Tails of edge scaling limits of Hermite matrix ensembles

Dionisios Margetis, Brian Sutton, and Alan Edelman

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139
(Dated: April 7, 2005)

Asymptotic formulas are derived for the distribution functions of the largest eigenvalue of infinite-dimensional β-Hermite random matrices with β = 1, 2, and 4, which correspond to Dyson’s “three-fold way”. For this purpose use is made of formulas by Tracy and Widom that relate the cumulative distribution functions (cdf) to a particular second Painlevé transcendent.

I. INTRODUCTION

Random matrices arise in a variety of problems in physics and statistics, and their properties have been subject of intensive studies. In particular, the Gaussian or Hermite, ensembles of random matrices were studied by Dyson in connection with the quantum mechanical description of atomic spectra. Dyson identified the ensembles with three types of division algebra: real numbers, complex numbers, and quaternions.

The distribution functions of the largest eigenvalue of the random-matrix models in the limit of infinite dimension arise from the edge scaling limits of their ensembles. In this paper we derive asymptotic formulas for the tails of such distributions for β-Hermite matrix models having β = 1, 2, 4. We show analytically that the corresponding probability density functions (pdf), \( f_\beta(s) \), have highly asymmetric tails described by

\[
\begin{align*}
    f_1(s) &\sim \left\{ \begin{array}{ll}
    \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-\frac{3}{8} s^{3/2}} & , \quad s \to +\infty, \\
    (-s)^{-1/16} \left[ \frac{s^2}{4} + \frac{1}{3} (-2s)^{1/2} + \frac{8}{3} \right] \exp \left[ \frac{s^3}{32} - \frac{\sqrt{2}}{6} (-s)^{3/2} + \frac{8}{3} - \frac{1}{3} (A + C) \right] & , \quad s \to -\infty,
\end{array} \right. \\
    f_2(s) &\sim \left\{ \begin{array}{ll}
    \frac{1}{2\sqrt{\pi}} e^{-\frac{3}{8} s^{3/2}} & , \quad s \to +\infty, \\
    (-s)^{-1/8} \left( \frac{s^2}{4} + B \right) \exp \left( \frac{s^3}{72} + Bs - C \right) & , \quad s \to -\infty,
\end{array} \right. \\
    f_4(s) &\sim \left\{ \begin{array}{ll}
    \frac{1}{128\pi} s^{-5/2} e^{-\frac{3}{8} s^{3/2}} & , \quad s \to +\infty, \\
    (-2^{50/3} s)^{-1/16} \left[ \frac{s^2}{4} - (-2s)^{1/2} + B 2^{-1/3} \right] \exp \left[ \frac{s^3}{32} + \frac{1}{3} (2s)^{3/2} + B 2^{-1/3} s + A - \frac{1}{2} C \right] & , \quad s \to -\infty.
\end{array} \right.
\end{align*}
\]

The constants \( A, B \) and \( C \) are defined by

\[
\begin{align*}
    A &= \int_{-\infty}^{\infty} dx \left[ q(x) - \theta(-x) \sqrt{-x/2} \right] \approx 0.35,
\end{align*}
\]

\[
\begin{align*}
    B &= \int_{-\infty}^{\infty} dx \left[ q(x)^2 + \frac{x}{2} \theta(-x) \right] \approx 0,
\end{align*}
\]

\[
\begin{align*}
    C &= \frac{1}{8} + \int_{-\infty}^{\infty} dx \left[ xq(x)^2 - \theta(-x) \left( \frac{x^2}{2} - \frac{1}{8} \frac{1}{1+|x|} \right) \right] \approx 0.14,
\end{align*}
\]

where \( q(x) \) is a particular second Painlevé transcendent (PII), to be defined below, and \( \theta(x) \) is the unit step function, \( \theta(x) = 1 \) for \( x > 0 \) and \( \theta(x) = 0 \) for \( x < 0 \).

The paper is organized as follows. In Sec. II we derive asymptotic formulas for the cumulative distribution functions (cdf), \( F_\beta(s) \), of the classical β-Hermite ensembles (β = 2, 1 and 4) in the limit \( s \to \pm \infty \), using known, exact expressions in terms of the particular PII entering (4)-(6), \( q(x) \). Our calculation relies on the property that \( q(x) \) decays exponentially for large positive \( x \) but grows as \( \sqrt{-x/2} \) for large negative \( x \). In Sec. III we discuss aspects and possible extensions of our results.
II. ASYMPTOTIC FORMULAS FOR TAILS

