

Solving systems of linear equations

(Following Kincaid & Ch.)

Topics

1. The LU and Cholesky factorizations.
2. Pivoting
3. Norms and the analysis of errors.

1. The LU & Cholesky factorizations

A system of n equations in n unknowns: $Ax=b$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Examples of easy to solve systems:

1) Diagonal $\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

\Rightarrow n equations \Rightarrow decoupled $x = \begin{pmatrix} b_1/a_{11} \\ \vdots \\ b_n/a_{nn} \end{pmatrix}$ assuming $a_{ii} \neq 0$.

2) A is lower triangular

$$A = \begin{pmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \dots & & a_{nn} \end{pmatrix}$$

assume $a_{ii} \neq 0$.

Solution: Solve for x_1 from eq. 1.
 Solve for $x_2 \dots \dots 2$ given x_1
 etc.

\Rightarrow We can obtain the x_i 's one by one.

This is called Solution by forward substitution.

$$\left(\begin{array}{l} x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j}{a_{ii}} \\ i=2 \rightarrow n \end{array} \right)$$

3) A is upper triangular

$$\left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ & \ddots & & \vdots \\ & & & a_{nn} \end{array} \right)$$

$a_{ii} \neq 0.$

Solution for $x_n \rightarrow x_{n-1} \rightarrow \dots \rightarrow x_1.$

Back substitution

$$x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij} x_j}{a_{ii}}$$

4) Permuting the equations to get an upper or lower triangular matrix.

Example

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Rightarrow \begin{pmatrix} a_{31} & a_{12} & 0 \\ a_{11} & a_{22} & a_{23} \\ a_{21} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_3 \\ b_1 \\ b_2 \end{pmatrix}$$

LU factorization

Suppose A can be factored into a product:

$$A = LU$$

lower triangular upper triangular

Solving $Ax = b$ then means $LUx = b$

So the solution can be obtained in 2

Single stages: Solve for z : $Lz = b$.

Solve for x : $Ux = z$.

- A form $A = LU$ is called
LU-decomposition

- Not always possible

- When exists: not unique.

- Deriving an algorithm for the LU decomposition:

Let

$$L = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & & \ddots & \\ l_{n1} & \dots & & l_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & \dots & u_{1n} \\ & u_{22} & \dots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{pmatrix}$$

- if $A=LU$
 then $a_{ij} = \sum_{s=1}^n l_{is} u_{sj} = \sum_{s=1}^{\min(i,j)} l_{is} u_{sj}$. $\textcircled{*}$

($l_{is}=0$ for $s>i$ and $u_{sj}=0$ for $s>j$).

- In each step in this process:
 determine one new row of U
 & one new column in L .

At step k : assume that rows $1, \dots, k-1$
 have been computed in U
 & columns $1, \dots, k-1$ are computed in L .

Set $i=j=k$ in $\textcircled{*}$:

$$a_{kk} = \sum_{s=1}^{k-1} \underset{\substack{\uparrow \\ \text{known}}}{l_{ks}} \underset{\substack{\uparrow \\ \text{known}}}{u_{sk}} + \underset{\substack{\uparrow \\ \text{unknown}}}{l_{kk}} \underset{\substack{\uparrow \\ \text{unknown}}}{u_{kk}} \quad \textcircled{*}$$

- If u_{kk} or l_{kk} has been specified, use $(*)$ to determine the other.

- We now write the k^{th} row and k^{th} column:
 ($i=k$) ($j=k$)

$$a_{kj} = \sum_{s=1}^{k-1} l_{ks} u_{sj} + l_{kk} u_{kj}, \quad k+1 \leq j \leq n.$$

known \downarrow known $\overbrace{\hspace{100px}}$ now known \downarrow
 a_{kj} = + $l_{kk} u_{kj}$, $k+1 \leq j \leq n$.
 ↑
 unknown

$$a_{ik} = \sum_{s=1}^{k-1} l_{is} u_{sk} + l_{ik} u_{kk}, \quad k+1 \leq i \leq n.$$

\uparrow \swarrow ↑ \nwarrow
 known $S=1$ unknown now known

As long as $l_{kk} \neq u_{kk}$ are $\neq 0$, we can find the unknown elements.

Terminology:

- (1) Doolittle's factorization: $L =$ unit lower triangular
 obtain by specifying $\vec{d_{ii}} = \vec{1}$ \rightarrow ($d_{ii} = 1 \forall i$).
- (2) Croout's factorization: $U =$ unit upper triangular
 obtain by specifying $\vec{u_{ii}} = \vec{1}$ \rightarrow ($u_{ii} = 1 \forall i$).
- (3) Cholesky's factorization: $U = L^T$
 and then $l_{ii} = u_{ii} \forall i$.

