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SURVEY

Periodic orbits and dynamical spectra

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Abstract. Basic results in the rigorous theory of weighted dynamical zeta functions or dynamically defined generalized Fredholm determinants are presented. Analytic properties of the zeta functions or determinants are related to statistical properties of the dynamics via spectral properties of dynamical transfer operators, acting on Banach spaces of observables.

1. *Introduction*

Thirty years ago, Smale (1967, I.4) conjectured that the Artin–Mazur (Artin and Mazur 1965) dynamical zeta function

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \#\{x \mid f^n x = x\} \tag{1.1}$$

of an Anosov diffeomorphism f was rational. In the same paper, Smale (1967, II.4) asked whether a Selberg dynamical zeta function $Z(s)$ associated to some flows always possessed a meromorphic continuation to the whole complex plane when the flow satisfied Axiom A (admitting that 'a positive answer would be a little shocking'). The first question was settled positively by Guckenheimer (1970), and by Manning (1971) for all Axiom A diffeomorphisms (see Theorem 2.4 later). The second question proved to be more delicate (the reason, in a nutshell, being that it involved working with *weighted* zeta functions for maps): Ruelle (1976b) introduced a dynamical zeta function $\zeta(s)$ (see (2.14)) for flows (with $\zeta(s) = Z(s+1)/Z(s)$ in the constant negative curvature case). Gallavotti (1976) then found a differentiable Axiom A flow whose Ruelle dynamical zeta function $\zeta(s)$ had a non-polar singularity. Much more recently Fried (1995b) proved, combining Grothendieck techniques from the pioneering article of Ruelle (1976b) with novel ideas of Rugh (1994), that the dynamical zeta function of a real analytic Axiom A flow (without assuming smoothness of the stable and unstable bundles) could indeed be extended meromorphically to $\mathbb{C}$ (see Theorem 4.1 later).

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In this period of over 30 years, the rigorous theory of dynamical zeta functions (in particular, *weighted* dynamical zeta functions, introduced by Ruelle by analogy with the thermodynamic formalism of statistical mechanics) has expanded in many directions. One could argue that this theory still lacks a unifying framework: this is perhaps a motivation to gather some of the 'fundamental' results in order to make them more accessible to mathematicians and also physicists (see the remarks on quantum chaos later). Several surveys have already appeared (Parry and Pollicott 1990, Ruelle 1995, Baladi 1995a). We recommend particularly Ruelle's (1994, ch. I) short, very readable, and broadviewed introduction, which evokes also historical and mathematical connections with the Riemann zeta function, Dirichlet L-functions, and other arithmetic zeta functions, not to be covered here. Although we have tried to include some of the more recent developments, this text is not intended for specialists: we have striven to give the simplest possible version of the statements, referring to the original papers for the full power of the technical results. We have included sketches of some proofs, hoping to communicate the elegant simplicity of certain arguments.

We shall relate the weighted dynamical zeta functions, or the various dynamical determinants, to generalized Fredholm determinants

$$\text{`det'}\,(1 - z\mathcal{L}) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{`tr'}\,\mathcal{L}^n \tag{1.2}$$

for dynamically defined weighted transfer operators $\mathcal{L}$ (e.g. (2.3)), which are often non-compact and in particular are *not* trace-class. The game consists thus in *defining* a 'trace' for the operators $\mathcal{L}^n$ (usually a weighted sum over period-n orbits), and then proving a connection between zeros of (1.2) and inverse eigenvalues of $\mathcal{L}$ acting on a suitable Banach space. In §2.1 we shall consider a trivial occurrence of this phenomenon (2.5), moving then to more interesting situations. Many different techniques and ideas appear in the proofs. However, we would like to emphasize that, since the building blocks of transfer operators are maps

$$\varphi \mapsto (g \cdot \varphi) \circ f^{-1} \tag{1.3}$$

where g is a smooth weight function, f^{-1} is (an inverse branch of) a dynamical system, and the observables (or test functions) φ belong to a vector space of smooth functions, the operations involved are essentially *composition* by a ('smoothness improving') map and *multiplication* by a function. Therefore, the basic toolkit which will be used over and over again (together with combinatorics now well understood in the uniformly hyperbolic case) contains two instruments: the *chain rule* (or the change of variable in an integral) and the *Leibniz formula* (or integration by parts). Certainly, this caricatural description is so vague that it would apply to many fields of mathematics. We nevertheless believe that keeping it in mind can be a guide to the intuition.

