{e^{i\pi/6}}{2\pi} \left[\int_0^\infty d\xi \frac{e^{i(\theta/\theta_{cr,1})\xi}}{\text{Ai}(\xi e^{-i2\pi/3}) \text{Ai}(\xi)} + \int_0^{\infty+i\epsilon} d\zeta \frac{e^{-i(\theta/\theta_{cr,1})\zeta}}{\text{Ai}(\zeta e^{i\pi/3}) \text{Ai}(-\zeta)} \right] \\ &= -\frac{i}{\pi} \left\{ \int_0^\infty dy \frac{\exp[-(\theta/\theta_{cr,1})y e^{i\pi/6}] + 2 \exp[(\theta/\theta_{cr,1})y e^{-i\pi/6}]}{\text{Ai}(y)^2 + \text{Bi}(y)^2} \right. \\ &\quad \left. + i \int_0^\infty dy \frac{\exp[-(\theta/\theta_{cr,1})y e^{i\pi/6}] \text{Bi}(y)}{\text{Ai}(y)^2 + \text{Bi}(y)^2} \right\}, \tag{4.31} \end{aligned}$$

by rotation of each integration path in the ξ - or ζ -plane by $2\pi/3$ or $\pi/3$ counterclockwise. The right-hand side of Eq. (4.31) involves exponentially converging integrals.

When $\theta \gg |\theta_{cr,1}|$, \mathcal{I}_1 is further simplified. By virtue of the equality

$$\text{Ai}(-\zeta) + e^{i2\pi/3} \text{Ai}(\zeta e^{-i\pi/3}) + e^{-i2\pi/3} \text{Ai}(\zeta e^{i\pi/3}) = 0,$$

it is deduced that

$$\frac{1}{\text{Ai}(\zeta e^{i\pi/3}) \text{Ai}(-\zeta)} = 2\pi e^{-i\pi/6} \frac{d}{d\zeta} \int_0^{\varpi(\zeta)} \frac{dy}{1+y}, \tag{4.32}$$

where

$$\varpi(\zeta) \equiv e^{-i2\pi/3} \frac{\text{Ai}(\zeta e^{-i\pi/3})}{\text{Ai}(\zeta e^{i\pi/3})}. \tag{4.33}$$

Note that $\varpi(\zeta)$ is bounded everywhere except near the zeros of $\text{Ai}(\zeta e^{i\pi/3})$. Substitution of Eq. (4.32) into (4.30) and application of integration by parts furnish

$$\mathcal{I}_1 = -i \frac{\theta}{\theta_{\text{cr},1}} \left[\sum_{p=0}^{P-1} \frac{(-1)^p}{p+1} \int_{\mathcal{L}} d\zeta e^{-i(\theta/\theta_{\text{cr},1})\zeta} \varpi(\zeta)^{p+1} + \mathcal{R}_p(\theta/\theta_{\text{cr},1}) \right], \tag{4.34}$$

where

$$\mathcal{R}_p(x) = (-1)^p \int_{\mathcal{L}} d\zeta e^{-ix\zeta} \int_0^{\varpi(\zeta)} dy \frac{y^p}{1+y}, \quad P \geq 1, \tag{4.35}$$

$$\mathcal{R}_p(x) \sim \frac{(-1)^p}{P+1} \int_{\mathcal{L}} d\zeta e^{-ix\zeta} \frac{\varpi(\zeta)^{P+1}}{1+\varpi(\zeta)}, \quad P \geq 1. \tag{4.36}$$

For $\theta \gg |\theta_{\text{cr},1}|$, the leading contributions in Eq. (4.34) come from points ζ_p that render the phase of $e^{-i(\theta/\theta_{\text{cr},1})\zeta} \varpi(\zeta)^{p+1}$ stationary. Consequently, ζ_p obey

$$\frac{\varpi'(\zeta_p)}{\varpi(\zeta_p)} = i \frac{\theta}{\theta_{\text{cr},1}(p+1)}, \tag{4.37}$$

or

$$\frac{1}{\pi} \frac{1}{\text{Ai}(-\zeta_p)^2 + \text{Bi}(-\zeta_p)^2} = \frac{\theta}{2\theta_{\text{cr},1}(p+1)} = \chi_p. \tag{4.38}$$

When $p < O(\theta/\theta_{\text{cr},1})$, ζ_p is positive and large,⁴⁰ viz.,

$$\zeta_p = \chi_p^2 + O(\chi_p^{-4}), \quad |\chi_p| \geq 1. \tag{4.39}$$

On the other hand, $\chi_p = O(1)$ implies $\zeta_p = O(1)$. In view of approximation (4.36), the remainder in Eq. (4.34) can be neglected if P is of the order of $\theta/\theta_{\text{cr},1}$. With

$$\varpi(\zeta)^{p+1} \sim (-i)^{p+1} \exp\left[i \frac{4}{3}(p+1)\zeta^{3/2} \right], \tag{4.40}$$

an ordinary stationary-phase calculation leads to

$$\begin{aligned} \mathcal{I}_1 &\sim -i \frac{\theta}{\theta_{\text{cr},1}} \sum_{p=0}^{O(\theta/\theta_{\text{cr},1})} \frac{(-1)^p}{p+1} \exp\left[-\frac{i}{12} \frac{\theta^3}{\theta_{\text{cr},1}^3 (p+1)^2} \right] (-i)^{p+1} \\ &\quad \times \int_{-\infty}^{\infty} d\zeta \exp\left\{ i \frac{\theta}{\theta_{\text{cr},1}} \left[\frac{\theta_{\text{cr},1}(p+1)}{\theta} \right]^2 (\zeta - \zeta_p)^2 \right\} \\ &\sim -e^{i\pi/4} \left(\frac{\theta}{\theta_{\text{cr},1}} \right)^{3/2} \sqrt{\pi} \mathcal{S}(k_1 a \theta^3), \end{aligned} \tag{4.41}$$

$$\mathcal{S}(z) = \sum_{p=0}^{\infty} \frac{i^p}{(p+1)^2} \exp\left[-\frac{i}{24} \frac{z}{(p+1)^2} \right]. \tag{4.42}$$

Finally, for $\theta \gg |\theta_{\text{cr},1}|$, $k_2^2 \ll |k_1^2|$ and $k_2 a \theta \gg 1$,

$$E_{2r,1}^{n=0} \sim \frac{\omega \mu_0 \theta}{4\pi a} e^{ik_1 a \theta} \mathcal{S}(k_1 a \theta^3) \cos \phi \sim i E_{2\theta,1}^{n=0}, \tag{4.43}$$

$$E_{2\phi,1}^{n=0} \sim \frac{i\omega\mu_0\theta}{4\pi a} e^{ik_1a\theta} \mathcal{S}(k_1a\theta^3) \sin\phi, \tag{4.44}$$

$$B_{2r,1}^{n=0} \sim \frac{i\mu_0k_1\theta}{4\pi a} e^{ik_1a\theta} \mathcal{S}(k_1a\theta^3) \sin\phi \sim iB_{2\theta,1}^{n=0}, \tag{4.45}$$

$$\begin{aligned} B_{2\phi,1}^{n=0} &\sim -\frac{i\mu_0k_2\theta}{4\pi a} e^{ik_1a\theta} \left(\frac{1}{k_2a\theta} - i\frac{k_2}{k_1} \right) \mathcal{S}(k_1a\theta^3) \cos\phi \\ &\sim -\frac{\mu_0k_2^2\theta}{4\pi k_1a} e^{ik_1a\theta} \mathcal{S}(k_1a\theta^3) \cos\phi, \quad k_2a\theta \gg |k_1|/k_2. \end{aligned} \tag{4.46}$$

V. FIELD IN THE RANGE $\mathcal{O}(\theta_{cr,j}) < \theta \leq \pi$

A. Formulation

Consider the identity

$$\begin{aligned} \mathcal{M}(\nu) - \frac{H_\nu^{(2)'}(k_1a)}{H_\nu^{(2)}(k_1a)} &= -\frac{W[H_\nu^{(1)}(k_1a), H_\nu^{(2)}(k_1a)]}{2H_\nu^{(2)}(k_1a)J_\nu(k_1a)} - \frac{k_2}{k_1} \frac{H_\nu^{(1)'}(k_2a)}{H_\nu^{(1)}(k_2a)} \\ &= \frac{2i}{\pi k_1a H_\nu^{(2)}(k_1a)J_\nu(k_1a)} - \frac{k_2}{k_1} \frac{H_\nu^{(1)'}(k_2a)}{H_\nu^{(1)}(k_2a)}, \end{aligned} \tag{5.1}$$

which is implied from Eq. (4.32) and leads to the decomposition

$$\frac{1}{\mathcal{M}(\nu)} = \frac{1}{\mathcal{D}(\nu)} - \mathcal{F}(\nu) \sum_{p=0}^{P-1} \mathcal{G}(\nu)^p - \mathcal{F}(\nu) \frac{\mathcal{G}(\nu)^P}{1-\mathcal{G}(\nu)}, \tag{5.2}$$

where

$$\mathcal{D}(\nu) = \frac{H_\nu^{(2)'}(k_1a)}{H_\nu^{(2)}(k_1a)} - \frac{k_2}{k_1} \frac{H_\nu^{(1)'}(k_2a)}{H_\nu^{(1)}(k_2a)}, \tag{5.3}$$

$$\mathcal{F}(\nu) = \frac{4i}{\pi k_1a} \frac{1}{\mathcal{D}(\nu)^2 H_\nu^{(2)}(k_1a)^2}, \tag{5.4}$$

$$\mathcal{G}(\nu) = \frac{-H_\nu^{(1)}(k_1a)}{H_\nu^{(2)}(k_1a)} - \frac{4i}{\pi k_1a} \frac{1}{\mathcal{D}(\nu) H_\nu^{(2)}(k_1a)^2}. \tag{5.5}$$

When $P \rightarrow \infty$, Eq. (5.2) reduces to an expansion of the Debye type,⁷ also employed by Nussenzveig.^{29,32} In the lower ν -plane slightly below the positive real axis, the limit $P \rightarrow \infty$ in Eq. (5.2) is meaningful because $|\mathcal{G}(\nu)| < 1$. However, care should be exercised in taking this limit under the integral sign.