Example:

$$A = \begin{pmatrix} 60 & 30 & 20 \\ 30 & 20 & 15 \\ 20 & 15 & 12 \end{pmatrix}$$

$$L = \begin{pmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{pmatrix}$$

(1) Doolittle Set $l_{11} = 1 \Rightarrow u_{11} = 60$

~~$$a_{12} = l_{11} u_{12}$$~~

$$a_{12} = l_{11} u_{12}$$

$$30 = 1 \cdot u_{12} \Rightarrow u_{12} = 30.$$

$$a_{13} = l_{11} u_{13}$$

$$20 = 1 \cdot u_{13} \Rightarrow u_{13} = 20.$$

$$a_{21} = l_{21} u_{11}$$

$$30 = l_{21} \cdot 60 \Rightarrow l_{21} = \frac{1}{2}.$$

$$a_{31} = l_{31} u_{11}$$

$$20 = l_{31} \cdot 60 \Rightarrow l_{31} = \frac{1}{3}.$$

Set $l_{22} = 1$

$$a_{22} = l_{21} \cdot u_{12} + l_{22} \cdot u_{22}$$

$$20 = \frac{1}{2} \cdot 30 + 1 \cdot u_{22} \Rightarrow u_{22} = 5$$

$$a_{23} = l_{21} \cdot u_{13} + l_{22} \cdot u_{23}$$

$$15 = \frac{1}{2} \cdot 20 + 1 \cdot u_{23} \Rightarrow u_{23} = 5$$

$$a_{32} = l_{31} \cdot u_{12} + l_{32} \cdot u_{22}$$

$$15 = \frac{1}{3} \cdot 30 + l_{32} \cdot 5 \Rightarrow l_{32} = 1$$

Set $l_{33} = 1$

$$a_{33} = l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + l_{33} \cdot u_{33}$$

$$12 = \frac{1}{3} \cdot 20 + 1 \cdot 5 + 1 \cdot u_{33} \Rightarrow u_{33} = \frac{1}{3}$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix}}_L \underbrace{\begin{pmatrix} 60 & 30 & 20 \\ 0 & 5 & 5 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}}_U$$

Crout factorization

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix} \begin{pmatrix} 60 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Combine

$$\Rightarrow A = \begin{pmatrix} 60 & 0 & 0 \\ 30 & 5 & 0 \\ 20 & 5 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Cholesky factorization

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{60} & & \\ & \sqrt{5} & \\ & & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} \sqrt{60} & & \\ & \sqrt{5} & \\ & & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{60} & 0 & 0 \\ \frac{1}{2}\sqrt{60} & \sqrt{5} & 0 \\ \frac{1}{3}\sqrt{60} & \sqrt{5} & \frac{1}{3}\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{60} & \frac{1}{2}\sqrt{60} & \frac{1}{3}\sqrt{60} \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & \frac{1}{3}\sqrt{3} \end{pmatrix}$$

L L^T

In this case,
A is
- real
- symmetric
- positive definite.

This is also a unique factorization.

2. Pivoting

- Gaussian elimination:

$$\begin{pmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 34 \\ 27 \end{pmatrix}$$

Subtract $2 \times R_1$ from R_2
 Subtract $\frac{1}{2} \times R_1$ from R_3 .

$2, \frac{1}{2} =$ multipliers

Pivot element. The 1^{st} row: pivot row

After the 1^{st} step:

$$\begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & -12 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 21 \end{pmatrix}$$

2^{nd} row pivot row

The pivot element

Subtract $3 \times R_2$ from R_3

$$\begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \end{pmatrix}$$

upper triangular.

An equivalent system to the original one.

Solving by backward substitution $\Rightarrow X = (\dots)$

The multipliers we arrange in a unit lower triangular matrix

$$L = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ \frac{1}{2} & 3 & 1 \end{pmatrix}$$

Together with: $U = \begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

We get

$$\begin{pmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 9 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ \frac{1}{2} & 3 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\boxed{A = LU}$$

Comment: It is clear why this is correct by going back in how U was obtained. For example

$$R_2 \rightarrow R_2 - 2R_1$$

So $R_2^{(0)}$ = $R_2 + 2R_1$, which explains the 2nd row of the matrix L (2 1 0).
etc.

Comment: Clearly, for everything to work, the pivot elements must be nonzero.

Problems:

1) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The simple version of the algorithm fails. (Can't add a multiple of the 1st row to R₂ to zero the coefficient of x₁).

2) $\begin{pmatrix} \epsilon & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Small ϵ .