In many cases, the *dynamical spectra*, i.e. poles and/or zeros of suitably weighted dynamical zeta functions or dynamical determinants, contain essential information on the statistical behaviour of the dynamical system: the leading pole (or zero) is often the topological pressure (for example, topological entropy) and the first gap, if it exists, may correspond to the exponential rate of decay of correlation functions for the equilibrium state associated to the weight and smooth test functions. The dynamical spectrum beyond the first gap can sometimes be interpreted as (Ruelle) resonances of the dynamical system (Eckmann (1989), see §2.1), connected to geometric or topological properties of a manifold (see §4), or have some more unexpected meaning (see §3.1 on connections with the Feigenbaum spectrum). This interpretation of the dynamical spectrum follows from connecting the poles and/or zeros of the zeta function or dynamical determinant with the spectrum of suitable transfer operators. We would like to point out that, although many proofs of such connections involve non-canonical constructions and choices (Markov partitions, tower extensions, local transfer operators, 'artificial' Banach spaces, etc.) which may cause an important loss of information (such as creation of spurious poles and zeros), the dynamical zeta function is 'just there'. Indeed, it is given by the periodic orbits of the map f, and the often canonical choice of a weight (such as $|\operatorname{Det} Df|^\beta$, or $|\operatorname{Det} Df|_{E_u}|^\beta$ with β a parameter). A specific example is the situation of Theorem 5.2 on Collet–Eckmann-type unimodal interval maps f. The statement and proof regarding the transfer operator involves a complicated construction, but the result on the zeta function itself is quite simple: the zeta function associated to the pair $(f, 1/|f'|)$ admits a meromorphic extension to a larger disc than its disc of convergence, where its only singularity is a simple pole at $z = 1$. (Of course, one uses the transfer operators to prove that these properties of the zeta function mirror the uniqueness of the SRB measure and exponential decay of correlations.) In §5.2, partial results indicate that the branch cut type of zeta function may describe the non-exponential decay of correlations for some intermittent maps. There are few rigorous results in such 'gapless' situations, where one can expect to discover phase transitions, as in statistical physics.

We know by now that it is not possible to hear the shape of a drum (Kac (1966), Gordon *et al* (1992)). It would be naive to expect to hear the statistical properties of a dynamical system. We should also keep in mind that some dynamical systems do not admit any periodic orbit. However, we do believe that weighted dynamical zeta functions are 'interesting invariants' (Smale (1967, p. 764)!) and that a good understanding of their qualitative analytic properties should play a significant part in the classification of differentiable dynamics.

1.1. *Applications of dynamical determinants in physics and mathematics.* In these notes we have limited ourselves to rigorous mathematical statements. However, the (long, but incomplete) bibliography includes some references to the rich physical literature, which we believe to be a potential source of interesting mathematical conjectures. We mention, in particular, the book in preparation by Cvitanović (1997) and co-workers (see also Artuso *et al* (1990)), which contains a wealth of results and insightful definitions, as well as pointers to computer programs that are able to effectively compute zeta functions of non-trivial systems.

We refer to Fried (1986b, 1995b) and references therein for the mathematical connection between Selberg and (Ruelle) dynamical zeta functions (see also §4). We shall not discuss physical applications of Selberg or dynamical zeta functions to quantum chaos (see the reviews of Eckhardt (1988) and Hurt (1993), the monographs of Gutzwiller (1990) and Knauf and Sinai (1997), and references therein, e.g. Bogomolny *et al* (1995), Voros (1988, 1993), Cartier and Voros (1988)). Recent *rigorous* results on Selberg functions connected with quantum chaos, starting from Mayer's (1991b) important study of the Selberg zeta function $Z(s)$ of the modular surface, via a thermodynamic formalism for the Gauss map (using a method due to Series, respectively Adler–Flatto, and expressing the Selberg function as a product $Z(s) = \mathrm{Det}(1 - \mathcal{L}_s)\,\mathrm{Det}(1 + \mathcal{L}_s)$ of Fredholm determinants), include Efrat (1993) (with later developments by Eisele and Mayer), Lewis (1997), and Chang and Mayer (1996). Some of these results have connections with the study of the Riemann zeta function (see also Knauf's (1993, 1994) statistical mechanics approach of the Riemann zeta function). See Pollicott (1991a, 1994) for extensions of the Bowen–Series approach. The background for this is discussed in §§3.1 and 4.

We shall not present Patterson's (1990) dynamical approach to understand the connection of the divisor (zeros and poles) of the Selberg zeta function associated to certain Kleinian groups with the cohomology of the group (see Deitmar (1996), Juhl (1995), and Patterson and Perry (1996) for recent results). A significant breakthrough in Patterson's program has recently been accomplished by Bunke and Olbrich (1996).