A corresponding decomposition for $\mathcal{K}^{-1}(\nu)$ reads as

$$\frac{1}{\mathcal{K}(\nu)} = \frac{1}{\mathcal{A}(\nu)} + \mathcal{B}(\nu) \sum_{p=0}^{P-1} \mathcal{C}(\nu)^p + \mathcal{B}(\nu) \frac{\mathcal{C}(\nu)^P}{1-\mathcal{C}(\nu)}, \tag{5.6}$$

where

$$\mathcal{A}(\nu) = \frac{H_\nu^{(1)'}(k_2a)}{H_\nu^{(1)}(k_2a)} - \frac{k_2}{k_1} \frac{H_\nu^{(2)'}(k_1a)}{H_\nu^{(2)}(k_1a)} + \frac{1}{2k_2a} - \frac{k_2^2}{k_1^2} \frac{1}{2k_2a}, \tag{5.7}$$

$$\mathcal{B}(\nu) = \frac{4i}{\pi k_1 a} \frac{k_2}{k_1} \frac{1}{\mathcal{A}(\nu)^2 H_\nu^{(2)}(k_1 a)^2}, \tag{5.8}$$

$$\mathcal{C}(\nu) = \frac{-H_\nu^{(1)}(k_1 a)}{H_\nu^{(2)}(k_1 a)} + \frac{4i}{\pi k_1 a} \frac{k_2}{k_1} \frac{1}{\mathcal{A}(\nu) H_\nu^{(2)}(k_1 a)^2}. \tag{5.9}$$

Expressions (5.6)–(5.9) are also derived in Ref. 33 for the field of a vertical electric dipole over a spherical earth. $\mathcal{A}(\nu)$ and $\mathcal{D}(\nu)$ are entire functions of ν satisfying³⁹

$$\mathcal{A}(-\nu) = \mathcal{A}(\nu), \quad \mathcal{D}(-\nu) = \mathcal{D}(\nu). \tag{5.10}$$

A brief discussion on the location of their zeros is given in Appendix B.

Residues that are associated with the poles ν_j of $\mathcal{A}^{-1}(\nu)$ and $\tilde{\nu}_j$ of $\mathcal{D}^{-1}(\nu)$ in the upper ν -plane give rise to exponentially decreasing waves that propagate through air. On the other hand, stationary-phase contributions from $\mathcal{B}(\nu)\mathcal{C}(\nu)^p$ and $\mathcal{F}(\nu)\mathcal{G}(\nu)^p$, combined with $e^{i2\pi n\nu}$ and the Legendre functions, give rise to rays that travel in region 1. Contributions from these rays become significant when $\text{Im } k_1 a \ll 1$.

With Eqs. (C1) and (C5) of Appendix C,

$$\mathbf{E}_2 = \mathbf{E}_2^{\text{res}} + \mathbf{E}_2^{\text{ray}}, \quad \mathbf{B}_2 = \mathbf{B}_2^{\text{res}} + \mathbf{B}_2^{\text{ray}}, \tag{5.11}$$

where

$$E_{2r}^{\text{res}} = \frac{i\omega\mu_0}{2\pi k_1 k_2 a^3} \cos\phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty d\nu e^{i2\pi n\nu} \frac{\nu}{\mathcal{A}(\nu)} \left[\frac{1}{2k_1 a} + \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} \right] P_{\nu-1/2}^1(\cos\theta), \tag{5.12}$$

$$\begin{aligned} E_{2\theta}^{\text{res}} &= \frac{i\omega\mu_0}{2\pi k_1 a^2} \cos\phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \\ &\times \left\{ \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} + \frac{k_2}{k_1} \frac{1}{\mathcal{A}(\nu)} \left[\frac{1}{2k_1 a} + \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} \right]^2 \right\} \frac{\partial P_{\nu-1/2}^1(\cos\theta)}{\partial\theta} \\ &- \frac{i\omega\mu_0}{2\pi k_1 a^2 \sin\theta} \cos\phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{D}(\nu)} P_{\nu-1/2}^1(\cos\theta), \end{aligned} \tag{5.13}$$

$$\begin{aligned} E_{2\phi}^{\text{res}} &= -\frac{i\omega\mu_0}{2\pi k_1 a^2 \sin\theta} \sin\phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \\ &\times \left\{ \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} + \frac{k_2}{k_1} \frac{1}{\mathcal{A}(\nu)} \left[\frac{1}{2k_1 a} + \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} \right]^2 \right\} P_{\nu-1/2}^1(\cos\theta) \\ &+ \frac{i\omega\mu_0}{2\pi k_1 a^2} \sin\phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{D}(\nu)} \frac{\partial P_{\nu-1/2}^1}{\partial\theta}, \end{aligned} \tag{5.14}$$

$$B_{2r}^{\text{res}} = -\frac{\mu_0}{2\pi k_1 a^3} \sin\phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^\infty d\nu e^{i2\pi n\nu} \frac{\nu}{\mathcal{D}(\nu)} P_{\nu-1/2}^1(\cos\theta), \tag{5.15}$$

$$\begin{aligned}
 B_{2\theta}^{\text{res}} = & -\frac{\mu_0}{2\pi a^2} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{D}(\nu)} \left[\frac{1}{2k_1 a} + \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} \right] \\
 & \times \frac{\partial P_{\nu-1/2}^1}{\partial \theta} - \frac{\mu_0 k_2}{2\pi k_1 a^2 \sin \theta} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{A}(\nu)} \\
 & \times \left[\frac{1}{2k_1 a} + \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} \right] P_{\nu-1/2}^1(\cos \theta), \tag{5.16}
 \end{aligned}$$

$$\begin{aligned}
 B_{2\phi}^{\text{res}} = & -\frac{\mu_0}{2\pi a^2 \sin \theta} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{D}(\nu)} \\
 & \times \left[\frac{1}{2k_1 a} + \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} \right] P_{\nu-1/2}^1(\cos \theta) - \frac{\mu_0 k_2}{2\pi k_1 a^2} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \\
 & \times \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \frac{1}{\mathcal{A}(\nu)} \left[\frac{1}{2k_1 a} + \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} \right] \frac{\partial P_{\nu-1/2}^1}{\partial \theta}, \tag{5.17}
 \end{aligned}$$

and

$$\begin{aligned}
 E_{2r}^{\text{ray}} = & \frac{i\omega\mu_0}{2\pi a} \frac{1}{(k_2 a)^2} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n\nu} \nu \mathcal{B}(\nu) \mathcal{C}(\nu)^p \\
 & \times \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] P_{\nu-1/2}^1(\cos \theta), \tag{5.18}
 \end{aligned}$$

$$\begin{aligned}
 E_{2\theta}^{\text{ray}} = & \frac{i\omega\mu_0}{2\pi k_2 a^2} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{B}(\nu) \mathcal{C}(\nu)^p \\
 & \times \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right]^2 \frac{\partial P_{\nu-1/2}^1(\cos \theta)}{\partial \theta} + \frac{i\omega\mu_0}{2\pi k_1 a^2 \sin \theta} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \\
 & \times \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{F}(\nu) \mathcal{G}(\nu)^p P_{\nu-1/2}^1(\cos \theta), \tag{5.19}
 \end{aligned}$$

$$\begin{aligned}
 E_{2\phi}^{\text{ray}} = & -\frac{i\omega\mu_0}{2\pi k_2 a^2 \sin \theta} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{B}(\nu) \mathcal{C}(\nu)^p \\
 & \times \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right]^2 P_{\nu-1/2}^1(\cos \theta) - \frac{i\omega\mu_0}{2\pi k_1 a^2} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \\
 & \times \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{F}(\nu) \mathcal{G}(\nu)^p \frac{\partial P_{\nu-1/2}^1}{\partial \theta}, \tag{5.20}
 \end{aligned}$$

$$B_{2r}^{\text{ray}} = \frac{\mu_0}{2\pi k_1 a^3} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n\nu} \nu \mathcal{F}(\nu) \mathcal{G}(\nu)^p P_{\nu-1/2}^1(\cos \theta), \tag{5.21}$$

$$\begin{aligned}
 B_{2\theta}^{\text{ray}} &= \frac{\mu_0 k_2}{2\pi k_1 a^2} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n \nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{F}(\nu) \mathcal{G}(\nu)^p \\
 &\times \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] \frac{\partial P_{\nu-1/2}^1}{\partial \theta} - \frac{\mu_0}{2\pi a^2 \sin \theta} \sin \phi \sum_{n=-\infty}^{\infty} (-1)^n \\
 &\times \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n \nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{B}(\nu) \mathcal{C}(\nu)^p \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] P_{\nu-1/2}^1, \tag{5.22}
 \end{aligned}$$

$$\begin{aligned}
 B_{2\phi}^{\text{ray}} &= \frac{\mu_0 k_2}{2\pi k_1 a^2 \sin \theta} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n \nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{F}(\nu) \mathcal{G}(\nu)^p \\
 &\times \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] P_{\nu-1/2}^1(\cos \theta) - \frac{\mu_0}{2\pi a^2} \cos \phi \sum_{n=-\infty}^{\infty} (-1)^n \\
 &\times \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n \nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{B}(\nu) \mathcal{C}(\nu)^p \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] \frac{\partial P_{\nu-1/2}^1}{\partial \theta}. \tag{5.23}
 \end{aligned}$$

B. Residue series

The residue contributions are illustrated by

$$\begin{aligned}
 E_{2r}^{\text{res}} &= -\frac{i\omega\mu_0}{k_1 k_2 a^3} \cos \phi \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \left\{ e^{i(2n+1)\pi\nu} \nu \left[\frac{1}{2k_1 a} + \frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} \right] \right. \\
 &\times \left. P_{\nu-1/2}^1(-\cos \theta) \right\}_{\nu=\nu_j} \text{Res}_{\nu=\nu_j} \{ \mathcal{A}^{-1}(\nu) \}, \tag{5.24}
 \end{aligned}$$

$$B_{2r}^{\text{res}} = \frac{\mu_0}{k_1 a^3} \sin \phi \sum_{n=0}^{\infty} (-1)^n \sum_{j=1}^{\infty} [e^{i(2n+1)\pi\nu} \nu P_{\nu-1/2}^1(-\cos \theta)]_{\nu=\bar{\nu}_j} \text{Res}_{\nu=\bar{\nu}_j} \{ \mathcal{D}^{-1}(\nu) \}, \tag{5.25}$$

where the poles are numbered in order of ascending imaginary part.