Gaussian elimination:

$$\begin{pmatrix} \epsilon & 1 \\ 0 & 1 - \frac{1}{\epsilon} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 - \frac{1}{\epsilon} \end{pmatrix}$$

$$\Rightarrow x_2 = \frac{2 - \frac{1}{\epsilon}}{1 - \frac{1}{\epsilon}}$$

$$x_1 = (1 - x_2) \cdot \frac{1}{\epsilon}$$

- What is the real solution?

$$x_2 = \frac{2-\epsilon}{2-1} \approx 1.$$

$$x_1 = \frac{1}{\epsilon} \left(1 - \frac{2\epsilon-1}{\epsilon-1} \right) = \frac{1}{\epsilon} \frac{\epsilon-1-2\epsilon+1}{\epsilon-1} = \frac{1}{1-\epsilon} \approx 1.$$

But on a computer:

$$2 - \frac{1}{\epsilon} \approx -\frac{1}{\epsilon}$$

$$1 - \frac{1}{\epsilon} \approx -\frac{1}{\epsilon}$$

$$\Rightarrow x_2 \approx 1$$

and

$$x_1 = (1 - x_2) \frac{1}{\epsilon} \approx 0.$$

The problem:
the smallness of a_{11} relative to the other elements in the row

$$B) \begin{pmatrix} 1 & \frac{1}{\epsilon} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon} \\ 2 \end{pmatrix}$$

Columns distinct:

$$\begin{pmatrix} 1 & \frac{1}{\epsilon} \\ 0 & 1 - \frac{1}{\epsilon} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon} \\ 2 - \frac{1}{\epsilon} \end{pmatrix}$$

The solution is:

$$x_2 = \frac{2 - \frac{1}{\epsilon}}{1 - \frac{1}{\epsilon}} \approx 1$$

$$x_1 = \frac{1}{\epsilon} - \frac{1}{\epsilon} x_2 \quad \begin{cases} \text{on a computer } \approx 0 \\ \text{real: } \frac{1}{\epsilon} \left[1 - \frac{2\epsilon - 1}{\epsilon - 1} \right] = \frac{1}{1 - \epsilon} \approx 1. \end{cases}$$

② If we change the order of the equations, the problem disappears:

$$\begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 - \epsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 - 2\epsilon \end{pmatrix}$$

$$\text{Solution: } x_2 = \frac{1 - 2\epsilon}{1 - \epsilon} \approx 1$$

$$x_1 = 2 - x_2 \approx 1.$$

Conclusion: The algorithm must allow for interchanging rows when necessary.

Gaussian elimination with scaled row pivoting

Key: Select the order.

Step I: Compute the "scale" of each row:

$$s_i = \max_{1 \leq j \leq n} |a_{ij}| = \max \{ |a_{i1}|, \dots, |a_{in}| \}, \quad 1 \leq i \leq n.$$

Example: $A = \begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ 3 & 2 & 1 \end{pmatrix}, \quad s = (6, 8, 3).$

Step II: Look at the ratios $\left\{ \frac{|a_{11}|}{s_1}, \frac{|a_{21}|}{s_2}, \frac{|a_{31}|}{s_3} \right\}$

pick the row with the largest scale (ratio) as the pivot row.

In this case $\left(\frac{2}{6}, \frac{1}{8}, \frac{3}{3} \right)$ the largest \downarrow R3 is the pivot

\Rightarrow Exchange R3 with R1.

New elimination:

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & -6 + \frac{2}{3} & 8 - \frac{1}{3} \\ 0 & 3 + \frac{4}{3} & -6 - \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & -\frac{16}{3} & \frac{23}{3} \\ 0 & \frac{13}{3} & -\frac{20}{3} \end{pmatrix}$$

Now we compare $\left\{ \frac{|a_{22}|}{s_2}, \frac{|a_{32}|}{s_3} \right\}$

In this case $\left\{ \frac{\frac{16}{3}}{8}, \frac{\frac{13}{3}}{6} \right\}$

← the original value of the pivot

↑ the largest.

So we exchange $R_2 \leftrightarrow R_3$.
etc.

Comment:

The scale we divide by is always the original calculated number.

Q: When can Gaussian elimination be used safely w/o pivoting?

Example: If the matrix is diagonally dominant.

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|, \quad 1 \leq i \leq n.$$

Thm: Gaussian elimination preserves the d.d. of a matrix (w/o pivoting).

Thm: Every diagonally dominant matrix is nonsingular and has an LU-decomposition.

Comment:

Scaled Gaussian elimination with pivoting when applied to a d.d. matrix doesn't change the order of the rows.

Norms and the analysis of errors

Norm - vectors

Def: On a vector space V a norm is a function $\|\cdot\|: V \rightarrow \mathbb{R}^+$ st.