Besides the Selberg zeta function, other counting functions may be connected to, or expressed as, dynamical zeta functions. Llibre and co-workers have used various Lefschetz zeta functions to obtain Sharkovskii-type (Block *et al* (1980)) 'forcing' results, see e.g. Casasayas *et al* (1994), Guillamon *et al* (1995). For Reidemeister and Nielsen zeta functions see Fel'shtyn and Hill (1995). Sometimes counting functions (such as Poincaré series appearing in hyperbolic groups, see Pollicott and Sharp (1994, 1995)) may be studied with tools from the thermodynamic formalism, such as the transfer operator techniques described in these notes. A very rich line of research is centered around the theme of zeta functions and closed orbits associated to homology classes (Parry and Pollicott 1986, Phillips and Sarnak 1987, Lalley 1989, Katsuda and Sunada 1990, Pollicott 1991b, Sharp 1993, Babillot and Ledrappier 1996).

A more unexpected application of dynamical Fredholm determinants appears in a study of the smoothness of scaling functions in the construction of multiresolution analysis and wavelets (Cohen and Daubechies 1996). Other applications are mentioned throughout the text.

2. *Symbolic dynamics and counting traces*

With the notable exception of some recent results for uniformly hyperbolic flows (see §2.2), most of the material in this section has been reviewed elsewhere, for example in the monograph of Parry and Pollicott (1990) and in the survey Baladi (1995a). For the convenience of the reader we nevertheless recall the most salient facts.

2.1. *Axiom A maps.* Consider a two-sided subshift of finite type on $d \geq 2$ symbols given by a $d \times d$ transition matrix A (with $A_{ij} \in \{0, 1\}$), i.e. let $\mathcal{S} = \{1, \ldots, d\}$ and set

$$\Sigma_A = \{(x_i)_{i\in\mathbb{Z}} \mid x_i \in \mathcal{S}, A_{x_i x_{i+1}} = 1, \ \forall i \in \mathbb{Z}\}. \tag{2.1}$$

The invertible dynamical system $\sigma : \Sigma_A \to \Sigma_A$ is defined by $(\sigma(x))_i = x_{i+1}$. We also consider the (non-invertible) one-sided shift σ^+ defined on the space of one-sided sequences Σ_A^+ with $\mathbb{Z}$ replaced by $\mathbb{Z}_+$ in (2.1). Let g be a bounded complex-valued function on Σ_A or Σ_A^+, set $f = \sigma$, and define the *weighted dynamical zeta function* of the pair (f, g) to be the formal power series

$$\zeta_g(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}\, f^n} \prod_{k=0}^{n-1} g(f^k(x)) \tag{2.2}$$

where $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ for any map $f : Y \to X$ with $Y \subset X$. In the case $g \equiv 1$, one recovers the Artin–Mazur *unweighted dynamical zeta function* (1.1). Define the *transfer operator* associated to the pair (σ^+, g) acting on (say, bounded) functions $\varphi : \Sigma_A^+ \to \mathbb{C}$ by

$$\mathcal{L}_g \varphi(x) = \sum_{\substack{y \in \Sigma_A^+ \\ \sigma^+(y) = x}} g(y)\varphi(y). \tag{2.3}$$

The sum

$$\sum_{x \in \mathrm{Fix}\,(\sigma^+)^n} \prod_{k=0}^{n-1} g((\sigma^+)^k(x))$$

is called the *counting trace* of the operator $\mathcal{L}_g^n$.

Clearly, *when the weight g is positive*, the logarithm of the spectral radius of $\mathcal{L}_g$ acting on the Banach space of bounded functions (with the supremum norm) is just

$$P := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x \in \Sigma_A^+} (\mathcal{L}_g^n 1) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x \in \Sigma_A^+} \sum_{(\sigma^+)^n(y) = x} \prod_{k=0}^{n-1} g((\sigma^+)^k(y)). \tag{2.4}$$

If the positive weight g is continuous, one can prove that the real number P defined by (2.4) coincides with the *topological pressure* $P(\log g)$ of the function $\log g$. (See e.g. Ruelle (1978), or Walters (1982) for the notion of pressure.) For a general continuous, complex, but non-vanishing g, one shows that the spectral radius of $\mathcal{L}_g$ acting on bounded functions is not larger than the spectral radius $\exp P(\log |g|)$ of $\mathcal{L}_{|g|}$ acting on bounded functions.

