1. (a) [5 pts] Solve exactly the integral equation

\[ u(x) = 1 + \int_0^1 (1 + x + y + xy)^\nu u(y) \, dy, \quad 0 \leq x \leq 1, \quad \nu : \text{real}. \]

(b) [5 pts] Solve exactly the nonlinear equation \( u(x) - \lambda \int_0^1 dy \, u^2(y) = 1 \). In particular, identify the bifurcation points of this equation. What kind of point is \( \lambda = 0 \)? What are the non-trivial solutions of the corresponding homogeneous equation?

2. (a) [5 pts] Find a Green’s function for the time-dependent, one-dimensional Schrödinger equation

\[ i\Psi_t(x,t) + \Psi_{xx}(x,t) = V(x,t) \Psi(x,t) \]

with the initial condition \( \Psi(x,0) = a(x) \).

(b) [5 pts] Express the above initial-value problem in terms of an integral equation. How would you proceed to solve this equation approximately? Explain.

3. (10 pts) Consider the (general) linear Volterra integral equation of second kind,

\[ u(x) = f(x) + \lambda \int_0^x K(x,y) u(y) \, dy, \quad 0 \leq x \leq a, \]

and assume that \( f, K \) and \( u \) are square integrable, and \( \lambda \) is real. Prove that a solution \( u \) exists for all \( \lambda \), and that this \( u \) is unique. In particular, show how one can construct the \( u \), given \( \lambda \).

4. (10 pts) Solve exactly the linear Volterra equation of the first kind

\[ f(x) = \int_0^x dy \, \ln(x-y) u(y); \quad f(0) = 0. \]

**Hint:** \( \int_0^\infty dx \, e^{-x} \ln x = -\gamma \), where \( \gamma = 0.5772156649 \ldots \) (Euler’s constant).

5. (10 pts) Antennas fed by transmission lines are often modeled by tubular dipoles with a current \( I(x) \) that satisfies the Hallén integral equation:

\[ \int_{-h}^{h} dy \, K(x-y) I(y) = A \sin(k|x|) + C \cos(kx), \quad |x| < h, \]
where $A$ and $C$ are constants, $h$ is the length of the dipole and $k > 0$ is proportional to frequency. The kernel $K(x)$ is a known yet very complicated function. In order to apply numerical methods to this equation, the exact kernel $K$ is sometimes replaced by the simpler (approximate) kernel

$$K_{ap}(x) = \frac{1}{4\pi} \frac{e^{ik \sqrt{x^2 + a^2}}}{\sqrt{x^2 + a^2}},$$

where $a$ is the radius of the dipole tube.

Develop an argument to show that, with this approximate kernel, the corresponding integral equation for $I(x)$ has no solution. **Hint:** You should use your ingenuity and some elementary calculus. A correct answer to this problem need not rely on prior knowledge of any theory of integral equations.