If both θ and $\pi - \theta$ are $O(1)$,³⁶

$$P_{\nu-1/2}^1(-\cos \theta) \sim \sqrt{\frac{\nu}{2\pi \sin \theta}} \sum_{s=\pm} e^{-is[\nu(\pi-\theta) + \pi/4]}. \tag{5.26}$$

Hence, each n in Eqs. (5.24) and (5.25) represents the “winding number” of two wave paths in air, namely, one of length $\rho_n^+(\theta) = (2\pi n + \theta)a$ and another of length $\rho_n^- = \rho_n^+(2\pi - \theta)$. Both paths originate from the source and reach the observation point clockwise (+) or counterclockwise (-) in the plane determined by the point source, center of sphere and observation point. This plane is henceforth called the meridian plane. The configuration is shown in Fig. 4.

The approximations for the Hankel functions yield³⁹

$$\mathcal{A}(\nu) \sim -\left(\frac{2}{k_2 a}\right)^{1/3} [\mathcal{H}(\xi; 2\pi/3) - i\alpha], \quad \alpha = (k_2/k_1)(k_2 a/2)^{1/3}, \tag{5.27}$$

$$\mathcal{D}(\nu) \sim \frac{k_2}{k_1} \left(\frac{2}{k_2 a}\right)^{1/3} \mathcal{H}(\xi; 2\pi/3) - i, \quad \xi = (\nu - k_2 a)(k_2 a/2)^{-1/3}, \tag{5.28}$$

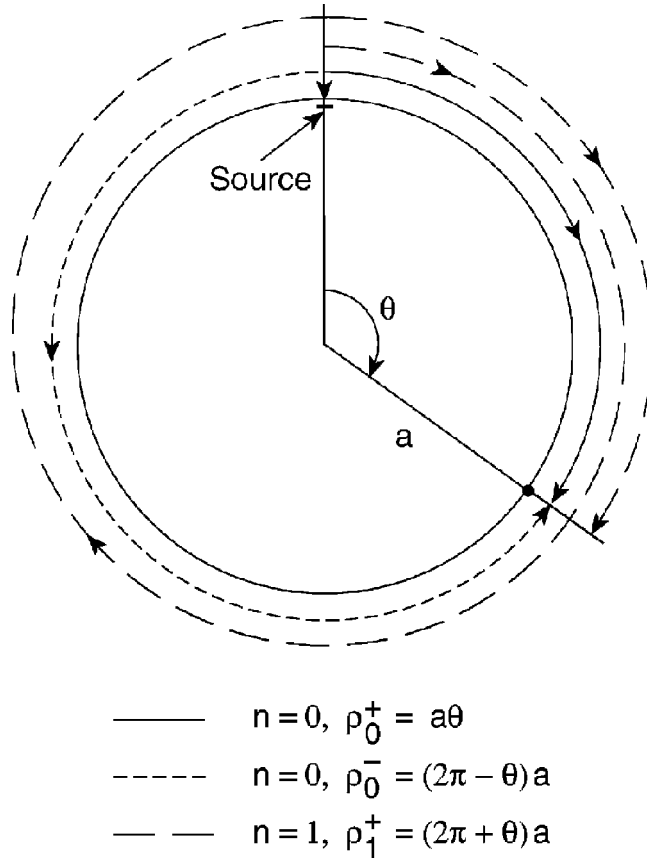


FIG. 4. Paths of lengths ρ_n^\pm ($n \geq 0$) traveled by exponentially decreasing waves that reach the observation point through the air.

where $\mathcal{H}(\xi; \psi)$ is defined by Eq. (4.3). Retainment of the $n=0$ terms in the residue series (5.24) and (5.25) and use of approximation (5.26) recover formulas (4.5a) and (4.8a), when $O(\theta_{cr,2}) \leq \theta < \pi$ and $k_2 a(\pi - \theta) \gg 1$. This procedure is also applied to the tangential field components. Care should be exercised when (i) $\theta < O(\theta_{cr,2})$, because the relevant residue series converge slowly, calling for the procedure of Sec. III, and (ii) $\pi - \theta \leq O[(k_2 a)^{-1}]$, where the Legendre functions must be replaced by Bessel functions of argument $\nu(\pi - \theta)$.

By following point (ii) above and extending MacDonald's formulas³⁶ to the range $O(\theta_{cr,2}) < \theta \leq \pi$, one gets⁴³

$$E_{2r}^{res} \sim -\frac{\omega \mu_0 k_2}{a k_1} \left(\frac{k_2 a}{2}\right)^{2/3} \frac{1}{\bar{\xi} + a^2} \sqrt{\frac{\pi - \theta}{\sin \theta}} e^{i\nu_1 \pi} J_1(\nu_1(\pi - \theta)) \cos \phi, \tag{5.29}$$

$$B_{2r}^{res} \sim -\frac{\mu_0 k_2}{a k_1^2} \sqrt{\frac{\pi - \theta}{\sin \theta}} e^{i\pi \tilde{\nu}_1} J_1(\tilde{\nu}_1(\pi - \theta)) \sin \phi, \tag{5.30}$$

where

$$\nu_1 \sim k_2 a + (k_2 a/2)^{1/3} \bar{\xi}, \quad \tilde{\nu}_1 \sim k_2 a + (k_2 a/2)^{1/3} |a_1| e^{i\pi/3}. \tag{5.31}$$

Bear in mind that $\bar{\xi}$ is introduced in Eq. (4.22) and a_1 is the first zero of $\text{Ai}(z)$.

Evidently, for fixed $\alpha = (k_2/k_1)(k_2 a/2)^{1/3}$ and $\theta \gg \theta_{cr,2}$, the H -type wave attenuates faster than the E -type one. Due to the factors of $P_{\nu-1/2}^1(-\cos \theta)/\sin \theta$, the latter prevails for all $O(\theta_{cr,2}) < \theta \leq \pi$. For example,

$$\begin{aligned}
 E_{2\phi}^{\text{res}} &\sim -\frac{i\omega\mu_0}{a} e^{ik_2a\pi} \frac{k_2^2}{k_1^2} \sqrt{\frac{\pi-\theta}{\sin\theta}} \left[\frac{1}{2\theta_{\text{cr},2}} e^{i(\pi/\theta_{\text{cr},2})\bar{\xi}} \frac{J_1(\nu_1(\pi-\theta))}{\sin\theta} \frac{1}{\bar{\xi}+\alpha^2} \right. \\
 &\quad \left. + e^{i(\pi/\theta_{\text{cr},2})|a_1|e^{i\pi/3}} J_1'(\tilde{\nu}_1(\pi-\theta)) \right] \sin\phi \\
 &\sim -\frac{i\omega\mu_0}{2\rho_{\text{cr},2}} e^{ik_2a\pi} \sqrt{\frac{\pi-\theta}{\sin\theta}} \frac{k_2^2}{k_1^2} \frac{J_1(\nu_1(\pi-\theta))}{\pi-\theta} \frac{e^{i(\pi/\theta_{\text{cr},2})\bar{\xi}}}{\bar{\xi}+\alpha^2} \sin\phi. \tag{5.32}
 \end{aligned}$$

C. Ray representations

1. Case $\theta=O(1)$, $\pi-\theta=O(1)$

Attention is now turned to Eqs. (5.18)–(5.23). Approximations of the Bessel functions outside the transitional regions for $k_2^2a/|k_1| \gg 1$ yield³⁹

$$\mathcal{A}(\nu) \sim i \sqrt{1 - \left(\frac{\nu}{k_2a}\right)^2} + i \frac{k_2}{k_1} \sqrt{1 - \left(\frac{\nu}{k_1a}\right)^2}, \tag{5.33}$$

$$\mathcal{B}(\nu) \sim -2i \frac{k_2}{k_1} \left[1 - \left(\frac{\nu}{k_2a}\right)^2 \right]^{-1/2} \mathcal{T}_E(\nu/a) \exp \left[2i\sqrt{(k_1a)^2 - \nu^2} - 2i\nu \arccos\left(\frac{\nu}{k_1a}\right) - i\frac{\pi}{2} \right], \tag{5.34}$$

$$\mathcal{C}(\nu) \sim R_E(\nu/a) \exp \left[2i\sqrt{(k_1a)^2 - \nu^2} - 2i\nu \arccos\left(\frac{\nu}{k_1a}\right) - i\frac{\pi}{2} \right], \tag{5.35}$$

$$\mathcal{D}(\nu) \sim -i \sqrt{1 - \left(\frac{\nu}{k_1a}\right)^2} - i \frac{k_2}{k_1} \sqrt{1 - \left(\frac{\nu}{k_2a}\right)^2}, \tag{5.36}$$

$$\mathcal{F}(\nu) \sim -2i \mathcal{T}_H(\nu/a) \exp \left[2i\sqrt{(k_1a)^2 - \nu^2} - 2i\nu \arccos\left(\frac{\nu}{k_1a}\right) - i\frac{\pi}{2} \right], \tag{5.37}$$

$$\mathcal{G}(\nu) \sim R_H(\nu/a) \exp \left[2i\sqrt{(k_1a)^2 - \nu^2} - 2i\nu \arccos\left(\frac{\nu}{k_1a}\right) - i\frac{\pi}{2} \right], \tag{5.38}$$

where

$$R_E(\lambda) = -\frac{\sqrt{1 - (\lambda/k_2)^2} - (k_2/k_1)\sqrt{1 - (\lambda/k_1)^2}}{\sqrt{1 - (\lambda/k_2)^2} + (k_2/k_1)\sqrt{1 - (\lambda/k_1)^2}}, \tag{5.39}$$

$$R_H(\lambda) = \frac{\sqrt{1 - (\lambda/k_1)^2} - (k_2/k_1)\sqrt{1 - (\lambda/k_2)^2}}{\sqrt{1 - (\lambda/k_1)^2} + (k_2/k_1)\sqrt{1 - (\lambda/k_2)^2}}, \tag{5.40}$$

$$\mathcal{T}_E(\lambda) = \frac{\sqrt{1 - (\lambda/k_1)^2} \sqrt{1 - (\lambda/k_2)^2}}{[\sqrt{1 - (\lambda/k_2)^2} + (k_2/k_1)\sqrt{1 - (\lambda/k_1)^2}]^2}, \tag{5.41}$$

$$\mathcal{T}_H(\lambda) = \frac{\sqrt{1 - (\lambda/k_1)^2}}{[\sqrt{1 - (\lambda/k_1)^2} + (k_2/k_1)\sqrt{1 - (\lambda/k_2)^2}]^2}. \tag{5.42}$$