We now turn to a trivial but very enlightening example. In the special case when the weight g is *locally constant*, i.e. if there is $M \geq 1$ so that $g(x)$ depends only on x_i for $0 \leq i < M$, the zeta function $\zeta_g(z)$ can be expressed in terms of the determinant of a finite matrix (see e.g. Bowen and Lanford (1970), Parry and Williams (1977)): indeed, after reducing to the case when $g(x) = g_{x_0, x_1}$, one introduces the $d \times d$ matrix $A(g)$ by setting $A(g)_{ij} = A_{ij} g_{ji}$. It is then easy to check that $\mathrm{Tr}\, A(g) = \sum_{x \in \mathrm{Fix}\, \sigma} g(x)$, and more generally

$$\mathrm{Tr}\, A^n(g) = \sum_{x \in \mathrm{Fix}\, \sigma^n} \prod_{k=0}^{n-1} g(\sigma^k(x))$$

(where $A^n(g)$ is the nth power of the matrix A). Therefore, using the formula $\mathrm{Tr} \log B = \log \mathrm{Det}\, B$ (for a finite matrix B) we find

$$\zeta_g(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}\, A^n(g) = \frac{1}{\mathrm{Det}(1 - zA(g))}. \tag{2.5}$$

In particular, the zeta function is rational, and its poles are exactly the inverses of the non-zero eigenvalues of the matrix $A(g)$ (the order of the poles coinciding with the multiplicity of the eigenvalues). Since $A(g)$ is the matrix of the operator $\mathcal{L}_g$ acting on the d-dimensional vector space of functions $\varphi(x)$ depending only on x_0 (in the canonical basis), the interpretation of the zeta function as an inverse Fredholm determinant is trivially true for locally constant weights. Finally, when the locally constant weight g is positive, the classical Perron–Frobenius theorem for finite matrices (see e.g. Walters (1982) for a statement) says that $A(g)$ admits a real positive eigenvalue e^P equal to its spectral radius, that whenever the matrix A is irreducible (i.e. for any i, j there is an n so that $(A^n)_{ij} > 0$) this eigenvalue has algebraic multiplicity equal to one, and that if A is additionally irreducible and aperiodic (i.e. there is $N > 0$ such that $A^N_{ij} > 0$ for all i, j) then e^P is the only eigenvalue of maximal modulus. To relate the conditions on A with topological properties of the dynamical system, recall (see e.g Walters (1982)) that a subshift of finite type is one-sided topologically transitive if and only if the transition matrix A is irreducible, and topologically mixing if and only if the transition matrix A is irreducible and aperiodic.

The observations in the previous paragraph do not apply directly to more general, non-locally constant, weights g. One of the first successes of the theory of dynamical zeta functions was the proof of an analogue of (2.5) for weights g which are *Lipschitz* with respect to a metric $d_\theta(x, y) = \sum_{k\in\mathbb{Z}} \theta^{|k|}(1 - \delta(x_k, y_k))$, for some fixed $0 < \theta < 1$, where δ denotes the Kronecker delta. (Note that each inverse branch of the one-sided shift σ^+ is a θ-contraction for the metric d_θ.) To state precisely this result, we need more notation. Write $\mathcal{F}^+_\theta$ for the Banach space of Lipschitz functions $\varphi : \Sigma^+_A \to \mathbb{C}$ (for d^+_θ, the one-sided version of d_θ), endowed with the norm $\|\varphi\| = \sup|\varphi| + \mathrm{Lip}(\varphi)$, where $\mathrm{Lip}(\varphi)$ is the smallest Lipschitz constant for φ. We first consider the spectral properties of the transfer operator $\mathcal{L}_g$, recalling that the *essential spectral radius* of a bounded linear operator acting on a Banach space is the smallest $\rho > 0$ so that the spectrum of the operator outside of the disc of radius ρ consists in a finite or countable set of isolated eigenvalues of finite multiplicity.

THEOREM 2.1. (Quasicompactness) *Assume that $g = \exp G$ where $G \in \mathcal{F}^+_\theta$.*

(1) (Ruelle 1968, 1976a, 1978) *The spectral radius of $\mathcal{L}_g : \mathcal{F}^+_\theta \to \mathcal{F}^+_\theta$ is bounded above by $e^{P(\log|g|)}$ and coincides with $e^{P(\log g)}$ if $g > 0$. If σ^+ is topologically mixing on Σ^+_A and $\mathcal{L}_g$ has an eigenvalue of modulus $e^{P(\log|g|)}$ then this eigenvalue is simple and the rest of the spectrum lies in a disc of strictly smaller radius.*

(2) (Pollicott 1986) *The essential spectral radius ρ_{ess} of $\mathcal{L}_g : \mathcal{F}^+_\theta \to \mathcal{F}^+_\theta$ is equal to $\theta \cdot e^{P(\log|g|)}$. Every point in the open disc of radius $\theta \cdot e^{P(\log|g|)}$ is an eigenvalue of infinite multiplicity of $\mathcal{L}_g$.*

