

On the Wiener–Hopf Method for Surface Plasmons: Diffraction from Semiinfinite Metamaterial Sheet

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By formally invoking the Wiener–Hopf method, we explicitly solve a one-dimensional, singular integral equation for the excitation of a slowly decaying electromagnetic wave, called surface plasmon-polariton (SPP), of small wavelength on a semiinfinite, flat conducting sheet irradiated by a plane wave in two spatial dimensions. This setting is germane to wave diffraction by edges of large sheets of single-layer graphene. Our analytical approach includes (i) formulation of a functional equation in the Fourier domain; (ii) evaluation of a *split function*, which is expressed by a contour integral and is a key ingredient of the Wiener–Hopf factorization; and (iii) extraction of the SPP as a simple-pole residue of a Fourier integral. Our analytical solution is in good agreement with a finite-element numerical computation.

1. Introduction

Surface plasmon-polaritons (SPPs) are evanescent electromagnetic waves that propagate on the surface of conductors and arise from the coupling of the incident radiation with the electron plasma [1,2]. The study of surface waves in the radio-frequency regime dates back to Sommerfeld [3,4]; for later advances, see [5]. Recently, it has been realized that SPPs with novel features can be generated on a class of atomically thick, conducting materials, e.g., graphene, in the infrared spectrum [6,7]. The dielectric

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permittivity of these materials can have a negative real part at terahertz frequencies. Accordingly, slowly decaying SPPs may emerge, confined near the surfaces of this class of metamaterials¹ with a wavelength much smaller than the free-space wavelength. These fine-scale waves form parts of solutions to boundary value problems for Maxwell's equations [1, 8].

In this article, we solve a prototypical scattering problem to analytically show how the SPP is excited via wave diffraction by the edge of a flat sheet with finite conductivity in the two-dimensional (2D) space, \mathbb{R}^2 . By using boundary conditions of a resistive half plane² and the Sommerfeld radiation condition for Maxwell's equations with an $e^{-i\omega t}$ time dependence ($\omega > 0$, $i^2 = -1$), we state the problem in the form (see the Appendix)³

$$u(x) = u^{\text{in}}(x) + \frac{i\tilde{\sigma}}{k} \left(\frac{d^2}{dx^2} + k^2 \right) \int_0^\infty \mathcal{K}(x-x') u(x') dx', \quad x > 0;$$

$$\mathcal{K}(x) = \frac{i}{4} H_0^{(1)}(k|x|), \quad -\infty < x < \infty. \quad (1)$$

The kernel, \mathcal{K} , of integral equation (1) is expressed in terms of the first-kind, zeroth-order Hankel function, $H_0^{(1)}$ [12], which comes from the Green function for the scalar Helmholtz equation on \mathbb{R}^2 [13]. Physically, (1) accounts for the continuity of the tangential electric field and a jump of the tangential magnetic field across a resistive sheet, consistent with the modeling of *isotropic*, one-layer graphene [1]. By (1), the sheet is the half line $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y = 0\}$. The solution $u : [0, \infty) \rightarrow \mathbb{C}$ expresses the (tangential) x -component of the electric field on Ω (where \mathbb{C} denotes the complex plane); u^{in} is the x -component of the *incident* electric field on Ω under transverse-magnetic (TM) polarization; k is the wave number of the ambient medium ($\text{Re } k > 0$, $\text{Im } k \geq 0$); and $\tilde{\sigma}$ is a dimensionless parameter for the strength of the jump discontinuity across the sheet, with

$$\tilde{\sigma} = \frac{\omega\mu\sigma}{k},$$

where σ is the sheet conductivity and μ is the magnetic permeability of the ambient medium. Although (1) can be cast into a dimensionless form via the scaling of x with $1/k$, we adhere to (1) for later convenience.

¹Here, we use the term “metamaterial” to mean any atomically thick, conducting material whose optical conductivity allows the generation of SPPs with sufficiently small wavelength; see (2) later.

²The terms “conducting” and “resistive” sheet or half plane are used interchangeably. The “top Riemann sheet,” which is invoked below for a particular branch of a multiple-valued function in the complex plane, should not be confused with the physical sheet causing wave diffraction.

³The placement of the 1D Helmholtz operator, $(d^2/dx^2) + k^2$, outside the integral of (1) affords a kernel, \mathcal{K} , that is (logarithmically) *integrable* through $x = x'$; cf. Pocklington's integral equation for wire antennas [9,10]. The problem can be stated in alternate forms, e.g., via direct application of the Fourier transform to a boundary value problem [11], which circumvents (1).

Our goal is to obtain a classical solution to (1). Our analysis is formal and consists of the following steps. (a) The conversion of (1) into a functional equation with two unknown functions, one of which is the Fourier transform, \hat{u} , of u [14, 15]; (b) the Wiener–Hopf factorization [16, 17] to solve the functional equation; (c) the explicit, approximate evaluation, when u^{in} is a plane wave, of a *split function* (defined in Section 2) that is a key ingredient of the factorization process; and (d) the derivation of a 1D Fourier integral for u . Note that we first apply the Wiener–Hopf method exactly, for reasonably arbitrary $\tilde{\sigma}$, and then approximate the requisite split function for physically motivated values of $\tilde{\sigma}$; see (2) below. In this procedure, the SPP is defined as a pole contribution to the Fourier integral of the exact result. The SPP propagates with a wave number k_{sp} , $|k_{\text{sp}}| \gg |k|$, along the sheet if $\tilde{\sigma}$ satisfies restrictions (2), which are identical to conditions for a fine-scale SPP in the 3D setting [8].

We relax mathematical rigor, avoiding a precise description of the space of solutions for (1). We seek an integrable u on $(0, \infty)$, $u \in L^1(0, \infty)$, assuming that u^{in} is also integrable if the ambient space is lossy ($\text{Im } k > 0$). The solution for $\text{Im } k = 0$ is then obtained via analytic continuation.

We also demonstrate that our solution is in good agreement with a numerical computation based on the finite element method. Details of the underlying numerical method lie beyond our present scope; the interested reader may consult [18].

1.1. Physical relevance and assumptions

Physically, the problem at hand is motivated by the need to understand how an SPP can be excited by a TM plane wave incident upon a conducting planar sheet with a defect, i.e., an edge. Our approach yields an integral that connects the relatively rapid spatial variation of the SPP with the behavior of the total diffracted field near the edge. In 2D, the edge acts as an induced point source, which radiates a superposition of plane waves with a wide range of wave numbers; depending on $\tilde{\sigma}$, these plane waves may selectively enable the requisite phase matching for the excitation of the SPP. The Wiener–Hopf method singles out their contribution. In contrast, a plane wave incident on an *infinite*, flat conducting sheet *cannot* excite an SPP [1].

To solve the Wiener–Hopf functional equation, we use assumptions consistent with low-dimensional metamaterials such as graphene. Specifically, the surface impedance, σ^{-1} , of the sheet is much larger than the intrinsic impedance of the ambient space [1]; and the imaginary part of σ is positive. For a slightly lossy ambient medium, these hypotheses amount to having [8]

$$|\tilde{\sigma}| \ll 1 \quad \text{and} \quad \text{Im } \tilde{\sigma} > 0, \quad (2)$$

and $\text{Re } \tilde{\sigma} > 0$. We show that the u solving (1) contains a wave, to be defined as the SPP, that has a wavelength much smaller than $(\text{Re } k)^{-1}$ if (2) holds and the imaginary part of σ is large enough compared to its real part.

1.2. On the mathematical approach

In view of (2), one wonders if the Wiener–Hopf method can be replaced by a simpler scheme in seeking a physically transparent approximation for the tangential electric field which would reveal the SPP. Let us heuristically entertain such a scenario of solving (1). The first condition in (2) suggests that (1) is amenable to regular perturbations, because K and u^{in} do not depend on $\tilde{\sigma}$; it is then tempting to solve (1) by successive iterations, starting with the replacement of u by u^{in} under the integral sign. This approach has the flavor of the Neumann expansion for second-kind Fredholm equations [17]; and produces mathematically meaningful terms for u if u^{in} is a plane wave.

However, this regular perturbation fails to *directly* produce the SPP.⁴ The main reason is that the SPP is expected to have a spatial microstructure, which is proportional to the small parameter, $\tilde{\sigma}$ [1, 8]. This feature calls for applying singular perturbations, e.g., a suitable two-scale expansion, for u . Alternatively, it is plausible to seek approximations of physical optics for the diffracted field; see Section 1.3 for a discussion of related past works.

The Wiener–Hopf method captures the SPP by a closed-form expression which is amenable to transparent approximations. In our analysis, we exploit the scale separation implied by $|k_{sp}| \gg k$, or, equivalently, conditions (2), to approximate the split function of the Wiener–Hopf factorization in the Fourier domain. Thus, this method does not require any unnecessary, a priori assumption about the structure of the SPP. At the risk of redundancy, we repeat that this wave naturally arises as a pole contribution to a 1D Fourier integral. On the other hand, more direct asymptotic methods may prove useful for a *finite* strip of graphene [19, 20]. This problem is not studied in this article.

1.3. On past works

There is a vast body of works in the diffraction of an electromagnetic or acoustic wave by a half plane. It is impossible to list them exhaustively.

Exact solutions to diffraction problems have been obtained in a limited number of cases. For example, Sommerfeld first solved completely the diffraction of a plane wave from a perfectly conducting half plane via the

⁴A plausible question is whether the SPP can be captured by a resummation of the ensuing regular-perturbation expansion (in the spirit, e.g., of the Poincaré–Lindstedt method for solutions to a class of ordinary differential equations). An answer lies beyond our present scope.