R_E and R_H are the usual Fresnel reflection coefficients. The corresponding integrands are separated into two groups as follows.

(i) The first group contains terms of the radial components along with integrands proportional to $\partial P_{\nu-1/2}^1/\partial\theta$. For instance,

$$\begin{aligned} & \nu \mathcal{B}(\nu) \mathcal{C}(\nu)^p \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] P_{\nu-1/2}^1(\cos \theta) e^{i2\pi n \nu} \\ & \sim \frac{k_2}{k_1} \nu \sqrt{\frac{2\nu}{\pi \sin \theta}} \mathcal{T}_E(\nu/a) R_E(\nu/a)^p (e^{i\Phi_{pn+}} + e^{i\Phi_{pn-}}), \end{aligned} \tag{5.43}$$

$$\begin{aligned} & \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{B}(\nu) \mathcal{C}(\nu)^p \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right]^2 \frac{\partial P_{\nu-1/2}^1}{\partial \theta} e^{i2\pi n \nu} \\ & \sim i \frac{k_2}{k_1} \sqrt{\frac{2\nu}{\pi \sin \theta}} \left[1 - \left(\frac{\nu}{k_2 a} \right)^2 \right]^{1/2} \mathcal{T}_E(\nu/a) R_E(\nu/a)^p [e^{i(\Phi_{pn+} + \pi/2)} + e^{i(\Phi_{pn-} - \pi/2)}], \end{aligned} \tag{5.44}$$

where

$$\Phi_{pn\pm}(\nu; \theta) = 2(p+1)\sqrt{(k_1 a)^2 - \nu^2} - 2(p+1)\nu \arccos\left(\frac{\nu}{k_1 a}\right) - (p+1)\frac{\pi}{2} + 2\pi n \nu \pm \left(\nu\theta + \frac{\pi}{4}\right). \tag{5.45}$$

The phase $\Phi_{pn\pm}(\nu; \theta)$ becomes stationary at

$$\nu = \nu_{pn\pm} = k_1 a \cos \psi_{pn\pm}, \quad \psi_{pn\pm} = \frac{2\pi n \pm \theta}{2(p+1)}, \quad 0 \leq \psi_{pn\pm} \leq \pi/2, \tag{5.46}$$

where the “+” sign holds if $0 \leq 2n \leq p$ and the “-” sign holds if $0 < 2n \leq p+1$. The integrals are calculated by the stationary-phase method with

$$\left. \frac{d^2 \Phi_{pn\pm}}{d\nu^2} \right|_{\nu = \nu_{pn\pm}} = \frac{2(p+1)}{k_1 a \sin \psi_{pn\pm}}. \tag{5.47}$$

The radial components involve the integrals

$$\begin{aligned} & \int_0^\infty d\nu e^{i2\pi n \nu} \nu \mathcal{B}(\nu) \mathcal{C}(\nu)^p \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] P_{\nu-1/2}^1(\cos \theta) \\ & \sim e^{-i\pi/4} \frac{k_1 k_2 a^2}{\sqrt{\sin \theta}} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \cos \psi_{pns} \sqrt{\sin 2\psi_{pns}} \mathcal{T}_E(k_1 \cos \psi_{pns}) R_E(k_1 \cos \psi_{pns})^p \\ & \quad \times \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4], \end{aligned} \tag{5.48}$$

$$\begin{aligned} & \int_0^\infty d\nu e^{i2\pi n \nu} \nu \mathcal{F}(\nu) \mathcal{G}(\nu)^p P_{\nu-1/2}^1(\cos \theta) \\ & \sim -e^{i\pi/4} \frac{(k_1 a)^2}{\sqrt{\sin \theta}} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \cos \psi_{pns} \sqrt{\sin 2\psi_{pns}} \mathcal{T}_H(k_1 \cos \psi_{pns}) R_H(k_1 \cos \psi_{pns})^p \\ & \quad \times \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4]. \end{aligned} \tag{5.49}$$

(ii) Terms of the second group pertain to the tangential components containing the factor $P_{\nu-1/2}^1(\cos \theta)$. The respective integrals are treated similarly, but the leading contributions stem from the endpoint $\nu=0$ with width $O(1)$ and from the stationary-phase points $\nu_{pn\pm}$ with width $O(\sqrt{k_1 a})$. The former contributions are cancelled. The surviving terms are corrections $O[(k_1 a)^{-1}]$ when $\psi_{pn\pm} = O(1)$ and $\pi/2 - \psi_{pn\pm} = O(1)$. For instance,

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i2\pi n\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{F}(\nu) \mathcal{G}(\nu)^p P_{\nu-1/2}^1(\cos \theta) \\ & \sim -e^{i\pi/4} \frac{1}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} \sqrt{2 \tan \psi_{pns}} \\ & \quad \times \mathcal{T}_H(k_1 \cos \psi_{pns}) R_H(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4], \end{aligned} \tag{5.50}$$

where

$$S_{p+} = \{n: 0 \leq 2n \leq p\}, \quad S_{p-} = \{n: 0 < 2n \leq p+1\}. \tag{5.51}$$

The preceding considerations lead to the rays

$$\begin{aligned} E_{2r}^{\text{ray}} & \sim e^{i\pi/4} \frac{\omega \mu_0 k_1}{2\pi k_2 a} \frac{\cos \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \cos \psi_{pns} \sqrt{\sin 2\psi_{pns}} \\ & \quad \times \mathcal{T}_E(k_1 \cos \psi_{pns}) R_E(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4], \end{aligned} \tag{5.52}$$

$$\begin{aligned} E_{2\theta}^{\text{ray}} & \sim e^{-i\pi/4} \frac{\omega \mu_0}{2\pi a} \frac{\cos \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \sqrt{\sin 2\psi_{pns}} \\ & \quad \times \bar{\mathcal{T}}_E(k_1 \cos \psi_{pns}) R_E(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} - is\pi/4] \\ & \quad + \frac{1}{k_1 a} e^{-i\pi/4} \frac{\omega \mu_0}{2\pi a \sin \theta} \frac{\cos \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \sqrt{2 \tan \psi_{pns}} \\ & \quad \times \mathcal{T}_H(k_1 \cos \psi_{pns}) R_H(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4], \end{aligned} \tag{5.53}$$

$$\begin{aligned} E_{2\phi}^{\text{ray}} & \sim e^{-i\pi/4} \frac{\omega \mu_0}{2\pi a} \frac{\sin \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \sqrt{\sin 2\psi_{pns}} \\ & \quad \times \mathcal{T}_H(k_1 \cos \psi_{pns}) R_H(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} - is\pi/4] \\ & \quad + \frac{1}{k_1 a} e^{-i\pi/4} \frac{\omega \mu_0}{2\pi a \sin \theta} \frac{\sin \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \sqrt{2 \tan \psi_{pns}} \\ & \quad \times \bar{\mathcal{T}}_E(k_1 \cos \psi_{pns}) R_E(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4], \end{aligned} \tag{5.54}$$

$$\begin{aligned} B_{2r}^{\text{ray}} & \sim -e^{i\pi/4} \frac{\mu_0 k_1}{2\pi a} \frac{\sin \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \cos \psi_{pns} \sqrt{\sin 2\psi_{pns}} \\ & \quad \times \mathcal{T}_H(k_1 \cos \psi_{pns}) R_H(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4], \end{aligned} \tag{5.55}$$

$$\begin{aligned}
 B_{2\theta}^{\text{ray}} \sim & -e^{-i\pi/4} \frac{\mu_0 k_2}{2\pi a} \frac{\sin \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \sqrt{\sin 2\psi_{pns}} \\
 & \times \bar{\mathcal{T}}_H(k_1 \cos \psi_{pns}) R_H(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} - is\pi/4] \\
 & - \frac{1}{k_1 a} e^{-i\pi/4} \frac{\mu_0 k_2}{2\pi a \sin \theta} \frac{\sin \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \sqrt{2 \tan \psi_{pns}} \\
 & \times \mathcal{T}_E(k_1 \cos \psi_{pns}) R_E(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4], \quad (5.56)
 \end{aligned}$$

$$\begin{aligned}
 B_{2\phi}^{\text{ray}} \sim & e^{-i\pi/4} \frac{\mu_0 k_2}{2\pi a} \frac{\cos \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \sqrt{\sin 2\psi_{pns}} \\
 & \times \mathcal{T}_E(k_1 \cos \psi_{pns}) R_E(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} - is\pi/4] \\
 & + \frac{1}{k_1 a} e^{-i\pi/4} \frac{\mu_0 k_2}{2\pi a \sin \theta} \frac{\cos \phi}{\sqrt{\sin \theta}} \sum_{p=0}^{\infty} \frac{e^{-ip\pi/2}}{\sqrt{p+1}} \sum_{s=\pm} \sum_{n \in S_{ps}} (-1)^n \sqrt{2 \tan \psi_{pns}} \\
 & \times \bar{\mathcal{T}}_H(k_1 \cos \psi_{pns}) R_H(k_1 \cos \psi_{pns})^p \exp[2i(p+1)k_1 a \sin \psi_{pns} + is\pi/4], \quad (5.57)
 \end{aligned}$$

where

$$\bar{\mathcal{T}}_F(\lambda) = \sqrt{1 - (\lambda/k_2)^2} \mathcal{T}(\lambda), \quad F = E, H. \quad (5.58)$$

Note that corrections to the leading terms of the first group introduced above are omitted.