To prove the upper bound on the essential spectral radius in Theorem 2.1(2), one shows that the iterates $\mathcal{L}^n_g$ can be exponentially well approximated by a sequence of finite rank operators. The key ingredient used to obtain the required bounds is the existence of a constant $C > 0$ such that for all $\varphi \in \mathcal{F}^+_\theta$ and all $n \in \mathbb{Z}^+$

$$\mathrm{Lip}(\mathcal{L}^n_g\varphi) \le \theta^n \,\mathrm{Lip}(\varphi) + C \sup|\varphi|. \tag{2.6}$$

(See Ionescu Tulcea and Marinescu (1950) for early occurrences of similar bounds.) The proof of the bound (2.6) in the normalized case $\mathcal{L}_{|g|}1 \equiv 1$ is by induction. It is based on the fact that whenever $x_0 = y_0$ (writing (jx) for the concatenation of the one-sided sequence $x \in \Sigma_A^+$ with the single symbol $j \in \mathcal{S}$) we have

$$\begin{aligned}\frac{|(\mathcal{L}_g\varphi)(x) - (\mathcal{L}_g\varphi)(y)|}{d_\theta(x,y)} &\leq \sum_{\substack{j\in\mathcal{S}\\ A_{jx_0}=1}} |g(jy)|\frac{|\varphi(jx)-\varphi(jy)|}{d_\theta(x,y)} \\ &\quad + \sum_{\substack{j\in\mathcal{S}\\ A_{jx_0}=1}} \frac{|g(jx)-g(jy)|}{d_\theta(x,y)}|\varphi(jx)| \\ &\leq \theta\,\mathrm{Lip}(\varphi)(\mathcal{L}_{|g|}1) + d\theta\,\mathrm{Lip}(g)\sup|\varphi| \\ &= \theta\,\mathrm{Lip}(\varphi) + C\sup|\varphi|. \qquad (2.7)\end{aligned}$$

The two terms on the right-hand side of (2.7) can be viewed as coming from an application of the Lipschitz version of the Leibniz upper bound for the differentiation of a product. The factor θ in front of the Lipschitz constant of φ is due to the composition of φ with the contracting inverse branches of σ^+.

Note that 'smoothness' in the function space is essential in order to prove quasicompactness. Replacing Lipschitz functions by α-Hölder functions one gets a factor θ^α instead of θ in Theorem 2.1(2). One can prove that each point in the the disk of radius $e^{P(\log|g|)}$ is an eigenvalue of $\mathcal{L}_g$ acting on continuous functions. In particular, $\mathcal{L}_g$ will *not* have a gap when acting on the Hilbert space $L^2(d\mu)$, for any probability measure μ such that $\mathcal{L}_g$ is defined on $L^2(d\mu)$. This is basically the reason why one is often forced to do spectral theory on Banach spaces.

THEOREM 2.2. (Zeta function) (Pollicott 1986, Haydn 1990b) *Assume that $g = \exp G$, where $G \in \mathcal{F}_\theta^+$. The zeta function $\zeta_g(z)$ (2.2) is analytic in the disc of radius $e^{-P(\log|g|)}$, and admits a meromorphic and zero-free extension to the disc of radius $\theta^{-1}e^{-P(\log|g|)}$. Its poles in this disc are exactly the inverses of the eigenvalues of $\mathcal{L}_g : \mathcal{F}_\theta^+ \to \mathcal{F}_\theta^+$ in the corresponding annulus (the order of each pole coinciding with the algebraic multiplicity of the eigenvalue).*

We refer to Parry and Pollicott (1990, ch. 10) for a proof of Theorem 2.1 and Theorem 2.2 (see also Baladi (1995a, 1.2) for a short sketch). In §3.2 we shall briefly describe the slightly more sophisticated proof of similar but more powerful results in a differentiable setting.

The introduction of the one-sided spaces $\mathcal{F}_\theta^+$ was useful to work with transfer operators associated to one-sided shifts with contracting inverse branches. When the weight g is two-sided, one can study the zeta function (2.2) with the help of the following lemma.

LEMMA 2.3. (Two-sided to one-sided) (Sinai 1972, Bowen 1975) *Let $G \in \mathcal{F}_\theta$. There exist G^+ and ψ in $\mathcal{F}_{\sqrt{\theta}}$ such that $G = G^+ + \psi - \psi\circ\sigma$, and $G^+(x) = G^+(y)$, whenever $x_i = y_i$ for all $i \geq 0$ (abusing notation: $G^+ \in \mathcal{F}^+_{\sqrt{\theta}}$).*

Indeed, whenever two functions differ by a coboundary $\varphi_1 = \varphi_2 + \psi - \psi\circ\sigma$, the sums $\sum_{k=0}^{n-1}\varphi_i(\sigma^k(x))$ coincide whenever $\sigma^n(x) = x$. If the functions are additionally real valued, one checks that the pressures $P(\varphi_1)$ and $P(\varphi_2)$ coincide.