Fresnel integral [3]; see also [21]. This technique has been extended to half planes and wedges with the “impedance boundary condition,” i.e., a linear relation between the tangential electric and magnetic field components, separately on each face of the plane or wedge [22–25]. Alternate approaches to this type of problem via the Wiener–Hopf functional equation can be found, e.g., in [26–28]. For comprehensive reviews, the reader may consult [29–33].

In principle, the impedance condition expresses a local physical effect different from the condition for a resistive sheet, which underlies (1). The latter condition expresses transmission of the tangential field components across the sheet, thus connecting values of the fields on the two sides of the sheet. However, as pointed in [34], the solution to the problem with impedance conditions can be viewed as a linear superposition of the solution for a resistive sheet and its electromagnetic dual, of the “magnetically conductive” sheet, with the magnetic and electric fields interchanged. Thus, the electric current, which is proportional to our u , induced on the resistive half plane can be extracted as a particular physical contribution to the current of a suitably chosen problem with impedance boundary conditions. This approach is pursued in [35] with recourse to [26].

Our treatment differs from [26,35] in the following three main aspects. First, our formulation spells out the structure of the corresponding boundary value problem for a conducting sheet via integral equation (1), without direct invocation of an impedance condition. Second, we obtain the solution directly, by means amenable to rigor. In contrast, in [26,35] the author employs a different integral-equation formulation, which amounts to impedance conditions; and extracts a solution for the resistive sheet implicitly. Hence, in [26,35] the analytical linkage of the derived solution to the underlying boundary value problem for a resistive sheet is not evident.

Third, the Wiener–Hopf method here invokes assumptions tailored to the physics of metamaterial sheets such as graphene; cf. (2). As a result, a principal ingredient (split function), which depends on the kernel and is needed in the factorization process, is evaluated approximately in terms of a known transcendental function, Euler’s dilogarithm. In addition, the SPP emerges as part of our solution. On the other hand, the requisite split function is expressed in terms of an unresolved integral in past works [26,35]; also, in those works the SPP is apparently not part of the solution.

We should mention disparate studies that make use of Keller’s geometric theory of diffraction; see, e.g., [36,37]. In a similar vein, approximations of physical optics focus on the construction of high-frequency solutions to Maxwell’s equations away from the edge [38–41]. We believe that incorporating the SPP into this physical-optics framework requires some prior knowledge of properties of this wave. Our method amply provides this clue.

1.4. Article organization and conventions

The remainder of this article is organized as follows. In Section 2, we derive an exact solution to (1) via the Wiener–Hopf method. In Section 3, we approximately evaluate a requisite contour integral (split function) by using (2), thus simplifying our analytical solution. Section 4 focuses on the comparison of our solution to a numerical computation.

Throughout this article, the hat on top of a symbol denotes the Fourier transform of the respective function with respect to x . We write $f = \mathcal{O}(g)$ ($f = o(g)$) to mean that $|f/g|$ is bounded by a nonzero constant (approaches zero) in a prescribed limit; accordingly, $f \sim g$ implies that $f - g = o(g)$. The terms *analytic* and *holomorphic* function are used interchangeably. The \pm subscript for a function indicates that the function is analytic in the upper (+) half, \mathbb{C}_+ , or lower (–) half, \mathbb{C}_- , of the complex Fourier domain.

2. Wiener–Hopf factorization: Exact solution, u

In this section, we formally seek an exact solution to (1) via the Wiener–Hopf factorization when u^{in} is a plane wave. In the Appendix, this integral equation is derived from the requisite boundary value problem. The basic ingredients of the Wiener–Hopf method are reviewed in the Appendix.

2.1. Main result and definition of SPP

The main result of this section is the Fourier integral

$$u(x) = \frac{e^{-Q_+(k_{\parallel})}}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\xi - k_{\parallel}} \frac{e^{Q_+(\xi)}}{\mathcal{P}(\xi)} e^{i\xi x} d\xi, \quad x > 0. \quad (3)$$

Let us clarify the notation. This formula accounts for $u^{\text{in}}(x) = e^{ik_{\parallel}x}$. We consider $k_{\parallel} = k \sin \theta$, $0 < \theta < \pi/2$, $\text{Im } k > 0$, $\text{Re } k > 0$; and extend the results to other values of θ or positive k (as $\text{Im } k \downarrow 0$) by analytic continuation. The integration path in the ξ -plane lies on the real axis, indented below possible singularities, e.g., $\xi = k_{\parallel}$, that fall on the real axis from \mathbb{C}_+ . In (3), the denominator is

$$\mathcal{P}(\xi) = 1 - i \frac{\tilde{\sigma}}{k} (k^2 - \xi^2) \widehat{\mathcal{K}}(\xi), \quad (4)$$

where $\widehat{\mathcal{K}}(\xi)$ is the Fourier transform of the kernel, namely (see the Appendix),

$$\widehat{\mathcal{K}}(\xi) = \int_{-\infty}^{\infty} \mathcal{K}(x) e^{-i\xi x} dx = \frac{i}{2\sqrt{k^2 - \xi^2}}, \quad \text{Im } \sqrt{k^2 - \xi^2} > 0, \quad (5)$$

in the top Riemann sheet for $\sqrt{k^2 - \xi^2}$ ($\xi \in \mathbb{C}$). The function $Q_+(\xi)$ entering (3) is analytic in the upper half, \mathbb{C}_+ , of the ξ -plane; and is defined by the (absolutely) convergent integral

$$Q_+(\xi) = \frac{\xi}{\pi i} \int_0^\infty \frac{Q(\zeta)}{\zeta^2 - \xi^2} d\zeta, \quad \xi \in \mathbb{C}_+; \quad Q(\zeta) := \ln \mathcal{P}(\zeta). \quad (6)$$

This Q_+ is a “+” split function. Note that $\mathcal{P}(\zeta)$ and $Q(\zeta)$ are even in the top Riemann sheet. We have assumed that (see Definition 2 in the Appendix)

$$\text{ind}(\mathcal{P}) = 0,$$

which means that $Q(\zeta) = \ln \mathcal{P}(\zeta)$ does *not* pick up any phase change of $\mathcal{P}(\zeta)$ as ζ moves from $-\infty$ to $+\infty$ in (6). The $Q_+(\xi)$ is evaluated in Section 3.

By resorting to (3), we can now define the SPP more precisely.

DEFINITION 1 (SPP). *The SPP is the contribution to (3) from the (simple) zero, $\xi = k_{\text{sp}}$, of $\mathcal{P}(\xi)$ in the upper half of the ξ -top Riemann sheet. If this pole contributes, the SPP is described as part of u by*

$$u^{\text{SP}}(x) = e^{ik_{\text{sp}}x} \frac{e^{-Q_+(k_{\parallel})+Q_+(k_{\text{sp}})}}{k_{\text{sp}} - k_{\parallel}} \frac{1}{\mathcal{P}'(k_{\text{sp}})}, \quad x > 0, \quad (7)$$

where $\mathcal{P}(k_{\text{sp}}) = 0$ and $\text{Im } k_{\text{sp}} > 0$ with $\text{Im} \sqrt{k^2 - \xi^2} \Big|_{\xi=k_{\text{sp}}} > 0$. (Here, $\mathcal{P}'(\xi)$ denotes the first derivative of $\mathcal{P}(\xi)$.)

It can be readily verified that, for sufficiently small $\text{Im } k$, this k_{sp} is present in the top Riemann sheet with $|k_{\text{sp}}| \gg |k|$ provided (2) hold. By (4), we find

$$k_{\text{sp}} = i \frac{2k}{\tilde{\sigma}} \sqrt{1 - \tilde{\sigma}^2} \sim i \frac{2k}{\tilde{\sigma}} \quad \text{if } |\tilde{\sigma}| \ll 1. \quad (8)$$

This k_{sp} is identical with the SPP wave number in the 3D setting [1, 8].

Remark 1. It is of interest to comment on the contribution of the pole $\xi = k_{\parallel}$ to integral (3). The respective residue is the “direct” field

$$u^{\text{dir}}(x) = \frac{e^{ik_{\parallel}x}}{\mathcal{P}(k_{\parallel})}, \quad x > 0, \quad (9)$$

which is the sum of the incident field, u^{in} , and its direct reflection from an infinite conducting plane. Thus, the remaining, diffracted field for $x > 0$ is

$$u^{\text{df}}(x) = u(x) - u^{\text{dir}}(x) = \frac{e^{-Q_+(k_{\parallel})}}{2\pi i} \int_{-\infty}^\infty \frac{1}{\xi - k_{\parallel}} \left[\frac{e^{Q_+(\xi)}}{\mathcal{P}(\xi)} - \frac{e^{Q_+(k_{\parallel})}}{\mathcal{P}(k_{\parallel})} \right] e^{i\xi x} d\xi, \quad (10)$$

which contains the SPP by Definition 1. In Section 3, we show that the SPP is the dominant contribution to u^{df} under (2) for a certain range of distances, x , from the edge.