An inspection of the summands for $p \gg 2n > 1$, $\theta = O(1)$ and $\pi - \theta = O(1)$ shows that their magnitudes decrease as $|R|^p/p^2$ when medium 1 is lossless. Of course, convergence is improved when $\text{Im} k_1$ is positive and $O(1)$. In general, $|R(\lambda)| \leq 1$ for complex λ while the condition $k_2^2 \ll |k_1^2|$ forces $|R(k_1 \cos \psi_{pn\pm})|$ to be nearly 1.⁴⁴

The rays described by formulas (5.52)–(5.57) circulate around the origin in the meridian plane while they are multiply reflected at the spherical boundary, as depicted in Fig. 5; p is the number of reflections, n is the number of circulations, the sign $s = \pm$ specifies the sense of circulation, and $\psi_{pn\pm}$ is the angle between the incident ray and the corresponding local tangent. The overall phase of each ray undergoes a change of $-\pi/2$ at each reflection (see also Ref. 33).

2. Reduction to a wave through region 1, $\theta \ll 1$

When $\theta = O[(k_1 a)^{-1/3}]$ and $n = 0$, the width $O(\sqrt{k_1 a \theta})$ of the stationary-phase contribution above becomes comparable to the width $O[(k_1 a)^{1/3}]$ of the transitional region about $\nu = k_1 a$. This suggests the transition of rays to the wave propagating in region 1 according to the integral \mathcal{I}_1 in Sec. IV. The two asymptotic formulas connect smoothly if $O[(k_1 a)^{-1/3}] < \theta < O[(k_1 a)^{-1/5}]$. The approximation

$$\sin \psi_{p0+} \sim \psi_{p0+} - \psi_{p0+}^3/6, \quad |k_1| a \theta^5 \ll 2^4 5!,$$

in the phase, along with $k_2^2 a \theta / |k_1| \gg 1$, $\sin(2\psi_{p0+}) \sim 2\psi_{p0+}$, and

$$R(k_1 \sin \psi_{p0+}) \sim -1,$$

$$\mathcal{T}_E(k_1 \sin \psi_{p0+}) \sim -i(k_2/k_1) \psi_{p0+}, \quad \mathcal{T}_H(k_1 \sin \psi_{p0+}) \sim -\psi_{p0+}$$

in the amplitude reduce the $n = 0$ terms in (5.52)–(5.57) to formulas (4.42)–(4.45).

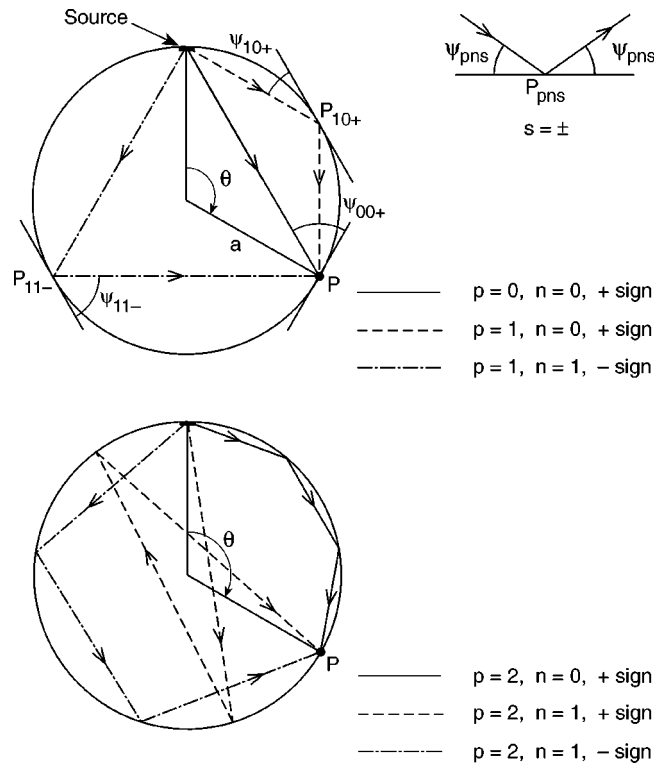


FIG. 5. Geometry of rays bouncing and circulating in the interior of the sphere. P is the observation point.

3. Field close to the antipodes, $\pi - \theta \ll 1$

Care should be exercised when $\theta \sim \pi$. This condition implies that a combination of exponentials may no longer be a reasonable approximation for the Legendre function.³⁶ In fact, in order to overcome this difficulty, one has to seek an alternative representation *a priori*. The identity $P_l^1(\cos \theta) = (-1)^{l+1} P_l^1(\cos(\pi - \theta))$ suggests the replacement

$$P_{\nu-1/2}^1(\cos \theta) \rightarrow i e^{i\pi\nu} P_{\nu-1/2}^1(\cos(\pi - \theta)). \tag{5.59}$$

The new representation is illustrated by

$$E_{2r}^{\text{ray}} = -\frac{\omega\mu_0}{2\pi a} \frac{1}{(k_2 a)^2} \cos \phi \sum_{n'=-\infty}^{\infty} (-1)^{n'} \sum_{p=0}^{\infty} \int_0^{\infty} d\nu e^{i\pi(2n'+1)\nu} \nu \mathcal{B}(\nu) \mathcal{C}(\nu)^p \times \left[\frac{1}{2k_2 a} + \frac{H_{\nu}^{(1)'}(k_2 a)}{H_{\nu}^{(1)}(k_2 a)} \right] P_{\nu-1/2}^1(-\cos \theta). \tag{5.60}$$

The ray $(n, p, s = \pm)$ of Sec. V C 1 is identified with the ray $(n' = n, p, s' = -)$ if $s = +$, or $(n' = n - 1, p, s' = +)$ if $s = -$ in the present formalism. The stationary-phase points are given by

$$\bar{\nu}_{pn'\pm} = k_1 a \cos \bar{\psi}_{pn'\pm}, \quad \bar{\psi}_{pn'\pm} = \frac{(2n'+1)\pi \pm \theta'}{2(p+1)}, \quad 0 \leq \bar{\psi}_{pn'\pm} \leq \pi/2, \tag{5.61a}$$

$$\theta' = \pi - \theta, \tag{5.61b}$$

with the sign convention $\{+ : 0 \leq 2n' \leq p - 1\}$ and $\{- : 0 \leq 2n' \leq p\}$.

An interesting case arises when $\pi - \theta \leq O[(k_1 a)^{-1/2}]$ and $2n' = p$ ($s' = -$), because the endpoint of integration, $\nu = 0$, then falls inside the critical vicinity of a stationary-phase point. The requisite integrals for $2n' = p$ are evaluated through MacDonald's formulas³⁶ with $|\lambda| = |\nu/a| \ll 1$ in the radicals of expressions (5.33)–(5.42). The phase of $e^{i\pi(2n'+1)\nu} \mathcal{B}(\nu) \mathcal{C}(\nu)^p$ and $e^{i\pi(2n'+1)\nu} \mathcal{F}(\nu) \mathcal{G}(\nu)^p$ is expanded about $\nu = 0$ up to $O(\nu^2)$. Let $\partial \mathbf{P}_{\nu-1/2}^1 / \partial \theta' = -\partial \mathbf{P}_{\nu-1/2}^1 (-\cos \theta) / \partial \theta$. The requisite integral for E_{2r} is

$$\begin{aligned}
 & i \int_0^\infty d\nu e^{i\pi(p+1)\nu} \nu \mathcal{B}(\nu) \mathcal{C}(\nu)^p \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right] \mathbf{P}_{\nu-1/2}^1(\cos \theta') \\
 & \sim -2 \frac{k_2}{k_1} e^{2i(p+1)k_1 a - ip\pi/2} \int_0^\infty d\nu \left(\nu^2 - \frac{1}{4} \right) J_1(\nu \theta') e^{i(p+1)\nu^2/(k_1 a)} \\
 & \sim \frac{k_1 k_2 a^2}{2} \frac{e^{-ip\pi/2}}{(p+1)^2} (\pi - \theta) \exp[2i(p+1)k_1 a - (i/4)k_1 a (\pi - \theta)^2 / (p+1)], \quad (5.62)
 \end{aligned}$$

provided that $|k_1 a|(\pi - \theta)^2 \leq O(1)$. Comparison with formula (5.48) shows that the character of this ray is not modified as θ approaches π , in contradistinction to the case of a vertical dipole.³³ Indeed, the approximation

$$\sin \psi_{(2n)n+} \sim 1 - \frac{(\pi - \theta)^2}{8(p+1)^2}, \quad \frac{|k_1 a|(\pi - \theta)^4}{(p+1)^4} \ll 1,$$

in the phase of formula (5.48), along with

$$\cos \psi_{(2n)n+} \sim \frac{\pi - \theta}{2(p+1)}, \quad \sin 2\psi_{(2n)n+} \sim \frac{\pi - \theta}{p+1}, \quad T_E \sim 1, \quad R_E \sim 1,$$

readily furnish formula (5.62). Similar considerations apply to B_{2r}^{ray} , with the requisite integral

$$\begin{aligned}
 & i \int_0^\infty d\nu e^{i\pi(p+1)\nu} \nu \mathcal{F}(\nu) \mathcal{G}(\nu)^p \mathbf{P}_{\nu-1/2}^1(\cos \theta') \\
 & \sim -i \frac{(k_1 a)^2}{2} \frac{e^{-ip\pi/2}}{(p+1)^2} (\pi - \theta) \exp[2i(p+1)k_1 a - (i/4)k_1 a (\pi - \theta)^2 / (p+1)]. \quad (5.63)
 \end{aligned}$$