- (i) $\|x\| \geq 0$ if $x \neq 0, x \in V$
- (ii) $\|\lambda x\| = |\lambda| \|x\| \quad \lambda \in \mathbb{R}, x \in V$
- (iii) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V.$

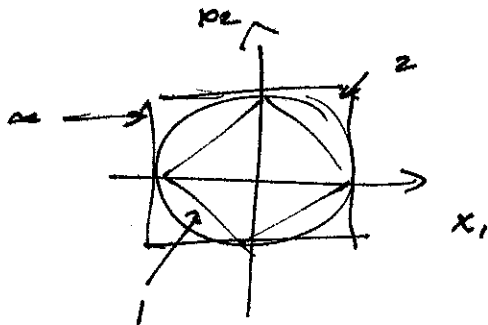
Examples: (1) in \mathbb{R}^n

$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

(2) The unit ball in \mathbb{R}^2



Matrix norm

Given a vector norm $\|\cdot\|$, the associated matrix norm is (for $A \in \mathbb{R}^{n \times n}$)

$$\|A\| = \sup \{ \|Ax\| : x \in \mathbb{R}^n, \|x\|=1 \}.$$

Thm: if $\|\cdot\|$ is a vector norm, then $\|A\|$ defined by \mathcal{R} is a norm on the space of $n \times n$ matrices.

Proof:

$$(i) \quad \|\lambda A\| = \sup \{ \|\lambda Ax\|, \|x\|=1 \} = |\lambda| \sup \{ \|Ax\|, \|x\|=1 \} = |\lambda| \|A\|.$$

$$\begin{aligned} (ii) \quad \|A+B\| &= \sup \{ \|(A+B)x\|, \|x\|=1 \} \\ &\leq \sup \{ \|Ax\| + \|Bx\|, \|x\|=1 \} \\ &\leq \sup \{ \|Ax\|, \|x\|=1 \} + \sup \{ \|Bx\|, \|x\|=1 \} \\ &= \|A\| + \|B\|. \end{aligned}$$

(iii) If $A \neq 0$, it has at least one nonzero element, say in column j . Let $u = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0)$. Set $x = \frac{u}{\|u\|}$. Then $\|x\|=1$.

$$\Rightarrow \|A\| \geq \|Ax\| = \frac{\|Au\|}{\|u\|} = \frac{\|A^j\|}{\|u\|} > 0.$$

Properties of matrix norms

1. $\|Ax\| \leq \|A\| \|x\|$
2. $\|I\| = 1$
3. $\|AB\| \leq \|A\| \|B\|$.

Example:

If $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

Then $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$.

Condition

Assume that A is invertible.

Then

$$\boxed{\kappa(A) = \|A\| \cdot \|A^{-1}\|}$$

The condition number of A .

any norm.

Question: Assume that $Ax=b$. A non invertible.

Perturb $A^{-1} \rightarrow B$

Then the solution $x=A^{-1}b$ becomes $\tilde{x}=Bb$.

How large is the perturbation in the solution?

Answer: any vector norm.

$$\|x - \tilde{x}\| = \|x - Bb\| = \|x - BAx\| = \|(\mathbb{I} - BA)x\|$$

$$\leq \|\mathbb{I} - BA\| \|x\|.$$

\Rightarrow The relative perturbation $\frac{\|x - \tilde{x}\|}{\|x\|} \leq \|\mathbb{I} - BA\|$

Question: Perturb $b \rightarrow \tilde{b}$.
If $Ax=b$ and $A\tilde{x}=\tilde{b}$, how close are x and \tilde{x} ?

Answer: $\|x - \tilde{x}\| = \|A^{-1}b - A^{-1}\tilde{b}\| = \|A^{-1}(b - \tilde{b})\| \leq \|A^{-1}\| \|b - \tilde{b}\|.$

To obtain relative estimate $\Rightarrow \|A^{-1}\| \|Ax\| \frac{\|b - \tilde{b}\|}{\|b\|} \leq \|A^{-1}\| (\|A\| \|x\|) \frac{\|b - \tilde{b}\|}{\|b\|}$

$$\Rightarrow \frac{\|x - \tilde{x}\|}{\|x\|} \leq \underbrace{\|A\| \|A^{-1}\|}_{K(A)} \frac{\|b - \tilde{b}\|}{\|b\|}.$$

- ⊛ The condition number depends on the norm.
- ⊛ If the c.n. is small then small perturbations in b will lead to small perturbations in x .
- ⊛ $K(A) \geq 1$.

Example: $A = \begin{pmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{pmatrix}$ $A^{-1} = \frac{1}{\epsilon^2} \begin{pmatrix} 1 & -1-\epsilon \\ -1+\epsilon & 1 \end{pmatrix}$.

In the ∞ norm: $\|A\|_{\infty} = 2 + \epsilon$
 $\|A^{-1}\|_{\infty} = \frac{1}{\epsilon^2} (2 + \epsilon)$

$\Rightarrow K(A) = \left(\frac{2+\epsilon}{\epsilon}\right)^2 > \frac{4}{\epsilon^2}$.

If $\epsilon = 0.01$ then $K(A) > 40,000$.

Hence a small perturbation in b may induce a relative perturbation 40,000 times greater in the solution of the system $Ax=b$.