2.2. Derivation of exact solution

This section is devoted to the derivation of (3) by virtue of (1). We assume that $\text{Im } k > 0$ and u is integrable on $(0, \infty)$. By setting $u(x) \equiv 0$ and $u^{\text{in}}(x) \equiv 0$ for $x < 0$, we write (1) as

$$u(x) = u^{\text{in}}(x) + g(x) + \frac{i\tilde{\sigma}}{k} \left(\frac{d^2}{dx^2} + k^2 \right) \int_{-\infty}^{\infty} \mathcal{K}(x - x') u(x') dx', \quad x \in \mathbb{R}, \tag{11}$$

where

$$g(x) := \begin{cases} -i \frac{\tilde{\sigma}}{k} \left(\frac{d^2}{dx^2} + k^2 \right) \int_0^{\infty} \mathcal{K}(x - x') u(x') dx', & x < 0, \\ 0, & x > 0. \end{cases}$$

The application of the Fourier transform to (11) yields the functional equation

$$\mathcal{P}(\xi) \widehat{u}(\xi) = \widehat{u}^{\text{in}}(\xi) + \widehat{g}(\xi), \quad \xi \in \mathbb{R}; \tag{12}$$

cf. (A.15) of the Appendix. Clearly, $\widehat{u}(\xi)$ is analytic in \mathbb{C}_- and $\widehat{g}(\xi)$ is analytic in \mathbb{C}_+ . The $\mathcal{P}(\xi)$ is dependent on the kernel and is defined in (4).

The task is to separate the terms in (12) into two parts; one part should be holomorphic in \mathbb{C}_+ and another in \mathbb{C}_- (see the Appendix). First, we need to determine split functions $Q_s(\xi)$, holomorphic in \mathbb{C}_s ($s = \pm$), such that

$$Q(\xi) = Q_+(\xi) + Q_-(\xi), \quad \xi \in \mathbb{R}; \quad \mathcal{P}(\xi) = e^{Q(\xi)}. \tag{13}$$

Because $Q(\xi)$ is analytic in a neighborhood of the real axis and an even function, with $\text{ind}(\mathcal{P}) = 0$, application of the Cauchy integral formula yields [16]

$$Q_s(\xi) = s \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Q(\zeta)}{\zeta - \xi} d\zeta = s \frac{\xi}{\pi i} \int_0^{\infty} \frac{Q(\zeta)}{\zeta^2 - \xi^2} d\zeta; \quad s \text{Im } \xi > 0 \quad (s = \pm). \tag{14}$$

It follows that (12) is recast to

$$e^{Q_-(\xi)} \widehat{u}(\xi) = e^{-Q_+(\xi)} \widehat{u}^{\text{in}}(\xi) + e^{-Q_+(\xi)} \widehat{g}(\xi), \quad \xi \in \mathbb{R}, \tag{15}$$

where the left-hand side is analytic in \mathbb{C}_- and the last term on the right-hand side is analytic in \mathbb{C}_+ . To split the remaining term, note that

$$\widehat{u}^{\text{in}}(\xi) = \frac{-i}{\xi - k_{\parallel}}, \quad \text{Im } k_{\parallel} > 0.$$

Thus, by setting

$$e^{-Q_+(\xi)}\widehat{u}^{\text{in}}(\xi) = \Lambda_+(\xi) + \Lambda_-(\xi),$$

we obtain the requisite split functions by inspection, namely,

$$\Lambda_+(\xi) = [e^{-Q_+(\xi)} - e^{-Q_+(k_{\parallel})}]\widehat{u}^{\text{in}}(\xi), \quad \Lambda_-(\xi) = e^{-Q_+(k_{\parallel})}\widehat{u}^{\text{in}}(\xi). \quad (16)$$

Consequently, (15) becomes

$$e^{Q_-(\xi)}\widehat{u}(\xi) - \Lambda_-(\xi) = \Lambda_+(\xi) + e^{-Q_+(\xi)}\widehat{g}(\xi), \quad \xi \in \mathbb{R}, \quad (17)$$

where the “+” and “−” parts are completely separated. Thus, we infer that the two sides of this equation together define an *entire* function, $E(\xi)$, in the complex ξ -plane. In particular, we have

$$E(\xi) = \begin{cases} e^{Q_-(\xi)}\widehat{u}(\xi) - \Lambda_-(\xi), & \text{Im } \xi < 0, \\ \Lambda_+(\xi) + e^{-Q_+(\xi)}\widehat{g}(\xi), & \text{Im } \xi > 0. \end{cases} \quad (18)$$

Once this $E(\xi)$ is determined, $\widehat{u}(\xi)$ is found via (18), namely,

$$\widehat{u}(\xi) = e^{-Q_-(\xi)}[\Lambda_-(\xi) + E(\xi)]. \quad (19)$$

Next, we show that $E(\xi) \equiv 0$ via the asymptotic behavior of (18) as $\xi \rightarrow \infty$. Because $u(x)$ is integrable on $(0, \infty)$ and vanishes identically on $(-\infty, 0)$, we assert that $\widehat{u}(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$ in \mathbb{C}_- . In a similar vein, $\widehat{g}(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$ in \mathbb{C}_+ . Hence, to extract $E(\xi)$, we need to know how $Q_{\pm}(\xi)$ behave as $\xi \rightarrow \infty$ in \mathbb{C}_{\pm} , respectively.

Let us focus on the $Q_+(\xi)$ by virtue of (14). Notice that

$$Q(\zeta) = \ln\left(\frac{i\tilde{\sigma}}{2k}\zeta\right) + \tilde{Q}(\zeta); \quad \tilde{Q}(\zeta) = \mathcal{O}(1/\zeta) \quad \text{as } \zeta \rightarrow +\infty.$$

Hence, if we naively approximate the denominator in (14) according to $\zeta^2 - \xi^2 \approx -\xi^2$ for large ξ , the resulting integral diverges. Therefore, to extract the leading-order term for $Q_+(\xi)$ as $\xi \rightarrow \infty$, we write

$$Q_+(\xi) = \frac{\xi}{i\pi} \int_0^{\infty} \frac{\ln\left(\frac{i\tilde{\sigma}}{2k}\zeta\right)}{\zeta^2 - \xi^2} d\zeta + \frac{1}{i\pi} \int_0^{\infty} \frac{\tilde{Q}(\zeta)}{(\zeta/\xi)^2 - 1} d(\zeta/\xi), \quad \text{Im } \xi > 0.$$

The second term approaches zero as $\xi \rightarrow \infty$; and the first term is an elementary integral, which is computed exactly by closing the path in the ζ -plane through the negative real axis and a large semicircle in \mathbb{C}_+ . Hence, we obtain

$$Q_+(\xi) = \frac{1}{2} \ln\left(\tilde{\sigma} \frac{\xi}{2k}\right) + o(1) \quad \text{as } \xi \rightarrow \infty \text{ in } \mathbb{C}_+. \quad (20)$$

By symmetry, we have $Q_-(\xi) = (1/2) \ln[-\tilde{\sigma}\xi/(2k)] + o(1)$ as $\xi \rightarrow \infty$ in \mathbb{C}_- .

Equations (16) then entail $\Lambda_{\pm}(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$ in \mathbb{C}_{\pm} . Thus, by (18) we assert that $E(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$ in \mathbb{C} . By Liouville’s theorem, $E(\xi)$ must be a constant everywhere, which has to be zero; $E(\xi) \equiv 0$. Consequently, formula (3) results from Fourier-inversion of (19) by use of (13) and (16).

3. Approximate split function and solution

In this section, we approximately evaluate $Q_+(\xi)$, which enters formula (3) for $u(x)$. To this end, we enforce conditions (2). Accordingly, we obtain an approximation for $u(x)$ by manipulating its 1D Fourier integral.

The main result of this section is the formula

$$Q_+(\xi) \sim \frac{1}{2\pi i} \frac{\tilde{\sigma}\xi}{k} [\mathcal{F}_1(\xi/k) + \mathcal{F}_2(\tilde{\sigma}\xi/k)], \quad |\tilde{\sigma}| \ll 1, \tag{21}$$

where the functions \mathcal{F}_j ($j = 1, 2$) are described in (24b) and (27b) below. Notably, this approximation for $Q_+(\xi)$ exhibits two distinct (fast and slow) scales in the complex ξ -plane.

3.1. Approximation for $Q_+(\xi)$

Equation (14), with $s = +$, is conveniently recast into the expression

$$Q_+(\xi) = \frac{\xi}{i\pi} \left\{ \int_0^\infty \frac{\ln\left(1 + \frac{\tilde{\sigma}}{2k} \sqrt{k^2 - \zeta^2}\right) - \ln\left(1 + \frac{\tilde{\sigma}}{2k} i\zeta\right)}{\zeta^2 - \xi^2} d\zeta + \int_0^\infty \frac{\ln\left(1 + \frac{\tilde{\sigma}}{2k} i\zeta\right)}{\zeta^2 - \xi^2} d\zeta \right\}.$$

The first integral is amenable to asymptotics if $|\tilde{\sigma}| \ll 1$. A key observation is that the integrand of this integral has a negligible contribution if $\zeta \gg |k|$. By expanding out its logarithms, we write

$$Q_+(\xi) \sim \frac{1}{2\pi i} \frac{\tilde{\sigma}\xi}{k} \{\mathcal{I}_1(\xi) + \mathcal{I}_2(\xi)\}, \quad |\tilde{\sigma}| \ll 1, \tag{22}$$

where

$$\mathcal{I}_1(\xi) = \int_0^\infty \frac{\sqrt{k^2 - \zeta^2} - i\zeta}{\zeta^2 - \xi^2} d\zeta = \int_0^{\infty e^{-i \arg k}} \frac{\sqrt{1 - \tau^2} - i\tau}{\tau^2 - (\xi/k)^2} d\tau, \tag{23a}$$

$$\begin{aligned} \mathcal{I}_2(\xi) &= \left(\frac{\tilde{\sigma}}{2k}\right)^{-1} \int_0^\infty \frac{\ln\left(1 + \frac{i\tilde{\sigma}}{2k}\zeta\right)}{\zeta^2 - \xi^2} d\zeta \\ &= \int_0^{\infty e^{i \arg(\tilde{\sigma}/k)}} \frac{\ln(1 + i\tau)}{\tau^2 - (\xi\tilde{\sigma}/(2k))^2} d\tau. \end{aligned} \tag{23b}$$

In the above, we applied transformations $\zeta \mapsto \tau$ to spell out the dependence of the integrals on the physical parameters k and $\tilde{\sigma}$. In (23a), $\tau = \zeta/k$; and in (23b), $\tau = (\tilde{\sigma}/(2k))\zeta$. Our next task is to compute \mathcal{I}_1 and \mathcal{I}_2 exactly.