For $E_{2\theta}^{\text{ray}}$, it suffices to compare the following integrals:³⁹

$$\begin{aligned}
 e_{\theta p} &= \frac{-i}{k_2 a} \int_0^\infty d\nu e^{i\pi(p+1)\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{B}(\nu) \mathcal{C}(\nu)^p \left[\frac{1}{2k_2 a} + \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} \right]^2 \frac{\partial \mathbf{P}_{\nu-1/2}^1}{\partial \theta'} \\
 & \sim \frac{-2i}{k_1 a} e^{2i(p+1)k_1 a - ip\pi/2} \frac{\partial}{\partial \theta'} \int_0^\infty d\nu J_1(\nu \theta') e^{i(p+1)\nu^2/(k_1 a)} \\
 & = - \left[\frac{e^{-(i/4)k_1 a (\pi - \theta)^2 / (p+1)}}{p+1} + 2i \frac{1 - e^{-(i/4)k_1 a (\pi - \theta)^2 / (p+1)}}{k_1 a (\pi - \theta)^2} \right] e^{2i(p+1)k_1 a - ip\pi/2} \quad (5.64)
 \end{aligned}$$

and

$$\begin{aligned} \tilde{e}_{\theta p} &= \frac{i}{k_1 a (\pi - \theta)} \int_0^\infty d\nu e^{i\pi(p+1)\nu} \frac{\nu}{\nu^2 - \frac{1}{4}} \mathcal{F}(\nu) \mathcal{G}(\nu)^p P_{\nu-1/2}^1(\cos \theta') \\ &\sim 2i \frac{1 - e^{-(i/4)k_1(\pi - \theta)^2/(p+1)}}{k_1 a (\pi - \theta)^2} e^{2i(p+1)k_1 a - ip\pi/2}. \end{aligned} \tag{5.65}$$

The three terms of formula (5.64) become of the same order in magnitude if $k_1 a (\pi - \theta)^2 = O(1)$. The first term is dominant if $|k_1 a|(\pi - \theta)^2 \gg 1$ and then recovers the corresponding geometrical ray with $2n = p$ and $s = +$. In contrast, in formula (5.65) all terms must be retained; the second term multiplied by $k_1 a (\pi - \theta)$ connects smoothly to the correction appearing in (5.53), while the first term is the contribution from the endpoint $\nu = 0$. This contribution is cancelled by the corresponding term in (5.64), viz.,

$$e_{\theta p} + \tilde{e}_{\theta p} \sim -i \frac{e^{-ip\pi/2}}{p+1} \exp[2i(p+1)k_1 a - (i/4)k_1 a (\pi - \theta)^2/(p+1)]. \tag{5.66}$$

These considerations can be repeated for the integral of $E_{2\phi}^{\text{ray}}$ with $P_{\nu-1/2}^1(\cos \theta')/\theta'$ and $\partial P_{\nu-1/2}^1/\partial \theta'$ interchanged. Along the same lines is the analysis for the components of the magnetic field, because the presence of the factor $[(2k_2 a)^{-1} + H_\nu^{(1)'}(k_2 a)/H_\nu^{(1)}(k_2 a)]$ amounts to multiplication by i .

The amplitudes of other rays with $2n' \leq p - 1$ are determined by noticing that the Bessel function varies slowly over the region of width $O(\sqrt{k_1 a})$ about each stationary-phase point, and can therefore be pulled out of the corresponding integrals. This program can be carried out straightforwardly; this case is not discussed further in this article.

VI. REMARKS AND DISCUSSION

Before closing this article, it is worthwhile making the following remarks.

(i) The order of magnitude of the critical distance $\rho_{\text{cr},j}$ in Eq. (3.36) can be obtained by postulating that, when $\rho = O(\rho_{\text{cr},j})$, the difference between the arc length $\rho = a\theta$ and its projection on the tangent at $\theta = 0$ becomes comparable to the wavelength in air ($j = 2$) or earth ($j = 1$).^{3,4}

(ii) It is tempting to compare the field of a horizontal dipole to that of the vertical dipole with equal moment, examined, for example, in Ref. 33. The dominant components of the former field are E_{2r} , $E_{2\theta}$ and $B_{2\phi}$. These are precisely the nonzero components of a vertical dipole. The corresponding maximum magnitudes in ϕ satisfy the relations

$$|E^{\text{hor}}|_m \sim \frac{k_2}{|k_1|} |E^{\text{ver}}|, \quad |B_\phi^{\text{hor}}|_m \sim \frac{k_2}{|k_1|} |B_\phi^{\text{ver}}|. \tag{6.1}$$

(iii) The present analysis offers some insight into the problem of a dipole elevated at a height h ($h = b - a \ll a$). A complication in this case stems from the additional transitional point in the integrands of the Poisson summation formula. For example,

$$\begin{aligned} E_{2r} &= \frac{i\mu_0\omega}{2\pi a} \left(\frac{a}{b}\right)^{1/2} \frac{1}{(k_2 a)^2} \cos \phi \sum_{n=-\infty}^\infty (-1)^n \int_0^\infty d\nu e^{i2\pi n\nu} \nu \frac{H_\nu^{(1)}(k_2 b)}{H_\nu^{(1)}(k_2 a)} \frac{1}{\mathcal{K}(\nu)} \\ &\times \left[\frac{1}{2k_2 b} + \frac{H_\nu^{(1)'}(k_2 b)}{H_\nu^{(1)}(k_2 b)} \right] P_{\nu-1/2}^1(\cos \theta). \end{aligned} \tag{6.2}$$

It is readily concluded that elevation of the dipole results in the increase of the critical angle $\theta_{\text{cr},j}$ by $O(\sqrt{2h/a})$. The analysis is simplified when $k_2 h \ll (k_2 a)^{1/3}$ so that the points $k_2 a$ and $k_2 b$ can be treated, in some sense, as a single transitional point.

For arbitrary h , the residue series for E_{2r} when $\theta > O(\sqrt{2h/a})$ contains the factors

$$f_j(h) = \frac{H_{\nu_j}^{(1)}(k_2 b)}{H_{\nu_j}^{(1)}(k_2 a)}, \quad j=1, 2, \dots, \tag{6.3}$$

where ν_j are zeros of the $\mathcal{A}(\nu)$ defined by Eq. (5.7) that lie in the vicinity of $\nu = k_2 a$. $f_j(h)$ is the ‘‘height-gain factor’’ introduced by Bremmer.¹⁴ In the corresponding factor $\tilde{f}_j(h)$ for B_{2r} , the ν_j need to be replaced by the zeros $\tilde{\nu}_j$ of the $\mathcal{D}(\nu)$ defined by Eq. (5.3). These factors express the dependence of the field beyond the horizon on the parameter $(k_2 a)^{1/3} \sqrt{2h/a}$. The height-gain factors for the θ - and ϕ -components are defined in a similar fashion. With regard to $E_{2\theta}$, one needs to consider the factor

$$\frac{H_{\nu_j}^{(1)}(k_2 b) (1/k_2 b) + [H_{\nu_j}^{(1)'}(k_2 b)/H_{\nu_j}^{(1)}(k_2 b)]}{H_{\nu_j}^{(1)}(k_2 a) (1/k_2 a) + [H_{\nu_j}^{(1)'}(k_2 a)/H_{\nu_j}^{(1)}(k_2 a)]}$$

(iv) The method of solution here needs to be modified when the medium in region 1 or 2 contains inhomogeneities, as is the case with ionospheric effects. The ionosphere can be modeled crudely via replacement of the air for $r > d$ ($d > a$) by a perfect conductor.

A problem of interest arises when the index of refraction near the earth’s surface exhibits variations due to high moisture. This phenomenon is called ‘‘ducting’’ and may cause super-refraction when rays emitted from the radiating source bend downwards.¹⁴ A model for the dielectric permittivity gives¹⁴

$$\epsilon_2(r) = \epsilon_0 \frac{A + B(r - r_0)^2}{r^2}. \tag{6.4}$$

(v) The method of stationary phase for the rays employed in Sec. V becomes questionable when $\nu_{pn\pm}$ lies in a neighborhood of width $O(\sqrt{k_1 a})$ of any pole of $\mathcal{A}^{-1}(\nu)$ or $\mathcal{D}^{-1}(\nu)$ close to the positive real axis. The value $\nu_{pn\pm} \sim k_2 a$ corresponds to a ray that undergoes total internal reflection.⁴⁵ Such a case follows, for instance, from taking $2n = p$, $s = +$ and

$$\nu_{pn+} = k_2 a (1 - \bar{\epsilon}), \quad \bar{\epsilon} = 1 - \frac{k_1}{k_2} \frac{\pi - \theta}{2(p+1)}, \quad |\bar{\epsilon}| \ll 1. \tag{6.5}$$

A remedy to this anomaly is quite elaborate, in principle involving sums of Fresnel integrals, and is provided elsewhere.³³

VII. SUMMARY AND CONCLUSIONS

The problem of the radiating electric dipole lying just below and tangential to the surface of an electrically large, homogeneous, isotropic and nonmagnetic sphere surrounded by air has been revisited. The present analysis, however, has a different perspective from previous works, since it was guided by the physical concept of the creeping wave.²² The Poisson summation formula, employed over 40 years ago in the study of plane-wave scattering by impenetrable objects,²² provided a useful starting point. In the present case the creeping wave, although evidently two-dimensional, has a more intricate structure being dependent on the nature and orientation of the source.