We now briefly recall how the above results are applied to C^1 Anosov diffeomorphisms, or more generally Axiom A diffeomorphisms, on compact manifolds. We refer to Bowen (1975) and Parry and Pollicott (1990) for details and references. A diffeomorphism f is called Axiom A if:

(1) the non-wandering set Λ of f coincides with the closure of the set of periodic points;

(2) there exist a decomposition $T\Lambda = E^u \oplus E^s$ of the tangent bundle over Λ, and constants $C > 0$, $0 < \theta < 1$ so that for all $x \in \Lambda$ and all $n \in \mathbb{Z}^+$,

$$\|Df_x^n v\| \le C\theta^n \|v\|, \quad \forall v \in E_x^s, \qquad \|Df_x^{-n} v\| \le C\theta^n \|v\|, \quad \forall v \in E_x^u.$$

By Smale's (1967) spectral decomposition, we may restrict f to a basic set $\Omega \subset \Lambda$ on which it is topologically transitive. Such a map can be modelled by a topologically transitive subshift of finite type via the use of Markov partitions. More precisely, there exist a subshift (σ, A) (with metric d_θ, for θ the contraction constant of f), and a Lipschitz surjective map $\pi : \Sigma_A \to \Omega$, such that $f \circ \pi = \pi \circ \sigma_A$. The non-injectivity of π is due to the fact that the rectangles of the Markov partition can meet on their boundaries. To cancel the overcounting of periodic points on these boundaries, Manning (1971) associated to f finitely many auxiliary subshifts of finite type $\{\sigma_i\}_{i=0,\dots,K}$ (with $\sigma_0 = \sigma$, the other shifts semi-conjugated with restrictions of f by projections π_i), and signs $\epsilon_i \in \{-1, 1\}$ such that we have the counting formula

$$\#\,\mathrm{Fix}\, f^n = \sum_{i=0}^{K} \epsilon_i \cdot \#\,\mathrm{Fix}\, \sigma_i^n$$

for each n. Therefore, writing ζ_i for the zeta function of σ_i, Smale's (1967) spectral decomposition together with the remark (2.5) of Bowen and Lanford (1970) give $\zeta(z) = \prod_{i=0}^{K} \zeta_i(z)^{\epsilon_i}$. This proves the following.

THEOREM 2.4. (Manning 1971) *The unweighted zeta function (1.1) of a C^1 Axiom A diffeomorphism is rational.*

For zeta functions weighted by an α-Hölder function $g : \Omega \to \mathbb{C}$, first lift g to functions $\bar{g}_i \in \mathcal{F}_{\theta^\alpha}$ via the projections π_i. Then, using Theorem 2.2, one can prove the following.

THEOREM 2.5. (Axiom A: weighted case) (Pollicott 1986, Haydn 1990b) *Let f be a C^1 Axiom A diffeomorphism on a transitive basic set Ω, with contraction coefficient $\theta < 1$, and let $g : \Omega \to \mathbb{C}$ be α-Hölder. Then the weighted zeta function $\zeta_g(z)$ is analytic and non-zero in the disc $|z| < e^{-P(\log|g|)}$, and admits a meromorphic extension to the disc $|z| < \theta^{-\alpha/2} \cdot e^{-P(\log|g|)}$, where its poles and zeros are a subset of the inverses of eigenvalues of each $\mathcal{L}_{\bar{g}_i}$ on $\mathcal{F}^+_{\theta^{\alpha/2}}$ outside of the disc of radius $\theta^{\alpha/2} \cdot e^{P(\log|g|)}$.*

To end this subsection on applications of symbolic dynamics to uniformly hyperbolic diffeomorphisms, we briefly discuss the important relationship between the poles of weighted zeta functions and the decay of *correlation functions* of the *equilibrium state* associated to the corresponding weight. Recall (Ruelle 1978, Walters 1982) that the set of equilibrium states associated to a pair (f, ψ), where $f : X \to X$ is a continuous map on a metric space and $\psi : X \to \mathbb{R}$ is a continuous function, is the (possibly empty) set of f-invariant Borel probability measures on X realizing the supremum

$$P(\psi) = \sup \left\{ h_\mu(f) + \int \psi \, d\mu \right\} \tag{2.8}$$

where $h_\mu(f)$ denotes the Kolmogorov–Sinai (measure-theoretical) entropy of the pair (f, μ). The variational principle of Walters asserts that the supremum in (2.8) coincides with the topological pressure $P(\psi)$ of (f, ψ).