3.1.1. *Integral $\mathcal{I}_1(\xi)$.* To simplify the derivation, first consider $\text{Re } k = 0$ with $\text{Im } k > 0$. We can continue analytically the result to the desired value of k in the end. By the change of variable $\zeta \mapsto \nu$ under $\zeta = ik \sinh(\ln \nu/2)$ and the substitution $\xi = -ik \sinh \varpi$, integral (23a) becomes

$$\begin{aligned} \mathcal{I}_1(\xi) &= i \int_0^1 \frac{1 + \nu}{\nu^2 - 2(\cosh(2\varpi))\nu + 1} d\nu \\ &= \frac{ie^\varpi}{2 \sinh \varpi} \ln(1 - e^{-2\varpi}) - \frac{ie^{-\varpi}}{2 \sinh \varpi} \ln(1 - e^{2\varpi}) = \mathcal{F}_1(\xi/k), \end{aligned} \tag{24a}$$

where

$$\mathcal{F}_1(w) = \frac{\sqrt{1 - w^2}}{w} \ln(w + i\sqrt{1 - w^2}) + i \ln(2w), \tag{24b}$$

which was computed by elementary methods. Note in passing that this result reads $\mathcal{I}_1(\xi) = i(1 - \varphi^2)^{-1/2} \varphi \arcsin \varphi + i \ln(2\sqrt{1 - \varphi^2})$, where $\varphi(\xi) = \sqrt{1 - (\xi/k)^2}$. The reader may observe that this function does not have any singularity, e.g., a branch point, at $\varphi = 0$. Thus, $\mathcal{I}_1(\xi)$ does not have any singularity at $\xi = \pm k$. Evidently, $\mathcal{I}_1(\xi)$ has a branch point at $\xi = 0$.

3.1.2. *Integral $\mathcal{I}_2(\xi)$.* We now turn our attention to (23b). Notice that

$$\mathcal{I}_2(\xi) = \frac{k}{\tilde{\sigma}\xi} \lim_{M \rightarrow +\infty} [F(\tilde{\sigma}\xi/k, M) - F(-\tilde{\sigma}\xi/k, M)], \tag{25}$$

where

$$F(w, M) := \int_0^M \frac{\ln(1 + i\zeta/2)}{\zeta - w} d\zeta. \tag{26}$$

This F can be evaluated in terms of Euler’s dilogarithm, L_2 , defined by [42]

$$L_2(-z) := - \int_0^z \frac{\ln(1 + u)}{u} du.$$

We directly compute

$$F(w, M) = \ln\left(1 + \frac{i}{2}w\right) \ln\left(\frac{M - w}{-w}\right) - L_2(-A(w, M)) + L_2(-A(w, 0)),$$

where

$$A(w, \ell) := \frac{i}{2}(\ell - w) \left(1 + \frac{i}{2}w\right)^{-1}.$$

In the limit $M \rightarrow \infty$, we need to invoke the asymptotic expansion of $L_2(z)$ for large $|z|$ [42]. Consequently, we are able to verify that the right-hand side of (25) indeed yields a finite limit for fixed $\tilde{\sigma}\xi/k$ as $M \rightarrow \infty$. After some algebra, the details of which we leave to the reader, we obtain

$$\mathcal{I}_2(\xi) = \mathcal{F}_2(\tilde{\sigma}\xi/k), \tag{27a}$$

where

$$\begin{aligned} \mathcal{F}_2(w) = \frac{1}{w} \left\{ \ln\left(\frac{1 - iw/2}{1 + iw/2}\right) \ln\left[\frac{iw}{2}(1 + w^2/4)^{-1/2}\right] + i\pi \ln(1 + iw/2) \right. \\ \left. + L_2(-A(w, 0)) - L_2(-A(-w, 0)) \right\}. \end{aligned} \tag{27b}$$

It can be readily checked that this $\mathcal{F}_2(w)$, for $w = \tilde{\sigma}\xi/k$, does not have any singularity, e.g., a branch point, at $\xi = i2k/\tilde{\sigma} \sim k_{\text{sp}}$ ($w = 2i$). In fact, the logarithmic singularity at $w = 2i$ that is present in the first line of (27b) turns out to be canceled by the asymptotic behavior of Euler’s dilogarithm as $A(w, 0) \rightarrow \infty$.

Equation (22) along with (24) and (27) yield the desired formula for $Q_+(\xi)$; cf. (21).

3.1.3. Particular values of $Q_+(\xi)$. It is of interest to compute two values of $Q_+(\xi)$, which are needed for the description of the solution, u ; see Section 3.2.1. We assume that $k_{\parallel} = \mathcal{O}(1)$ as $\tilde{\sigma} \rightarrow 0$. By (21), we find

$$\begin{aligned} Q_+(k_{\parallel}) = \frac{1}{2\pi} \frac{k_{\parallel}}{k} \tilde{\sigma} [\ln(4/\tilde{\sigma}) + 1] + \frac{\tilde{\sigma}}{2\pi} \left[\frac{\pi}{2} - \arcsin\left(\frac{k_{\parallel}}{k}\right) \right] \sqrt{1 - (k_{\parallel}/k)^2} \\ + \mathcal{O}(\tilde{\sigma}^2 \ln \tilde{\sigma}), \end{aligned} \tag{28a}$$

$$Q_+(k_{\text{sp}}) = \frac{1}{4\pi i} \left[-\ln^2 2 + 2\pi i \ln 2 - \frac{\pi^2}{3} - 2L_2(1/2) \right] + \mathcal{O}(\tilde{\sigma}^2 \ln \tilde{\sigma}), \tag{28b}$$

for $|\tilde{\sigma}| \ll 1$. Note that (28a) includes the case with small or vanishing k_{\parallel}/k , in which the second term of the asymptotic expansion may dominate; and the case $k_{\parallel} = k$ in which the first term prevails. Hence, we assert that

$$\begin{aligned} e^{-Q_+(k_{\parallel})} = 1 - \frac{1}{2\pi} \frac{k_{\parallel}}{k} \tilde{\sigma} [\ln(4/\tilde{\sigma}) + 1] - \frac{\tilde{\sigma}}{2\pi} \left[\frac{\pi}{2} - \arcsin\left(\frac{k_{\parallel}}{k}\right) \right] \sqrt{1 - (k_{\parallel}/k)^2} \\ + \mathcal{O}(\tilde{\sigma}^2 \ln \tilde{\sigma}). \end{aligned}$$

3.2. Approximations for u

Next, we heuristically discuss the relative importance of the SPP in the diffracted field, formula (10), in light of approximation (22). Furthermore, we describe the total field, $u(x)$, if $|k_{\text{sp}}x| \ll 1$, i.e., near the edge of the conducting sheet.

3.2.1. *On the diffracted field, u^{df} .* By formula (10), we write

$$u^{\text{df}}(x) = u^{\text{sp}}(x) + u^{\text{rad}}(x), \quad x > 0,$$

where, in view of Definition 1, we introduce the radiation field

$$u^{\text{rad}}(x) = \frac{e^{-Q_+(k_{\parallel})}}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\mathcal{P}(\xi)} \left\{ \left[e^{Q_+(\xi)} - \frac{\mathcal{P}(\xi)}{\mathcal{P}(k_{\parallel})} e^{Q_+(k_{\parallel})} \right] \frac{1}{\xi - k_{\parallel}} - \frac{e^{Q_+(k_{\text{sp}})}}{k_{\text{sp}} - k_{\parallel}} \frac{\mathcal{P}(\xi)}{\mathcal{P}'(k_{\text{sp}})(\xi - k_{\text{sp}})} \right\} e^{i\xi x} dx, \quad (29)$$

along with the SPP contribution, namely,

$$u^{\text{sp}}(x) = \left(\frac{2k}{\tilde{\sigma}} \right)^2 \frac{1}{k_{\text{sp}}(k_{\text{sp}} - k_{\parallel})} e^{-Q_+(k_{\parallel}) + Q_+(k_{\text{sp}})} e^{ik_{\text{sp}}x} \sim -e^{-Q_+(k_{\parallel}) + Q_+(k_{\text{sp}})} e^{ik_{\text{sp}}x}; \quad (30)$$

recall formulas (21) and (28) for $Q_+(\xi)$, and $k_{\text{sp}} = i(2k/\tilde{\sigma})\sqrt{1 - \tilde{\sigma}^2} \sim i2k/\tilde{\sigma}$.

Let us now focus on (29). The only singularity of the integrand in the upper half of the ξ -plane, \mathbb{C}_+ , is the branch point at $\xi = k$, because of $\mathcal{P}(\xi)$. Although we have not been able to express the requisite integral in terms of known transcendental functions for all x , it is possible to simplify it in certain regimes of kx .