We start with exact formulas obtained by Tracy and Widom\(^{8,9}\) for the cdf \(F_\beta(s)\) where \(\beta = 2, 1, 4,\)

\[
F_2(s) = \exp \left[ - \int_s^\infty (x-s)q(x)^2 \, dx \right],
\]

\[
F_1(s) = \sqrt{F_2(s)} \exp \left[ -\frac{1}{2} \int_s^\infty q(x) \, dx \right],
\]

\[
F_4 \left( \frac{s}{2^{2/3}} \right) = \frac{1}{2} \sqrt{F_2(s)} \left\{ \exp \left[ \frac{1}{2} \int_s^\infty dx \, q(x) \right] + \exp \left[ -\frac{1}{2} \int_s^\infty dx \, q(x) \right] \right\},
\]

where \(q(x)\) satisfies the ordinary differential equation\(^{6,7}\)

\[
q'' = qx + 2q^3,
\]

supplemented with the asymptotic formula

\[
q(x) \sim \text{Ai}(x), \quad x \to +\infty;
\]

\text{Ai}(x) is the well-known Airy function\(^{10}\).

Equations (7)–(9) are conveniently recast into the form

\[
F_2(s) = \exp[-I_2(s)],
\]

\[
F_1(s) = \sqrt{F_2(s)} \exp \left[ -\frac{1}{2} I_1(s) \right],
\]

\[
F_4(2^{-2/3}s) = \sqrt{F_2(s)} \cosh \left[ \frac{1}{2} I_1(s) \right],
\]

where

\[
I_1(s) = \int_s^\infty dx \, q(x),
\]

\[
I_2(s) = \int_s^\infty dx \, (x-s)q(x)^2.
\]

Thus, the task is to evaluate \(I_1(s)\) and \(I_2(s)\) in the limits \(s \to \pm \infty\). In order to study the limit \(s \to -\infty\) we need to use a property of \(q(x)\) that stems from (11), specifically from the fact that the coefficient multiplying Ai(x) is unity\(^{11}\):

\[
q(x) \sim \sqrt{-\frac{x}{2}}, \quad x \to -\infty.
\]

A more detailed, explicit calculation yields\(^{9,12}\)

\[
Q(x) \equiv q(-x) = \sqrt{\frac{x}{2}} - \frac{1}{8\sqrt{2}} x^{-5/2} - \frac{73}{128\sqrt{2}} x^{-11/2} + O(x^{-17/2}), \quad x \to +\infty.
\]
A. Integral $I_1(s)$

First, we derive an asymptotic formula for $I_1(s)$ for large positive $s$, $s \to +\infty$, starting from (15) and using (11). With the change of variable

$$1 + t = \left( \frac{2}{s} \right)^{3/2},$$

(19) reads

$$I_1(s) = \frac{2s}{3} \int_0^\infty dt (1 + t)^{-1/3} q[s(1 + t)^2/3] \sim \frac{2s}{3} \int_0^\infty dt (1 + t)^{-1/3} \text{Ai}(s(1 + t)^2/3)$$

$$\sim \frac{s^{3/4}}{3\sqrt{\pi}} \int_0^\infty dt (1 + t)^{-1/2} e^{-\frac{2}{3}s^{3/2}(1 + t)} \left[ 1 - \frac{5}{48} s^{-3/2}(1 + t)^{-1} \right]$$

$$\sim \frac{e^{-\frac{2}{3}s^{3/2}}}{3\sqrt{\pi}} \left( \frac{3}{2} s^{-3/4} - \frac{41}{32} s^{-9/4} \right), \quad s \to +\infty,$$

(20)

where we used the formula\(^\text{10}\)

$$\text{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3}s^{3/2} z^{-1/4}} \left( 1 - \frac{5}{48} z^{-3/2} \right), \quad z \to +\infty,$$

(21)

and treated the resulting expression for $I_1(s)$ as a Laplace integral\(^\text{13}\).

To evaluate $I_1(s)$ asymptotically for large negative $s$, $s \to -\infty$, we replace (15) by

$$I_1(s) = \int_{\varphi}^{\frac{\pi}{2}} dx \, Q(x) + \int_{-\frac{\pi}{2}}^{\varphi} dx \, q(x), \quad Q(x) = q(-x),$$