Let f be a C^1 Axiom A diffeomorphism on a transitive basic set Ω, and let $\bar{g}$ be a lift to Σ_A of a positive α-Hölder continuous weight $g : \Omega \to \mathbb{R}^+_*$. The maximal eigenfunction φ_0 for $\mathcal{L}_{\bar{g}}$, and the maximal eigenmeasure ν_0 for the dual of $\mathcal{L}_{\bar{g}}$ determine a σ-invariant ergodic probability measure $\bar{\mu}$ which is the unique equilibrium state for $\log \bar{g}$ and σ (Ruelle 1976a). The projection μ of $\bar{\mu}$ to Ω is the equilibrium state for $\log g$ and f. If $f|_\Omega$ is topologically mixing, A is irreducible and aperiodic and the measure μ is mixing.

Assume for a moment that Ω is an attractor for f (i.e. there is an open neighbourhood U of Ω with $f(U) \subset U$), that f is $C^{1+\epsilon}$, and consider the special weight $g(x) = 1/|\operatorname{Det} Df|_{E^u}(x)|$, where E^u is the unstable bundle of f (recall that E^u is Hölder continuous, but usually not C^1, so that $g(x)$ is usually only α-Hölder for some α, see e.g. Katok and Hasselblatt (1995, ch. 19) and references therein). Then $P(\log g) = 0$, and the projection μ of $\bar{\mu}$ to Ω is the *Sinai–Ruelle–Bowen (SRB) measure* for f, i.e. the unique probability measure whose conditionals on the unstable manifolds are absolutely continuous with respect to Lebesgue measure (see Bowen (1975, ch. 4) for proofs and references). This measure is the *physical measure* because for Lebesgue almost all x in a neighbourhood of the attracting basic set Ω, the time averages $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)}$ (where δ_y is the Dirac mass at y) weakly converge to μ when $n \to \infty$.

For an equilibrium state μ associated to a general positive α-Hölder g, define for any fixed pair of α-Hölder continuous observables $\varphi, \psi : \Omega \to \mathbb{C}$ the *correlation function* $C_{\varphi,\psi} : \mathbb{Z} \to \mathbb{C}$:

$$C_{\varphi,\psi}(k) = \int_\Omega (\varphi \circ f^k) \cdot \psi \, d\mu - \int_\Omega \varphi \, d\mu \cdot \int_\Omega \psi \, d\mu. \tag{2.9}$$

In view of studying the decay rate of $C_{\varphi,\psi}(k)$, it is natural to consider the formal Fourier transform $\hat{C}_{\varphi,\psi}(\omega) = \sum_{k\in\mathbb{Z}} e^{i\omega k} C_{\varphi,\psi}(k)$. If we can show that $\hat{C}_{\varphi,\psi}$ is meromorphic in a strip for all α-Hölder φ, ψ, it makes sense to define the *correlation spectrum* (or Ruelle resonances) of μ to be the union of the poles of the $\hat{C}_{\varphi,\psi}$.

Consider the lifts $\bar{\varphi}$ and $\bar{\psi}$ of φ, ψ to Σ_A and *assume that they only depend on* x_i for $i \geq 0$ (we call such observables *one-sided*). Using the notation above and assuming for simplicity that the spectral radius of $\mathcal{L}_g$ is 1, we have

$$\begin{aligned} \int_{\Sigma_A^+} (\bar{\varphi} \circ (\sigma^+)^k) \cdot \bar{\psi} \cdot \varphi_0 \, d\nu_0 &= \int_{\Sigma_A^+} \mathcal{L}^k_{\bar{g}}((\bar{\varphi} \circ (\sigma^+)^k) \cdot \bar{\psi} \cdot \varphi_0) \, d\nu_0 \\ &= \int_{\Sigma_A^+} \bar{\varphi} \cdot \mathcal{L}^k_{\bar{g}}(\bar{\psi} \cdot \varphi_0) \, d\nu_0. \end{aligned} \tag{2.10}$$

Therefore, there is a constant $C > 0$ so that for all α-Hölder φ and ψ

$$\begin{aligned} |C_{\varphi,\psi}(k)| &\leq \sup \left| \mathcal{L}_g^k(\bar{\psi} \cdot \varphi_0) - \varphi_0 \int_{\Sigma_A^+} \bar{\psi} \cdot \varphi_0 \, d\nu_0 \right| \cdot \int_{\Sigma_A^+} |\bar{\varphi}| \, d\nu_0 \\ &\leq C \int_{\Sigma_A^+} |\bar{\varphi}| \, d\nu_0 \cdot (\sup |\bar{\psi}| + \|\bar{\psi}\|_\alpha) \cdot \kappa^k, \end{aligned} \tag{2.11}$$

where $\kappa = \sup\{|z| \mid z \in \text{spectrum } (\mathcal{L}_{\bar{g}}), z \neq 1\}$. Since $\kappa < 1$ by Theorem 2.1, we have proved that the correlation function $C_{\varphi,\psi}(k)$ goes to zero exponentially fast with a rate independent of the *one-sided* α-Hölder observables φ and ψ.