By deforming the integration path in \mathbb{C}_+ for $x > 0$, and wrapping the path around the branch cut that emanates from $\xi = k$, we directly obtain

$$u^{\text{rad}}(x) \sim \frac{\tilde{\sigma}}{2\pi} e^{ikx - i\pi/4} \int_0^{\infty} \frac{e^{Q_+(k(1+i\tau))}}{1 - k_{\parallel}/k + i\tau} \frac{\sqrt{\tau}\sqrt{2+i\tau}}{1 + (i\tilde{\sigma}^2/4)\tau(2+i\tau)} e^{-(kx)\tau} d\tau. \quad (31)$$

This formula is amenable to comparisons to $u^{\text{sp}}(x)$.

First, consider $kx = \mathcal{O}(1)$. By treating τ as an $\mathcal{O}(1)$ variable, we can use $Q_+(k(1+i\tau)) = \mathcal{O}(\tilde{\sigma} \ln \tilde{\sigma})$ as $\tilde{\sigma} \rightarrow 0$; thus, we end up with the approximation

$$u^{\text{rad}}(x) \sim \frac{\tilde{\sigma}}{2\pi} e^{ikx - i\pi/4} h(kx), \quad (32)$$

where

$$h(z) := \int_0^\infty \frac{\sqrt{\tau}\sqrt{2+i\tau}}{i\tau + 1 - k_\parallel/k} e^{-z\tau} d\tau.$$

Clearly, by (30), the SPP dominates if

$$e^{\operatorname{Re}(2k/\tilde{\sigma}) - \operatorname{Im}k]x} \ll |\tilde{\sigma}|^{-1}; \quad kx = \mathcal{O}(1). \quad (33)$$

We can still use and simplify (32) when $|kx| \gg 1$, uniformly in the parameter $1 - k_\parallel/k$. For this purpose, consider the identity [42]

$$\frac{d}{dz} \left[e^{iz(1-k_\parallel/k)} h(z) \right] = e^{-i3\pi/4} \frac{\pi}{2} \frac{e^{-i(k_\parallel/k)z}}{z} H_1^{(1)}(z),$$

where $H_1^{(1)}(z)$ is the first-kind Hankel function of order one. Integration of the last equation entails

$$u^{\text{rad}}(x) \sim \frac{\tilde{\sigma}}{4} e^{ik_\parallel x} \int_{kx}^\infty \frac{H_1^{(1)}(t)}{t} e^{-i(k_\parallel/k)t} dt. \quad (34)$$

Hence, if $|kx| \gg 1$, we may use the asymptotic expansion of $H_1^{(1)}(t)$ for large $|t|$ [43]. After some algebra, we find

$$u^{\text{rad}}(x) \sim \frac{\tilde{\sigma}}{\sqrt{2\pi}} e^{ik_\parallel x - i3\pi/4} \left(\frac{k - k_\parallel}{k} \right)^{1/2} \left[\frac{e^{i(k-k_\parallel)x}}{\sqrt{(k-k_\parallel)x}} + \mathcal{F}((k-k_\parallel)x) \right], \quad (35)$$

where

$$\mathcal{F}(z) := i\sqrt{2\pi} \left[2^{-1/2} e^{i\pi/4} - C(z) - iS(z) \right],$$

and $C(z)$ and $S(z)$ are Fresnel integrals, defined by [42]

$$C(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\cos t}{\sqrt{t}} dt, \quad S(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\sin t}{\sqrt{t}} dt.$$

Note the expansion

$$\mathcal{F}(z) \sim e^{iz} \left(-\frac{1}{\sqrt{z}} + \frac{i}{2z^{3/2}} \right) \quad \text{as } |z| \rightarrow \infty,$$

which is useful if $|(k - k_\parallel)x| \gg 1$, along with $|kx| \gg 1$.

Next, consider $k_{\text{sp}}x = \mathcal{O}(1)$, i.e., $kx = \mathcal{O}(\tilde{\sigma})$. The major contribution to integration in (31) comes from values of τ such that $\tilde{\sigma}\tau = \mathcal{O}(1)$. By (21), we assert that

$$Q_+(k(1+i\tau)) \sim (2\pi i)^{-1} \tilde{\sigma}(1+i\tau) \mathcal{F}_2(\tilde{\sigma}(1+i\tau)).$$

Indeed, this formula suggests that $\tilde{\sigma}\tau$ is the natural, $\mathcal{O}(1)$ variable in this regime; then, $Q_+(k(1+i\tau)) = \mathcal{O}(1)$. The scaling of τ by $\tilde{\sigma}^{-1}$ implies that

$$e^{-ikx} u^{\text{rad}}(x) = \mathcal{O}(1) \quad \text{if } k_{\text{sp}}x = \mathcal{O}(1). \quad (36)$$

Hence, in this regime, $|u^{\text{rad}}|$ can be comparable to or smaller than $|u^{\text{sp}}|$.

Now let us assume that $1 \ll |k_{\text{sp}}x| < \mathcal{O}(|\tilde{\sigma}|^{-1})$, so that the major contribution to integration for $u^{\text{rad}}(x)$ comes from $\tilde{\sigma}\tau = \mathcal{O}((k_{\text{sp}}x)^{-1})$. It follows that

$$e^{-ikx} u^{\text{rad}}(x) = \mathcal{O}((k_{\text{sp}}x)^{-1}). \tag{37}$$

The SPP dominates if

$$e^{[\text{Re}(2k/\tilde{\sigma}) - \text{Im}k]x} \ll |2kx| |\tilde{\sigma}|^{-1}.$$

3.2.2. *Near-edge field*, $|k_{\text{sp}}x| \ll 1$. In this case, we need to examine the total field, $u(x)$. As $x \downarrow 0$, this solution is expected to be singular, i.e., the first derivative of $u(x)$ blows up.

First, by (3) we notice that $\lim_{x \downarrow 0} u(x) = 0$. Accordingly, to derive the asymptotic behavior of $u(x)$ as $x \downarrow 0$, it suffices to compute du/dx for small x and then integrate the result. Bear in mind that

$$\frac{du}{dx} = \frac{e^{-Q_+(k_{\parallel})}}{2\pi} \int_{-\infty}^{\infty} \frac{\xi}{1 + \frac{\tilde{\sigma}}{2k}\sqrt{k^2 - \xi^2}} \frac{e^{Q_+(\xi)}}{\xi - k_{\parallel}} e^{i\xi x} d\xi. \tag{38}$$

Second, as $x \downarrow 0$, in (38) the major contribution to integration arises from large ξ , $|\xi| \gg |k_{\text{sp}}|$, in \mathbb{C}_+ . Thus, by properly deforming the path in \mathbb{C}_+ , we simplify $Q_+(\xi)$ according to

$$Q_+(\xi) \sim \frac{1}{2\pi i} \frac{\tilde{\sigma}\xi}{k} F_2(\tilde{\sigma}\xi/k) = \frac{1}{2} \ln\left(\frac{\tilde{\sigma}\xi}{2k}\right) + \mathcal{O}(1/\xi),$$

in agreement with (20). Consequently, we compute

$$\begin{aligned} \frac{du}{dx} &\sim \frac{e^{-Q_+(k_{\parallel})}}{2\pi} \int_{-\infty}^{\infty} \frac{\xi}{\frac{\tilde{\sigma}}{2k}i\xi} \frac{e^{(1/2)\ln[\tilde{\sigma}\xi/(2k)]}}{\xi} e^{i\xi x} d\xi \\ &\sim e^{-i\pi/4 - Q_+(k_{\parallel})} \sqrt{\frac{2}{\pi}} \frac{k}{\sqrt{\tilde{\sigma}kx}} \quad \text{as } x \downarrow 0. \end{aligned}$$

Hence, we obtain

$$u(x) = \int_0^x \frac{du(z)}{dz} dz \sim 2e^{-i\pi/4 - Q_+(k_{\parallel})} \sqrt{\frac{2kx}{\pi\tilde{\sigma}}} \quad \text{as } x \downarrow 0. \tag{39}$$

Notice that the singularity is strengthened by the factor $\tilde{\sigma}^{-1/2}$ and $e^{-Q_+(k_{\parallel})} \sim 1$ ($|\tilde{\sigma}| \ll 1$).

Table 1
Comparison of Approximations for $Q_+(\xi)$, Particularly Formulas (28) and (21), Against the Defining, Exact Integral (6)

	$Q_+(k_{\parallel})$	$Q_+(k_{\text{sp}})$
Approx. (28)	0.017793 + 0.079874 i	0.34657 + 0.39270 i
Approx. (21)	0.017507 + 0.079995 i	0.34793 + 0.39902 i
Definition (6)	0.019864 + 0.079563 i	0.34813 + 0.40071 i

4. A numerical computation

In this section, we compare our analytical findings against a direct numerical computation with curl-conforming finite elements. For details of the underlying numerical method, we refer the reader to [18]. For the numerical computation, we set $k = 1$ and fix the remaining parameters by choosing

$$\tilde{\sigma} = 0.002 + 0.2i, \quad \theta = \pi/9.$$

We consider the above value of $\tilde{\sigma}$ as prototypical for graphene [18].