(22)

where $\varphi$ is arbitrary and positive ($\varphi > 0$). The first integral is evaluated using expansion (18),

$$\int_{\varphi}^{\frac{\pi}{2}} dx \, Q(x) = \left( \int_{\varphi}^{\infty} - \int_{-\infty}^{-\varphi} \right) dx \left[ Q(x) - \sqrt{x/2} \right] \frac{\sqrt{\varphi}^{3/2} + \sqrt{2} |s|^{3/2}}{3}$$

$$= \int_{\varphi}^{\infty} dx \left[ Q(x) - \sqrt{x/2} \right] - \frac{\sqrt{2}}{3} \varphi^{3/2} + \frac{\sqrt{2}}{24} |s|^{-3/2} + \frac{73}{1152} |s|^{-9/2} + O(|s|^{-15/2})$$

$$= \int_{0}^{\infty} dx \left[ Q(x) - \sqrt{x/2} \right] + \frac{\sqrt{2}}{3} |s|^{3/2} + \frac{\sqrt{2}}{24} |s|^{-3/2} + \frac{73}{1152} |s|^{-9/2} + O(|s|^{-15/2}).$$

(23)

In the last line we allowed $\varphi \to 0^+$ for definiteness. The combination of (22) and (23) yields

$$I_1(s) \sim \frac{\sqrt{2}}{3} (-s)^{3/2} + A + \frac{\sqrt{2}}{24} (-s)^{-3/2} + \frac{73}{1152} (-s)^{-9/2}, \quad s \to -\infty,$$

(24)

where the constant $A$ is defined by the (absolutely convergent) integral of (4), and $\theta(x)$ is the unit step function. A numerical integration by MATLAB yields $A \approx 0.35$.

B. Integral $I_2(s)$

Next we focus on the integral $I_2(s)$ of (16). With the change of variable (19) and by virtue of (11) and (21) we have

$$I_2(s) = \frac{2s^2}{3} \int_0^\infty dt \, [(1 + t)^{1/3} - (1 + t)^{-1/3}] \left[ q(s(1 + t)^2/3) \right]^2$$

$$\sim \frac{s^{3/2}}{6\pi} e^{-\frac{2}{3}s^{3/2}} \int_0^\infty dt \, [1 - (1 + t)^{-2/3}] e^{-\frac{2}{3}s^{3/2} t} \left[ 1 - \frac{5}{24} s^{-3/2}(1 + t)^{-1} \right]$$

$$\sim \frac{1}{6\pi} e^{-\frac{2}{3}s^{3/2}} \left( \frac{3}{8} s^{-3/2} - \frac{35}{64} s^{-3} \right), \quad s \to +\infty.$$
In order to evaluate \( I_2(s) \) for large negative \( s \), we replace \( I_2(s) \) by the expression
\[
I_2(s) = -s I_{21}(s) + I_{22}(s); \quad I_{21}(s) = \int_{-\infty}^{\infty} dx \, q(x)^2, \quad I_{22}(s) = \int_{-\infty}^{\infty} dx \, x q(x)^2,
\]
and treat the integrals \( I_{2j} \) separately. Accordingly,
\[
I_{21}(s) = \int_{-\varphi}^{\varphi} dx \, Q(x)^2 + \int_{-\varphi}^{\infty} dx \, x q(x)^2,
\]
whereby, by using (18) and following the steps of Sec. II A, we obtain
\[
I_{21}(s) = \int_{-\varphi}^{\varphi} dx \left[ Q(x)^2 - \frac{x^2}{2} \right] - \frac{\varphi^2}{4} + \frac{1}{8} \ln(1 + \varphi) + \frac{1}{6} \varphi^3 - \frac{9}{64} s^{-4} + \int_{-\varphi}^{\infty} dx \, x q(x)^2 + O(|s|^{-7})
\]
\[
= \frac{s^2}{4} + B - \frac{1}{8}s + \frac{9}{64} s^{-4} + O(s^{-7}), \quad s \to -\infty,
\]
where the constant \( B \) is described by the integral of (5). Numerical integration by MATLAB indicates that \( |B| \) is likely less than \( 10^{-6} \). Therefore, it is tempting to take \( B \approx 0 \) for all practical purposes. We have not been able to show analytically that \( B = 0 \).

By following a procedure similar to that for \( I_{21}(s) \) we obtain
\[
I_{22}(s) = -\int_{-\varphi}^{\varphi} dx \, xQ(x)^2 + \int_{-\varphi}^{\infty} dx \, xq(x)^2
\]
\[
= -\int_{-\varphi}^{\infty} dx \left[ xQ(x)^2 - \frac{x^2}{2} + \frac{1}{8 s} \right] - \frac{1}{8} \ln(1 + \varphi) + \frac{3}{16} \frac{1}{s^3} + \frac{3}{16 s^3} + O(s^{-6}), \quad s \to -\infty,
\]
where
\[
C = \int_{-\infty}^{\infty} dx \left[ xq(x)^2 - \left( \frac{x^2}{2} - \frac{1}{8 s} \right) \theta(-x) \right].
\]
An asymptotic formula for \( I_2(s) \) is obtained via adding \(-s I_{21}\) and \( I_{22} \),
\[
I_2(s) \sim \frac{s^3}{12} - Bs + C + \frac{1}{8} \ln(-s) + \frac{3}{16 s^3} - \frac{9}{64 s^4} + O(s^{-6}), \quad s \to -\infty,
\]
where \( C = C + \frac{1}{8} \) is introduced by (6).