All six components of the electromagnetic field on the boundary were determined without recourse to scalar potentials. For an optically dense sphere, each component is decomposed into waves propagating through air and through the sphere. When the polar angle θ is sufficiently lower than a critical value given by Eq. (3.36), known planar-earth formulas are recovered³⁴ along with simplified corrections to account for the curved boundary. In particular, the electric-type wave in

the tangential components with the air phase velocity gives rise to Fresnel integrals and surface waves,³⁴ and therefore has a character distinctly different from solutions to plane-wave scattering within the scalar wave and Schrödinger's equations.²⁹

As θ progresses to values that are comparable to or exceed $\theta_{cr,2}$, the electric-type wave through air is described by series of exponentials decreasing in θ . The attenuation rates were determined to the lowest order in $(k_2 a)^{-1}$ and k_2/k_1 by solving a transcendental equation, also encountered in the problem of the vertical radiating dipole;⁴² its roots depend on the widely varying parameter $\alpha = (k_2 a/2)^{1/3} k_2/k_1$. On the other hand, lowest-order attenuation constants for the magnetic-type wave are fixed numbers, in agreement with early findings by Gray.¹² Higher-order corrections to these equations are easily obtained within this scheme. By starting with the zeroth-order equations, an argument was presented to show that the H -type wave attenuates faster than the E -type one when $|\alpha| \leq O(1)$; a conjecture made by Fock¹⁶ was therefore placed on firmer grounds. The electric field was found to have a dominant component perpendicular to the boundary, while the magnetic field has a dominant component in the azimuthal direction. The ensuing polarization resembles that of a vertical electric dipole, but the maximum magnitude of the field in this case is multiplied by the factor $k_2/|k_1|$. When $\theta = O(\theta_{cr,1})$ the wave inside the sphere is described by a well-converging integral of Airy functions.

A physical picture of waves exponentially decreasing in air and rays circulating in the interior of the sphere via their multiple reflections at the boundary was exposed when $\theta = O(1)$ and $\pi - \theta = O(1)$. These ray contributions are significant when $\text{Im } k_1 a \ll 1$. As expected from elementary geometrical optics, only one type of wave (electric or magnetic) prevails in each component, with the amplitude of the dominant ray being described by the corresponding Fresnel reflection coefficient. There are features of both the amplitude and phase of these rays, however, that are attributed to the nature of the source and are not fully predictable by standard geometrical optics. This ray picture breaks down at the antipodal point ($\theta \sim \pi$), or any point of total internal reflection.⁴⁵ In both cases, the modified analysis unveils characteristics due to the nature of the source. For instance, the two types of polarization in the tangential components can provide comparable ray contributions if $\theta \sim \pi$, when $2n = p$, $s = +$ in the notation of Sec. V; the total amplitude then recovers the Fresnel reflection coefficient of geometrical optics. This situation is to be contrasted with the case of a vertical dipole, where the ray amplitude changes drastically at the antipodes.

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APPENDIX A: ON THE ZEROS OF $\mathcal{K}(\nu)$, $\mathcal{M}(\nu)$

To simplify the calculations in this appendix, consider k_1 to be real, unless it is stated otherwise. For $k_2 a \gg 1$, $k_2^2 \ll k_1^2$ and $k_2^2 a/k_1 \gg 1$, the terms $(2k_2 a)^{-1}$ and $k_2 k_1^{-1} (2k_1 a)^{-1}$ in $\mathcal{K}(\nu)$ are neglected. Following Refs. 29 and 33, define

$$\mathcal{P}(\nu; \kappa) = \frac{J'_\nu(k_1 a)}{J_\nu(k_1 a)} - \kappa \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)}, \quad (\text{A1})$$

where $\kappa = k_1/k_2$ corresponds to $\mathcal{K}(\nu)$ and $\kappa = k_2/k_1$ corresponds to $\mathcal{M}(\nu)$. In this appendix, the task is to locate those ν that satisfy

$$\mathcal{P}(\nu; \kappa) = 0. \quad (\text{A2})$$

$$\text{For } 0 < \nu < k_1 a \text{ and } |\nu - k_{1,2} a| \gg (k_{1,2} a)^{1/3},$$

$$\frac{J'_\nu(k_1 a)}{J_\nu(k_1 a)} \sim -\sqrt{1-\left(\frac{\nu}{k_1 a}\right)^2} \tan\left[\sqrt{(k_1 a)^2-\nu^2}-\nu \arccos\left(\frac{\nu}{k_1 a}\right)-\frac{\pi}{4}\right], \tag{A3}$$

$$\frac{H^{(1)'}_\nu(k_2 a)}{H^{(1)}_\nu(k_2 a)} \sim i \sqrt{1-\left(\frac{\nu}{k_2 a}\right)^2}, \tag{A4}$$

and the branch cut emanating from $\nu=k_2 a$ lies in the upper ν -plane. Accordingly,

$$\begin{aligned} \mathcal{P}(\nu; \kappa) \sim & -\sqrt{1-\left(\frac{\nu}{k_1 a}\right)^2} \tan\left[\sqrt{(k_1 a)^2-\nu^2}-\nu \arccos\left(\frac{\nu}{k_1 a}\right)-\frac{\pi}{4}\right] \\ & -i \kappa \sqrt{1-\left(\frac{\nu}{k_2 a}\right)^2}. \end{aligned} \tag{A5}$$

No zeros of the right-hand side lie in $(0, k_2 a)$. The analytic continuation to $(k_2 a, k_1 a)$ through the lower ν -plane does not exhibit any zeros either. More precisely,

$$\frac{H^{(1)'}_\nu(k_2 a)}{H^{(1)}_\nu(k_2 a)} \sim \frac{Y'_\nu(k_2 a)}{Y_\nu(k_2 a)} \left\{ 1 + i \frac{W[J_\nu, Y_\nu]}{Y_\nu(k_2 a) Y'_\nu(k_2 a)} \right\}, \tag{A6}$$

where $W[J_\nu, Y_\nu] = (2/\pi)(k_2 a)^{-1}$ denotes the Wronskian of $J_\nu(k_2 a)$ and $Y_\nu(k_2 a)$. This approximation produces a recessive imaginary term for $\nu > k_2 a$. Hence, (A2) reads as

$$\begin{aligned} & \tan\left[\sqrt{(k_1 a)^2-\nu^2}-\nu \arccos\left(\frac{\nu}{k_1 a}\right)-\frac{\pi}{4}\right] \\ & \sim \kappa \sqrt{\frac{[\nu/(k_2 a)]^2-1}{1-[\nu/(k_1 a)]^2}} \left\{ 1 - i \exp\left[-2\nu \cosh^{-1}\left(\frac{\nu}{k_2 a}\right) + 2\sqrt{\nu^2-(k_2 a)^2}\right] \right\}. \end{aligned} \tag{A7}$$

This equation cannot be satisfied by any real ν . In fact, it is satisfied only in the upper ν -plane. Let $\nu = \nu_r + i\nu_i$, where ν_r and ν_i are real, $|\nu_i| \ll \nu_r$, and $k_2 a < \nu_r < k_1 a$. Then,

$$\begin{aligned} & \tan\left[\sqrt{(k_1 a)^2-\nu_r^2}-\nu_r \arccos\left(\frac{\nu_r}{k_1 a}\right)-i\nu_i \arccos\left(\frac{\nu_r}{k_1 a}\right)-\frac{\pi}{4}\right] \\ & \sim \kappa \sqrt{\frac{[\nu_r/(k_2 a)]^2-1}{1-[\nu_r/(k_1 a)]^2}} \left\{ 1 - i \exp\left[-2\nu_r \cosh^{-1}\left(\frac{\nu_r}{k_2 a}\right) + 2\sqrt{\nu_r^2-(k_2 a)^2}\right] \right\}. \end{aligned} \tag{A8}$$

For $\kappa = k_1/k_2$,

$$\sqrt{(k_1 a)^2-\nu_r^2}-\nu_r \arccos\left(\frac{\nu_r}{k_1 a}\right) \sim \arctan\left\{\frac{k_1}{k_2} \sqrt{\frac{[\nu_r/(k_2 a)]^2-1}{1-[\nu_r/(k_1 a)]^2}}\right\} + m\pi + \frac{\pi}{4}, \tag{A9}$$

$$\nu_i \arccos\left(\frac{\nu_r}{k_1 a}\right) \sim \frac{k_2}{k_1} \sqrt{\frac{1-[\nu_r/(k_1 a)]^2}{[\nu_r/(k_2 a)]^2-1}} \exp\left[-2\nu_r \cosh^{-1}\left(\frac{\nu_r}{k_2 a}\right) + 2\sqrt{\nu_r^2-(k_2 a)^2}\right], \tag{A10}$$

where m is any integer. In particular, if $k_2 a \ll \nu_r \ll k_1 a$,

$$\nu_r \sim \frac{2k_1 a}{\pi} - \left(2m + \frac{3}{2}\right), \quad \nu_i \sim \frac{4k_2}{\pi k_1} \left(\frac{k_2 a}{2\nu_r}\right)^{2\nu_r+1} e^{2\nu_r}. \tag{A11}$$

For $\kappa = k_2/k_1$,

$$\sqrt{(k_1 a)^2 - \nu_r^2} - \nu_r \arccos\left(\frac{\nu_r}{k_1 a}\right) \sim \arctan\left\{\frac{k_2}{k_1} \sqrt{\frac{[\nu_r/(k_2 a)]^2 - 1}{1 - [\nu_r/(k_1 a)]^2}}\right\} + m\pi + \frac{\pi}{4}, \tag{A12}$$

$$\begin{aligned} \nu_i \arccos\left(\frac{\nu_r}{k_1 a}\right) &\sim \frac{k_2}{k_1} \sqrt{[\nu_r/(k_2 a)]^2 - 1} \sqrt{1 - [\nu_r/(k_1 a)]^2} \\ &\times \exp\left[-2\nu_r \cosh^{-1}\left(\frac{\nu_r}{k_2 a}\right) + 2\sqrt{\nu_r^2 - (k_2 a)^2}\right]. \end{aligned} \tag{A13}$$