4.1. On the numerical computation of Q_+

Next, we indicate the accuracy of our approximate formulas for $Q_+(\xi)$ by selectively computing this function for $\xi = k_{\parallel}$ and $\xi = k_{\text{sp}}$. These values of ξ are widely separated; k_{\parallel} is of the order of k and $|k_{\text{sp}}| \gg k$ because $|\tilde{\sigma}|$ is small. Note that the corresponding values of $Q_+(\xi)$ enter some of the analytical expressions related to the solution $u(x)$, particularly formula (7) for the SPP.

In our computation, we compare approximate formulas (21) and (28) to a full numerical evaluation of integral (6) for $Q_+(\xi)$. The results are displayed in Table 1. Evidently, our approximations are in good agreement with the numerically computed value of the defining integral. In fact, the relative error is found to be around 2% for $Q_+(k_{\parallel})$ and less than 0.5% for $Q_+(k_{\text{sp}})$.

4.2. Numerically computed solution, u

Next, we perform a finite element computation for the electric field in the presence of the semiinfinite sheet in 2D [18]. The computational domain is chosen to be large enough to account for distances, x , from the edge such that $0 \leq kx \leq 30$. A numerical approximation, $u_h(x)$, of the tangential (x -directed) electric field, $u(x)$, on the sheet is thus extracted from the finite element computation.

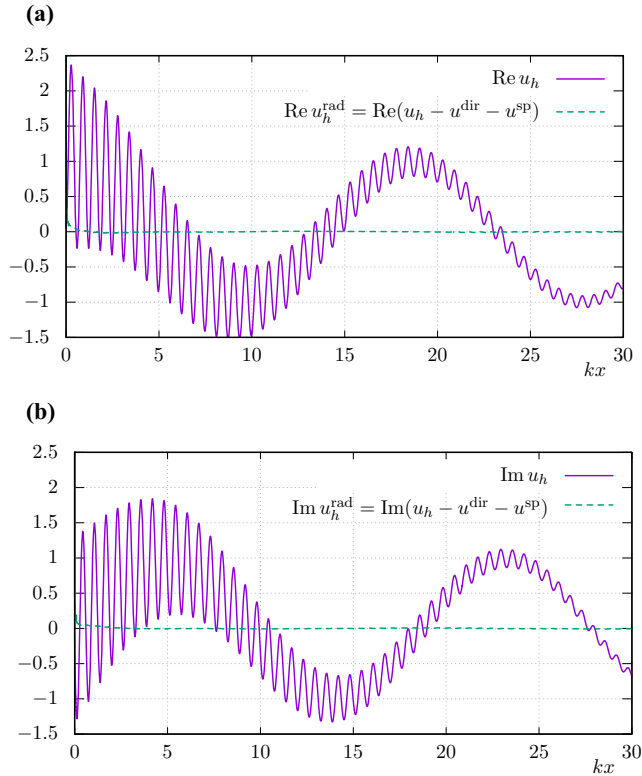


Figure 1. (Color online) Real part (a) and imaginary part (b), as a function of kx , of: the numerically computed solution, $u_h(x)$ (solid line); and the corresponding numerical approximation of the radiation field, $u_h^{\text{rad}}(x) = u_h(x) - u^{\text{dir}}(x) - u^{\text{sp}}(x)$ (dashes).

The result of this computation for $u(x)$ is shown in Fig. 1. In the same figure, we also present a plot for the quantity

$$u_h^{\text{rad}}(x) := u_h(x) - u^{\text{dir}}(x) - u^{\text{sp}}(x),$$

which pertains to the corresponding approximation of the radiation field, $u^{\text{rad}}(x)$. Note that the direct field, u^{dir} , is computed by (9) and the SPP contribution, u^{sp} , is evaluated by (7).

A few comments on Fig. 1 are in order. First, $u_h(x)$ contains a wave with a spatially fast scale, which can be identified with the SPP. Second, it is evident that, for all practical purposes, u_h^{rad} has a negligible contribution compared to the SPP and the direct field when $3 \leq kx \leq 30$. Furthermore, the radiation field decays algebraically and approaches zero for increasing and sufficiently large kx . This implies that the corresponding diffracted field, $u_h^{\text{df}} = u_h - u^{\text{dir}}$, is dominated by the SPP contribution, u^{sp} , unless x lies close enough to the edge ($kx \leq 3$); cf. Sections 3.2 and 4.3.

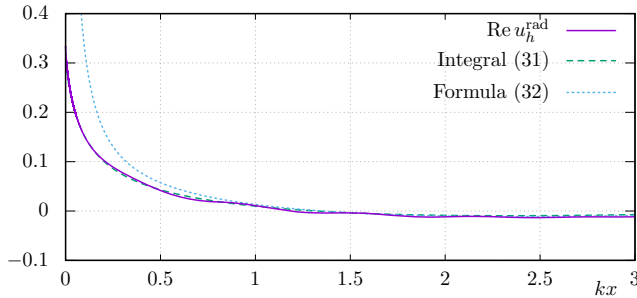


Figure 2. (Color online) Real part of numerical approximation, u_h^{rad} , of the radiation field (solid line); branch-cut integral (31) (dashes); and approximate formula (32) (dotted line).

4.3. Radiation field near edge: Numerical and analytical solutions

In light of the preceding comparison (Section 4.2), we now restrict attention to the radiation field, u^{rad} , close to the edge of the sheet, for $0 \leq kx \leq 3$. In this regime, $u^{\text{rad}}(x)$ may be comparable to the diffracted field, $u^{\text{df}}(x)$. Accordingly, we carry out another finite element computation to obtain a respective numerical resolution, in the proximity of the edge.

In Fig. 2, we indicate the accuracy of our analytical formulas for u^{rad} by showing the real parts of (i) the numerically computed radiation field, u_h^{rad} ; (ii) the numerically evaluated integral (31) for u^{rad} ; and (iii) the corresponding approximation (32), valid for $kx = \mathcal{O}(1)$. A high-order, adaptive quadrature rule is used to evaluate the requisite integral for u^{rad} , while $Q_+(\xi)$ is computed with recourse to approximate formula (21). Evidently, branch-cut integral (31) is in good agreement with the finite-element-based result. As we expected, formula (32) is reasonably accurate for $kx \geq 1$.

5. Conclusion

In this paper, we analyzed a prototypical scattering problem for the excitation by an incident plane wave of a fine-structure SPP on a semi-infinite metamaterial sheet. The governing integral equation (1) was solved explicitly in terms of a Fourier integral by use of the Wiener–Hopf method. The factorization process for the solution, u , was carried out exactly first, for reasonably arbitrary (scaled) sheet conductivity, $\tilde{\sigma}$. Then, we approximately evaluated the requisite split function, $Q_+(\xi)$, in the Fourier domain. Our approximation made use of the scale separation implied by conditions (2) for the conductivity of physically relevant sheets.

In our approach, we identified the SPP with a particular pole contribution to the Fourier representation of the diffracted field; the latter results by

subtraction from the solution of the incident and directly reflected fields. In addition, by removal of the SPP from the diffracted field, we derived approximations for the remaining, radiation field in certain regimes of kx .

Our approximate, analytical results were found to be in good agreement with a direct numerical computation based on the finite element method if $|\tilde{\sigma}|$ is sufficiently small; cf. (2). In particular, we compared a key integral formula and an ensuing approximate expression for the radiation field against the finite element, numerical computation. In this vein, we indicated that the radiation field is relatively small for a certain range of values of kx ($kx \geq 2$). Our approximate formula for the split function, $Q_+(\xi)$, is expected to break down if $|\tilde{\sigma}|$ becomes comparable to or larger than unity. In this case, one may resort to the numerical evaluation of the exact integral for the split function Q_+ .

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Appendix

Derivation of integral equation

In this appendix, we derive (1) from the appropriate boundary value problem for time-harmonic Maxwell's equations in the spirit of [19]. In 2D, we consider a conducting strip of length L , identified with the set $\Omega = \{(x, y) : 0 \leq x \leq L, y = 0\}$; ultimately, let $L \rightarrow \infty$.

For TM polarization, the magnetic field has only a component in the z -direction; thus, it can be fully described by a scalar function, $\Psi(x, y)$. The vector-valued electric field has components $E_x(x, y) = U(x, y)$ and $E_y(x, y) = V(x, y)$. The field (U, V, Ψ) is decomposed into the incident field, $(U^{\text{in}}, V^{\text{in}}, \Psi^{\text{in}})$, and the scattered field, $(U^{\text{s}}, V^{\text{s}}, \Psi^{\text{s}})$. The latter solves

$$\frac{\partial \Psi^{\text{s}}}{\partial y} = -i \frac{k^2}{\omega} U^{\text{s}}, \quad \frac{\partial \Psi^{\text{s}}}{\partial x} = i \frac{k^2}{\omega} V^{\text{s}}, \quad (\text{A.1})$$

$$\frac{\partial V^{\text{s}}}{\partial x} - \frac{\partial U^{\text{s}}}{\partial y} = i\omega \Psi^{\text{s}}; \quad (x, y) \in \mathbb{R}^2 \setminus \Omega. \quad (\text{A.2})$$

Thus, Ψ^s satisfies the Helmholtz equation outside Ω , namely,

$$\Delta_{x,y}\Psi^s + k^2\Psi^s = 0, \quad (x, y) \in \mathbb{R}^2 \setminus \Omega. \quad (\text{A.3})$$

In addition, Ψ^s obeys the following boundary conditions [1]:

$$\left[\frac{\partial \Psi^s}{\partial y} \right]_{\Omega} = \frac{k^2}{i\omega} [U^s]_{\Omega} = 0, \quad [\Psi^s]_{\Omega} = [\Psi]_{\Omega} = \mu\sigma U = \frac{i\omega\mu\sigma}{k^2} \frac{\partial \Psi}{\partial y} \quad \text{on } \Omega; \quad (\text{A.4})$$

and the Sommerfeld radiation condition as $\sqrt{x^2 + y^2} \rightarrow \infty$. Here, $\Psi = \Psi^s + \Psi^{\text{in}}$; $[Q]_{\Omega} = Q|_{y=0^+} - Q|_{y=0^-}$ denotes the jump of $Q(x, y)$ across Ω ($0 < x < L$); σ is the (constant) conductivity of the sheet; and μ is the magnetic permeability of the ambient medium. We also require that $\partial\Psi/\partial y$ be bounded at $x = 0, L$. The external source that produces $(U^{\text{in}}, V^{\text{in}}, \Psi^{\text{in}})$ lies outside Ω ; thus, the incident field is smooth in a neighborhood of Ω .

