6. [10pts] Consider the kernel $K$ of a 2nd-kind Fredholm equation, which is given by

$$K(x, y) = \begin{cases} 
3, & 0 \leq y < x \leq 1, \\
2, & 0 \leq x < y \leq 1.
\end{cases}$$

(a) **Find** the kernel eigenfunctions $u_n$ and corresponding eigenvalues $\lambda_n$.
(b) **Is** $K$ symmetric? **Determine** the transposed kernel $K^T$, and **find** its eigenfunctions $v_n$ with corresponding eigenvalues $\lambda_n$.
(c) **Show by an explicit calculation** that any $u_n$ is orthogonal to any $v_m$ if $m \neq n$.

7. [10pts] By using the Hilbert-Schmidt formula show that, if $\lambda$ is an eigenvalue of the symmetric $L^2$-kernel $K(x, y)$, then the integral equation

$$u(x) = f(x) + \lambda \int_a^b dy K(x, y) u(y), \quad a \leq x \leq b,$$

has no solution, unless $f(x)$ is orthogonal to all the eigenfunctions corresponding to $\lambda$.

8. (a)[8pts] Consider the integral equation for the scattering of a non-relativistic electron by a potential,

$$\psi(x) = e^{ikx} + \int_{-\infty}^{\infty} dy \frac{e^{ik|x-y|}}{2ik} V(y) \psi(y), \quad -\infty < x < \infty; \quad k > 0.$$

Symmetrize the kernel and find the first 2 terms of the Taylor series for the functions $D(\lambda)$ and $N(x, y; \lambda)$ defined in class. The ratio of these two series yields the so-called “improved Born series” for $\psi$. **Calculate** $\psi$ accordingly. **Note**: State assumptions on $k$, $V(x)$ and $\psi(x)$.

(b)[7pts] In the theoretical search for “supergain antennas,” maximizing the directivity in the far field of axially invariant currents $j = j(\phi)$ that flow along the surface of infinitely long, circular cylinders of radius $a$ leads to the following equation for the (unknown) density $j$:

$$j(\phi) = e^{ika} \sin \phi - \alpha \int_0^{2\pi} \frac{d\phi'}{2\pi} J_0 \left(2ka \sin \frac{\phi - \phi'}{2}\right) j(\phi'), \quad 0 \leq \phi < 2\pi;$$

$\phi$ is the polar angle of the circular cross section, $k$ is a positive constant proportional to frequency, $\alpha$ is a parameter (Lagrange multiplier) that expresses a constraint on the current magnitude, $\alpha \geq 0$, and $J_0(x)$ is the Bessel function of zeroth order. **Determine** the eigenvalues of the homogeneous equation. Then, **solve** the given inhomogeneous equation in terms of Fourier series. **Hint**: You may use the integral formula $J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} d\phi' e^{inx} \sin \phi' e^{-in\phi'},$ $n$: integer and $J_n$: Bessel function of $n$th order.