These expressions are trivially simplified if $k_2 a \ll \nu_r \ll k_1 a$, i.e.,

$$\nu_r \sim \frac{2k_1 a}{\pi} - \left(2m + \frac{1}{2}\right), \quad \nu_i \sim \frac{2\nu_r}{\pi k_1 a} \left(\frac{k_2 a}{2\nu_r}\right)^{2\nu_r} e^{2\nu_r}. \tag{A14}$$

Consider $0 < \text{Re } \nu < k_2 a$. Setting the right-hand side of (A5) equal to zero in the upper ν -plane yields

$$k_1 a - \frac{\nu\pi}{2} \sim m\pi + \frac{\pi}{4} - i \arctan\left[\frac{k_1}{k_2} \sqrt{1 - \left(\frac{\nu}{k_2 a}\right)^2}\right] \tag{A15}$$

for $\kappa = k_1/k_2$, or

$$k_1 a - \frac{\nu\pi}{2} \sim m\pi + \frac{\pi}{4} - i \frac{k_2}{k_1} \sqrt{1 - \left(\frac{\nu}{k_2 a}\right)^2} \tag{A16}$$

for $\kappa = k_2/k_1$. With $0 < |\nu| \ll k_2 a$,

$$\nu \sim \frac{2k_1 a}{\pi} - \left(2m + \frac{1}{2}\right) + i, \tag{A17}$$

or

$$\nu \sim \frac{2k_1 a}{\pi} - \left(2m + \frac{1}{2}\right) + i \frac{2k_2}{\pi k_1}. \tag{A18}$$

In consideration of the transitional region of $H_\nu^{(1)}(k_2 a)$,³⁹ (A2) becomes

$$\tan\left[\sqrt{(k_1 a)^2 - \nu^2} - \nu \arccos\left(\frac{\nu}{k_1 a}\right) - \frac{\pi}{4}\right] = \kappa \left(\frac{2}{k_2 a}\right)^{1/3} e^{i2\pi/3} \frac{\text{Ai}'(e^{i2\pi/3}\xi)}{\text{Ai}(e^{i2\pi/3}\xi)}. \tag{A19}$$

This is not satisfied by any real ν . In the lower ν -plane,

$$e^{i2\pi/3} \frac{\text{Ai}'(e^{i2\pi/3}\xi)}{\text{Ai}(e^{i2\pi/3}\xi)} = i\bar{\kappa}, \tag{A20}$$

where

$$\bar{\kappa} = \kappa^{-1} \left(\frac{k_2 a}{2}\right)^{1/3}. \tag{A21}$$

Use of the large-argument approximation for the Airy function when $|\bar{\kappa}| \gg 1$ evinces that no zeros exist for $-\pi < \text{Arg } \xi < 0$. In the upper ξ -plane, Eq. (A20) is satisfied at points ξ_s lying in the neighborhoods of zeros of the denominator:

$$\xi_s \sim (-a_s) e^{i\pi/3} [1 + e^{i\pi/6} a_s^{-1} \bar{\kappa}^{-1}], \quad |\bar{\kappa}| \gg 1, \quad s = 1, 2, \dots, \tag{A22}$$

where a_s are the zeros of $\text{Ai}(z)$ ($a_s < 0$) numbered in order of ascending magnitude.⁴⁰ On the other hand,

$$\xi_s \sim (-\check{a}_s) e^{i\pi/3} [1 + e^{i\pi/6} \check{a}_s^{-2} \bar{\kappa}], \quad |\bar{\kappa}| \ll 1, \quad s = 1, 2, \dots, \tag{A23}$$

where \check{a}_s are zeros of $\text{Ai}'(z)$.⁴⁰ A function can be constructed which is holomorphic in the sector $\{\bar{\kappa}: -\pi/4 < \text{Arg } \bar{\kappa} < \pi/4\}$ and whose values are determined by the zeros in the lower ξ -plane given by Eq. (A20). By starting with expression (A22), it can be shown via analytic continuation that no such zeros exist. Such a construction is given in Ref. 33.

APPENDIX B: ON THE ZEROS OF $\mathcal{A}(\nu)$ AND $\mathcal{D}(\nu)$

In the spirit of Appendix A, consider the equation

$$\frac{H_\nu^{(2)'}(k_1 a)}{H_\nu^{(2)}(k_1 a)} - \kappa \frac{H_\nu^{(1)'}(k_2 a)}{H_\nu^{(1)}(k_2 a)} = 0, \tag{B1}$$

where $\kappa = k_1/k_2$ for $\mathcal{A}(\nu)$ and $\kappa = k_2/k_1$ for $\mathcal{D}(\nu)$. The following conclusions are reached.

(i) $\mathcal{A}(\nu)$ and $\mathcal{D}(\nu)$ exhibit no zeros for $0 \leq \text{Re } \nu < k_2 a$ and $k_2 a < \text{Re } \nu < k_1 a$ outside the transitional regions associated with $\nu = k_1 a$ or $\nu = k_2 a$.

(ii) The zeros inside the transitional region of $H_\nu^{(1)}(k_2 a)$ are approximated by those of the corresponding $\mathcal{K}(\nu)$ and $\mathcal{M}(\nu)$ in the upper ν -plane according to (A20) of Appendix A.

(iii) $\mathcal{A}(\nu)$ and $\mathcal{D}(\nu)$ have zeros inside the transitional region of $H_\nu^{(2)}(k_1 a)$ in the lower ν -plane. In view of Eq. (4.3), the equation there is

$$\mathcal{H}(\xi; -2\pi/3) = \kappa \frac{k_1}{k_2} \left(\frac{k_1 a}{2}\right)^{1/3}, \quad \xi = (\nu - k_1 a)(k_1 a/2)^{-1/3}, \tag{B2}$$

which is in turn approximated by

$$\text{Ai}(e^{-i2\pi/3} \xi) = 0. \tag{B3}$$

APPENDIX C: AN INTEGRAL IDENTITY

In this appendix, it is shown that

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n \nu} f(\nu) P_{\nu-1/2}^1(\cos \theta) \\ &= -2\pi \sum_{n=0}^{\infty} (-1)^n \sum_{s=\pm} \sum_{j=1}^{\infty} s e^{is(2n+1)\pi \nu_j^s} P_{\nu_j^s-1/2}^1(-\cos \theta) \text{Res}\{f(\nu)\}|_{\nu=\nu_j^s}, \end{aligned} \tag{C1}$$

where $f(\nu)$ is a meromorphic function with simple poles $\{\nu_j^s\}$ in the first ($s = +$) and fourth ($s = -$) quadrant ($j = 1, 2, \dots$, in order of ascending imaginary part), and with no singularities in the imaginary and real axes other than poles that coincide with zeros of $P_{\nu-1/2}^1(\cos \theta)$. $f(\nu)$ is assumed to satisfy

$$f(-\nu) = -f(\nu) \quad \forall \nu, \quad f(\nu) = O(\nu^d) \quad \text{as } \nu \rightarrow \infty, \quad \text{Re } \nu > 0, \tag{C2}$$

where d is a real number that may depend on $\text{Arg } \nu$.

In the spirit of Ref. 9, the left-hand side of (C1) is written as an integral over a contour C encircling the positive real axis clockwise, as shown in Fig. 6. Then,

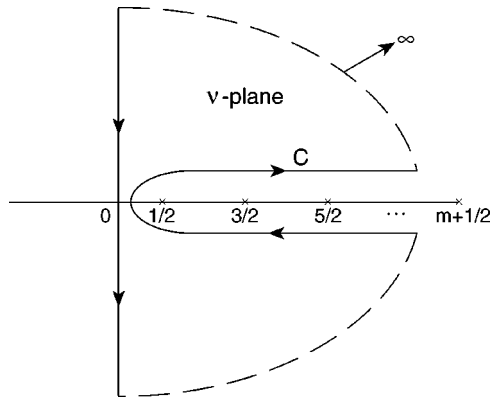


FIG. 6. Contour C that serves the derivation of identity (C1) of Appendix C; m is a non-negative integer.

$$\begin{aligned}
 & \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} f(\nu) P_{\nu-1/2}^1(\cos \theta) \\
 &= i \sum_{l=0}^{\infty} e^{i(l+1/2)\pi} f\left(l+\frac{1}{2}\right) P_l^1(-\cos \theta) \\
 &= -\frac{1}{2i} \int_C \frac{d\nu}{\cos \pi\nu} f(\nu) P_{\nu-1/2}^1(-\cos \theta) \\
 &= -\pi \sum_{s=\pm} \sum_{j=1}^{\infty} \frac{P_{\nu_j^s-1/2}^1(-\cos \theta)}{\cos \pi\nu_j^s} \text{Res}\{f(\nu)\}_{\nu=\nu_j^s} - \frac{1}{2i} \int_{-i\infty}^{i\infty} \frac{d\nu}{\cos \pi\nu} f(\nu) P_{\nu-1/2}^1(-\cos \theta),
 \end{aligned} \tag{C3}$$

by employing $P_l^1(\cos \theta) = (-1)^{l+1} P_l^1(\cos(\pi - \theta))$ and properly closing C at infinity. By virtue of (C2) and the identity $P_{-\nu-1/2}(x) = P_{\nu-1/2}(x)$, the integral in the right-hand side of (C3) is identically zero. The expansion

$$\frac{1}{\cos \pi\nu} = 2e^{\pm i\pi\nu} \sum_{n=0}^{\infty} (-1)^n e^{\pm i2\pi n\nu}, \tag{C4}$$

where $\text{Im } \nu > 0$ (+) or $\text{Im } \nu < 0$ (-), immediately yields (C1). As a corollary,

$$\sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} d\nu e^{i2\pi n\nu} f(\nu) P_{\nu-1/2}^1(\cos \theta) = 0, \tag{C5}$$

if $f(\nu)$ is holomorphic for $\text{Re } \nu > 0$.

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