At this stage, we choose to proceed by converting the boundary value problem for Ψ^s into an integral equation for U . To this end, consider the Green function, $G(x', y'; x, y)$, for the Helmholtz equation on \mathbb{R}^2 defined via

$$(\Delta_{x,y} + k^2)G = -\delta(x - x')\delta(y - y'); \quad (x, y) \in \mathbb{R}^2, (x', y') \in \mathbb{R}^2 \setminus \Omega, \quad (\text{A.5})$$

and imposition of the Sommerfeld radiation condition at infinity. For fixed (x', y') not in Ω , this $G(x', y'; \cdot)$ is infinitely differentiable in a neighborhood of Ω . Specifically, note that [12]

$$G(x', y'; x, y) = \frac{i}{4} H_0^{(1)}(k\sqrt{(x - x')^2 + (y - y')^2}), \quad (\text{A.6})$$

which has the Fourier transform

$$\widehat{G}(\xi, y; 0, 0) = \int_{-\infty}^{\infty} G(x, y; 0, 0) e^{-i\xi x} dx = \frac{i}{2\sqrt{k^2 - \xi^2}} e^{i\sqrt{k^2 - \xi^2}|y|}, \quad (\text{A.7})$$

with $\text{Im}\sqrt{k^2 - \xi^2} > 0$. The last inequality is dictated by the Sommerfeld radiation condition, and defines the top Riemann sheet.

By a standard procedure of scattering theory [13, 19], a combination of (A.3) and (A.5) is suitably integrated on $\mathbb{R}^2 \setminus \Omega$. The application of the divergence theorem then yields the integral relation (with primed and unprimed coordinates interchanged)

$$\Psi^s(x, y) = \int_0^L \frac{\partial G(x, y; x', y')}{\partial y'} \Big|_{y'=0} [\Psi^s(x', y' = 0^+) - \Psi^s(x', y' = 0^-)] dx',$$

where $(x, y) \in \mathbb{R}^2 \setminus \Omega$ and use was made of the continuity of $\partial\Psi^s/\partial y$ across Ω . By $\partial G/\partial y' = -\partial G/\partial y$ and the second condition of (A.4), we have

$$\Psi^s(x, y) = -\mu\sigma \frac{\partial}{\partial y} \int_0^L G(x, y; x', 0) U(x', 0) dx', \quad (x, y) \notin \Omega.$$

Now differentiate both sides of the last equation with respect to y to obtain

$$U^s(x, y) = \frac{i\omega}{k^2} \frac{\partial}{\partial y} \Psi^s(x, y) = -\frac{i\omega\mu\sigma}{k^2} \frac{\partial^2}{\partial y^2} \int_0^L G(x, y; x', 0) U(x', 0) dx'.$$

By using the equation

$$(\Delta_{x,y} + k^2) \int_0^L G(x, y; x', 0) U(x', 0) dx' = 0, \quad (x, y) \in \mathbb{R}^2 \setminus \Omega,$$

and then allowing (x, y) to approach the strip, Ω , we find that U satisfies

$$U(x, 0) - U^{\text{in}}(x, 0) = \frac{i\omega\mu\sigma}{k^2} \left(\frac{\partial^2}{\partial x^2} + k^2 \right) \int_0^L G(x - x', 0; 0, 0) U(x', 0) dx',$$

where $0 < x < L$. Equation (1) is thus recovered as $L \rightarrow \infty$, with $\mathcal{K}(x) := G(x, 0; 0, 0)$ and $u(x) := U(x, 0)$.

Wiener–Hopf method: A review

In this appendix, we formally outline the core ideas of the Wiener–Hopf method in some correspondence to [16, 17]. Although the starting point is (A.8), the factorization method can treat integral equation (1) on the basis of generic functional equation (A.15).

Consider the second-kind Wiener–Hopf integral equation

$$u(x) - \lambda \int_0^\infty K(x - y) u(y) dy = f(x), \quad x > 0, \tag{A.8}$$

where $f(x)$ is given and $u(x)$ must be determined; and λ is a given, complex parameter.⁵ Suppose that $u(x)$ and $f(x)$ are integrable on $[0, \infty)$. If $K(x)$ is integrable in $(-\infty, \infty)$, it has a continuous, single-valued Fourier transform, $\widehat{K}(\xi)$, for all real ξ . More generally, let us write

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \widehat{K}(\xi) e^{i\xi x} d\xi,$$

assuming that $\widehat{K}(\xi)$ is holomorphic in a region of the ξ -plane that contains the real axis.

⁵In integral equation (1), this λ is replaced by the 1D Helmholtz operator. However, the main steps of the Wiener–Hopf method described here remain essentially intact.

Next, we extend the domain of (A.8) to the whole real axis; thus, define

$$u(x) \equiv 0 \quad \text{and} \quad f(x) \equiv 0 \quad \text{if } x < 0. \tag{A.9}$$

Accordingly, (A.8) reads

$$u(x) - \lambda \int_{-\infty}^{\infty} K(x - y)u(y) dy = f(x) + g(x), \quad x \in \mathbb{R}, \tag{A.10}$$

where

$$g(x) := \begin{cases} -\lambda \int_0^{\infty} K(x - y)u(y) dy, & x < 0, \\ 0, & x > 0. \end{cases} \tag{A.11}$$

Evidently, this g is unknown.

We define the Fourier transform, \widehat{u} , of u by

$$\widehat{u}(\xi) = \int_{-\infty}^{\infty} u(x) e^{-i\xi x} dx = \int_0^{\infty} u(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}. \tag{A.12}$$

By our assumption for $u(x)$, especially its vanishing for $x < 0$, this $\widehat{u}(\xi)$ can be viewed as the analytic continuation to the real axis of a holomorphic function in the lower half-plane, $\mathbb{C}_- = \{\xi \in \mathbb{C} : \text{Im}\xi < 0\}$. We write

$$u(x) = \int_{\Gamma} \widehat{u}(\xi) e^{i\xi x} \frac{d\xi}{2\pi}; \quad \Gamma = \{\xi \in \mathbb{C} : \text{Im}\xi = \text{const.} < 0\}. \tag{A.13}$$

In a similar vein, the Fourier transform, $\widehat{f}(\xi)$, of f is holomorphic in \mathbb{C}_- . In regard to g , define

$$\widehat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-i\xi x} dx = \int_{-\infty}^0 g(x) e^{-i\xi x} dx. \tag{A.14}$$

If $g(x)$ is integrable in $(-\infty, 0]$ then (A.14) is defined in the upper half-plane, $\mathbb{C}_+ = \{\xi \in \mathbb{C} : \text{Im}\xi > 0\}$, where $\widehat{g}(\xi)$ is holomorphic; cf. (A.11). We analytically continue $\widehat{g}(\xi)$ from \mathbb{C}_+ to the real axis.

The Fourier transformation of (A.10) yields the functional equation

$$[1 - \lambda \widehat{K}(\xi)] \widehat{u}(\xi) = \widehat{f}(\xi) + \widehat{g}(\xi), \tag{A.15}$$

where $\xi \in \mathbb{R} \setminus \{\xi \in \mathbb{R} : \widehat{f}(\xi) = \infty \text{ or } \widehat{g}(\xi) = \infty\}$. We assume that λ is such that

$$1 - \lambda \widehat{K}(\xi) \neq 0 \quad \text{for all real } \xi;$$

also, we do not allow real limit points of complex zeros of $1 - \lambda \widehat{K}(\xi)$. Although we derived (A.15) from (A.8), it is important to consider (A.15) as a functional equation that may be a starting point in its own right. Notably, (A.15) can result directly from a class of linear boundary value problems [11].

Next, we focus on determining $\widehat{u}(\xi)$ from (A.15). The primary task is to explicitly separate (A.15) into two parts, one of which is holomorphic in \mathbb{C}_+ and another that is holomorphic in \mathbb{C}_- . For this purpose, define [16]

$$\widehat{Q}(\xi) = \ln[1 - \lambda\widehat{K}(\xi)]. \tag{A.16}$$

This $\widehat{Q}(\xi)$ is single valued and holomorphic in a region of the complex plane that includes the real axis. We need to find $\widehat{Q}_\pm(\xi)$ such that

$$\widehat{Q}(\xi) = \widehat{Q}_+(\xi) + \widehat{Q}_-(\xi), \quad \xi \in \mathbb{R}, \tag{A.17}$$

where $\widehat{Q}_s(\xi)$ is holomorphic in \mathbb{C}_s ($s = \pm$). We refer to each $Q_\pm(\xi)$ as a “ \pm ” split function.