**C. Distribution functions for \( \beta = 2, 1, 4 \)**

1. \( \beta = 2 \)

Next we derive asymptotic formulas for the cdf \( F_\beta(s) \) of (12), (13) and (14), and hence for their corresponding pdf, \( f_\beta(s) = \frac{d}{ds} F_\beta \). First, we evaluate \( F_2(s) \) to the leading order for large positive \( s \) by use of (12) and (25),
\[
F_2(s) \sim \exp \left( -\frac{1}{16 \pi} s^{-3/2} e^{-\frac{1}{4} s^{3/2}} \right) \sim 1 - \frac{1}{16 \pi} s^{-3/2} e^{-\frac{1}{4} s^{3/2}}, \quad s \to +\infty.
\]
For large negative \( s \), use of (26), (28) and (29) yields
\[
F_2(s) \sim (-s)^{-1/8} \exp \left( \frac{s^3}{12} + Bs - C \right), \quad s \to -\infty.
\]
An asymptotic formula for the pdf \( f_2(s) \) is thus obtained by differentiation of (32) and (33); see (2) of Sec. I.
2. $\beta = 1$

The cdf $F_1(s)$ is obtained from (13), (20) and (24),

$$F_1(s) \sim 1 - \frac{1}{4\sqrt{\pi}} s^{-3/4} e^{-\frac{8}{3} s^{3/2}}, \quad s \to +\infty,$$  \hspace{1cm} (34)

$$F_1(s) \sim (-s)^{-1/16} \exp \left[ \frac{s^3}{24} - \frac{\sqrt{2}}{6} (-s)^{3/2} + \frac{B}{2} s - \frac{1}{2} (A + C) \right], \quad s \to -\infty,$$  \hspace{1cm} (35)

where $A$ is defined by (4). It follows that the pdf $f_1(s) = \frac{d}{ds} F_1(s)$ is given by (1).

3. $\beta = 4$

From (14) with (12), (20) and (25) we obtain

$$F_4(s) \sim 1 - \frac{1}{512\pi s^3} e^{-\frac{4}{3} s^{3/2}}, \quad s \to +\infty.$$  \hspace{1cm} (36)

Use of (24), (26), (28) and (29) yields

$$F_4(s) \sim 2^{-25/24} (-s)^{-1/10} \exp \left[ \frac{s^3}{6} + \frac{2\sqrt{2}}{3} (-s)^{3/2} + \frac{B}{2} s^{1/3} + A - \frac{1}{2} C \right], \quad s \to -\infty.$$  \hspace{1cm} (37)

Differentiation of $F_4(s)$ to obtain $f_4(s)$ leads to (3).

III. DISCUSSION

In this paper we derived asymptotic formulas for the tails of distributions arising from the edge scaling limits of classical Gaussian matrix ensembles, namely, the Gaussian orthogonal (GOE, $\beta = 1$), Gaussian unitary (GUE, $\beta = 2$), and Gaussian symplectic (GSE, $\beta = 4$) via known formulas for the cdf in terms of a particular PII, $q(x)$. Our analytical approach was motivated by difficulties in the accurate numerical computation of the corresponding distributions for large values of their variable, due to the exceptional behavior of $q(x)$ under condition (11) for large negative $x^{11}$.

Our asymptotic formulas for the cdf and pdf $F_\beta(s)$ and $f_\beta(s)$ for $s \to -\infty$ involve three constants, $A$, $B$ and $C$ defined by (4), (5) and (6), which are given in terms of integrals involving $q(x)$ over the entire real axis. A numerical computation by MATLAB has produced the values $A \approx 0.35$, $C \approx 0.14$, while $B \approx 0$ for all practical purposes. More accurate values for these constants have not been possible by our current computations.

Finally, it is perhaps interesting to extend our calculations to the bulk scaling limits of GOE, GUE and GSE using other known formulas in terms of the fifth Painlevé transcendents$^2$.

Acknowledgments

We thank Professors Hung Cheng and Percy Deift for useful discussions.

---

13 Erdélyi, A., Asymptotic Expansions (Dover, New York, 1956), Chap. II.