We slightly digress to discuss particulars of (A.16) and introduce the notion of the *index* from Krein’s theory [16]. Bearing in mind the role of K in \widehat{Q} above, we expect that a continuous and single-valued branch of $\widehat{Q}(\xi)$ can be chosen such that $\arg[1 - \lambda\widehat{K}(\xi)] - \arg[1 - \lambda\widehat{K}(-\xi)] \rightarrow 0$ as $\xi \rightarrow +\infty$. This amounts to a zero *index*, $\text{ind}(1 - \lambda\widehat{K}) = 0$, by the following definition [16].

DEFINITION 2 (Index of functional equation). *The index of (A.15) is*

$$\text{ind}(1 - \lambda\widehat{K}) := \frac{1}{2\pi} \left[\arg(1 - \lambda\widehat{K}(\xi)) \right]_{\xi=-\infty}^{\infty}. \tag{A.18}$$

We mention in passing a rigorous result from [16] in relation to this index. Let \mathcal{E} denote a functional space from a family of spaces that includes, e.g., any space $L^p(0, \infty)$ with $p \geq 1$ and the space of all bounded continuous functions on $(0, \infty)$ [16]. Note the following excerpt of a theorem in [16].

THEOREM 1. *Let $K(x)$ be integrable in $(-\infty, \infty)$. Then, (A.8) has exactly one solution $u \in \mathcal{E}$ for an arbitrary $f \in \mathcal{E}$ if and only if the conditions*

$$1 - \lambda\widehat{K}(\xi) \neq 0 \quad (\xi \in \mathbb{R}) \quad \text{and} \quad \text{ind}(1 - \lambda\widehat{K}) = 0 \tag{A.19}$$

are satisfied.

Henceforth, we assume that (A.19) holds. We return to (A.17). To determine Q_s ($s = \pm$), consider the contour $\Gamma = \Gamma_+ \cup \Gamma_- \cup C_r \cup C_l$, as shown in Fig. A1. Let $\xi_* \in \mathbb{R}$ be an arbitrary point enclosed by Γ . By the Cauchy integral formula and the property $\text{ind}(1 - \lambda\widehat{K}) = 0$, we write

$$\begin{aligned} \widehat{Q}(\xi_*) &= \frac{1}{2\pi i} \lim_{M \rightarrow +\infty} \left(\int_{\Gamma_+(M)} + \int_{\Gamma_-(M)} + \int_{C_l(M)} + \int_{C_r(M)} \right) \frac{\widehat{Q}(\xi)}{\xi - \xi_*} d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma_+(\infty)} \frac{\widehat{Q}(\xi)}{\xi - \xi_*} d\xi + \left(-\frac{1}{2\pi i} \int_{-\Gamma_-(\infty)} \frac{\widehat{Q}(\xi)}{\xi - \xi_*} d\xi \right) \end{aligned}$$

$$+ \frac{1}{2\pi} \lim_{M \rightarrow \infty} \int_{-c_1}^{c_1} \left(\frac{\widehat{Q}(\xi)}{\xi - \xi_*} - \frac{\widehat{Q}(-\xi)}{-\xi - \xi_*} \right) \Big|_{\xi=M+i\varrho} d\varrho, \quad (\text{A.20})$$

where the integrals in the second line are presumed convergent; cf. (A.21). In regard to the third line of (A.20), we assert that the integral approaches zero as $M \rightarrow \infty$ if the growth of $\widehat{K}(\xi)$ is not faster than polynomial ($\xi = M + i\varrho$, $|\varrho| \leq c_1$). Now consider separately each of the integrals in the second line of (A.20) as a function of ξ_* ; and let ξ_* be moved slightly off the real axis, to \mathbb{C}_+ or \mathbb{C}_- . Each of these integrals can then be carried out on the real axis, with the appropriate indentation of the respective path around ξ_* , e.g., by use of a small circular arc. By typical analytic continuation, the sum of the resulting integrals furnishes $\widehat{Q}(\xi_*)$ for all complex ξ_* [17]. We write

$$\int_{\Gamma_s(\infty)} d\xi(\cdot) = \lim_{M \rightarrow +\infty} \int_{-M}^M d\xi(\cdot), \quad \text{if } s \operatorname{Im}\xi_* > 0.$$

Hence, we deduce that

$$\begin{aligned} \widehat{Q}_s(\xi_*) &= s \frac{1}{2\pi i} \lim_{M \rightarrow \infty} \int_{-M}^M d\xi \frac{\widehat{Q}(\xi)}{\xi - \xi_*} \quad (s = \pm), \\ &= s \frac{1}{2\pi i} \lim_{M \rightarrow \infty} \int_0^M \left[\frac{\widehat{Q}(\xi)}{\xi - \xi_*} - \frac{\widehat{Q}(-\xi)}{\xi + \xi_*} \right] d\xi, \end{aligned} \quad (\text{A.21})$$

where $s \operatorname{Im}\xi_* > 0$. Each of these two integrals ($s = \pm$) converges if $\widehat{K}(\xi)$ has at most polynomial growth. In fact, the condition $\operatorname{ind}(1 - \lambda\widehat{K}) = 0$ guarantees that $\widehat{Q}(\xi) - \widehat{Q}(-\xi)$ approaches 0 as $\xi \rightarrow +\infty$.

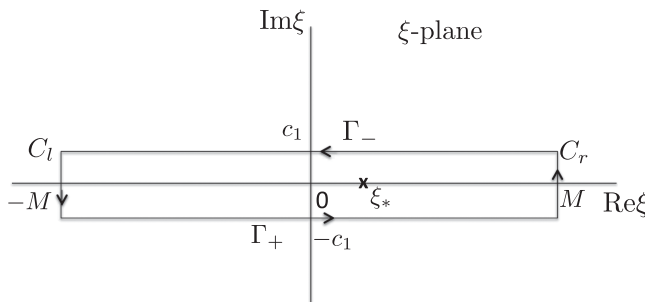


Figure A1. Integration part $\Gamma = \Gamma_+ \cup \Gamma_- \cup C_l \cup C_r$, for $Q_s(\xi)$ ($s = \pm$). The arrows indicate sense of circulation. The oriented path $\Gamma_s = \{\xi \in \mathbb{C} : \operatorname{Im}\xi = -s c_1, -M < \operatorname{Re}\xi < M\}$, for small c_1 , lies in \mathbb{C}_s ($s = \pm$). The path $C_p = \{\xi \in \mathbb{C} : \operatorname{Re}\xi = \mp M, |\operatorname{Im}\xi| < c_1\}$ ($p = l, r$) is a small segment connecting Γ_+ and Γ_- on the left ($p = l$) or right ($p = r$) of the imaginary axis; eventually, let $M \rightarrow +\infty$.

Accordingly, (A.15) is recast into

$$e^{\widehat{Q}_-(\xi)} \widehat{u}(\xi) = e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) + e^{-\widehat{Q}_+(\xi)} \widehat{g}(\xi). \tag{A.22}$$

The left-hand side is holomorphic in \mathbb{C}_- , while the second term on the right-hand side is holomorphic in \mathbb{C}_+ . By further decomposing $e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi)$ as

$$e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) = \left[e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) \right]_+ + \left[e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) \right]_- ,$$

we derive the expression

$$e^{\widehat{Q}_-(\xi)} \widehat{u}(\xi) - \left[e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) \right]_- = \left[e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) \right]_+ + e^{-\widehat{Q}_+(\xi)} \widehat{g}(\xi), \tag{A.23}$$

where

$$\left[e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) \right]_s = s \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\zeta \frac{e^{-\widehat{Q}_+(\zeta)} \widehat{f}(\zeta)}{\zeta - \xi} \quad (s = \pm), \tag{A.24}$$

with $s \operatorname{Im} \xi > 0$, and under the assumption that these integrals converge.⁶

In (A.23), the left-hand side defines a function holomorphic in \mathbb{C}_- while the right-hand side is holomorphic in \mathbb{C}_+ ; these two functions are equal on the real axis. Hence, these functions together define an *entire* function, $E(\xi)$. In particular, we have

$$e^{\widehat{Q}_-(\xi)} \widehat{u}(\xi) - \left[e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) \right]_- = E(\xi), \quad \xi \in \mathbb{C}_-, \tag{A.25}$$

and

$$\left[e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) \right]_+ + e^{-\widehat{Q}_+(\xi)} \widehat{g}(\xi) = E(\xi), \quad \xi \in \mathbb{C}_+. \tag{A.26}$$

Thus, we conclude that

$$\widehat{u}(\xi) = e^{-\widehat{Q}_-(\xi)} \left\{ \left[e^{-\widehat{Q}_+(\xi)} \widehat{f}(\xi) \right]_- + E(\xi) \right\}, \quad \xi \in \mathbb{C}_-. \tag{A.27}$$

The last formula can be Fourier-inverted to yield $u(x)$ according to (A.13).

The function $E(\xi)$ can in principle be determined by its behavior in \mathbb{C}_\pm as $\xi \rightarrow \infty$. This behavior may be extracted from (A.25) and (A.26). If the left-hand sides of these equations approach 0 as $\xi \rightarrow \infty$ in \mathbb{C}_- and \mathbb{C}_+ , then, by Liouville’s theorem, $E(\xi) \equiv 0$. This result is consistent with the existence of a unique solution u in the case of Theorem 1.

⁶If $\widehat{f}(\xi)$ has singularities on the real axis, the path of integration in (A.24) is indented below ($s = -$) or above ($s = +$) these singularities, with the point ξ keeping its position *relative* to the path.

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