This property of exponential decay of correlations (or the exponential cluster property) of Hölder equilibrium states of Axiom A diffeomorphisms, with uniform rate, was proved 20 years ago for general *two-sided* Hölder test functions (Ruelle (1976a), see also Bowen (1975, §1.E)), but the relationship between the rate of decay and the spectral gap of $\mathcal{L}_{\bar{g}}$ (*a fortiori* a 'polar gap' for a zeta function) was not established at that time. A new proof of exponential decay of correlations of two-sided Hölder observables for the SRB measure of Axiom A attractors has been obtained recently using very elegant Birkhoff cone techniques (Liverani (1995) introduced the method in the Anosov area-preserving case, and Viana (1997) later extended his strategy), bypassing Markov partitions and symbolic dynamics. However, the exact value of the rate of decay, and *a fortiori* the rest of the correlation spectrum, do not seem to be accessible by these methods.

Before we state the most precise result available in the symbolic dynamics setting, we go back to the one-sided observables φ and ψ, and observe that formally evaluating a geometric series gives

$$\begin{aligned} \hat{C}_{\varphi,\psi}(\omega) &= \nu_0(\bar{\varphi}(1 - e^{-i\omega}\rho_g^{-1}\mathcal{L}_{\bar{g}})^{-1}(\varphi_0\bar{\psi})) \\ &\quad + \nu_0(\bar{\psi}(1 - e^{i\omega}\rho_g^{-1}\mathcal{L}_{\bar{g}})^{-1}(\varphi_0\bar{\varphi})) - \nu_0(\varphi_0\bar{\varphi}\bar{\psi}), \end{aligned} \tag{2.12}$$

where $\rho_g = e^{P(\log g)}$ denotes the spectral radius of $\mathcal{L}_{\bar{g}}$ on $\mathcal{F}_\theta^+$. Using the results mentioned above and handling carefully the transition from two-sided to one-sided observables, one obtains the following theorem.

THEOREM 2.6. (Axiom A: correlation spectrum) (Pollicott 1985, Ruelle 1987a, Haydn 1990a) *Let f be a C^1 Axiom A diffeomorphism on a transitive basic set Ω, with contraction coefficient θ, and let μ be an equilibrium state for an α-Hölder weight $g > 0$ on Ω. For α-Hölder observables φ, ψ on Ω, the Fourier transform $\hat{C}_{\varphi,\psi}(\omega)$ of the correlation function for μ extends to a meromorphic function in the strip $|\Im\omega| < \log\theta^{-(\alpha/2)}$, regular at $\omega = 0$. The position of the poles is independent of φ, ψ (although residues can vanish). More precisely there is a holomorphic function $N_{\varphi,\psi}$ on the strip $|\Im\omega| < \log\theta^{-(\alpha/2)}$ such that*

$$\hat{C}_{\varphi,\psi}(\omega) = N_{\varphi\psi}(e^{i\omega})\zeta_{\bar{g}}(e^{i\omega - P(\log g)}) + N_{\psi\varphi}(e^{-i\omega})\zeta_{\bar{g}}(e^{-i\omega - P(\log g)}), \tag{2.13}$$

with $\zeta_{\bar{g}}$ the weighted zeta function of a subshift of finite type modeling f via a Markov partition, and the corresponding lift $\bar{g}$ of g. If the basic set Ω is mixing, then $\hat{C}_{\varphi,\psi}(\omega)$ admits an analytic extension to a strip $|\Im\omega| < \log(1/\kappa)$ with $\kappa = \max_i(\kappa_i) < 1$ the smallest spectral gap of the $\mathcal{L}_{\bar{g}_i}$.

The poles of $\hat{C}_{\varphi,\psi}(\omega)$ are called the *resonances* (or *correlation spectrum*) of f for μ. In the mixing case, the poles with smallest possible imaginary part in absolute value correspond to the rate of decay of correlations for generic α-Hölder observables. The real part of these first resonances indicates how the decay is modulated (see Eckmann (1989)). The next resonances correspond to the decay rate of observables in subspaces of finite codimension.

Although Theorem 2.6 represents the optimal result for subshifts of finite type, the boundaries of the Markov partitions are a source of problems when translating back the results to Ω. In particular, the quotient $\zeta_g^f/\zeta_{\tilde{g}}^\sigma$ could in principle have 'spurious' zeros and poles in the disc of radius $\theta^{-(\alpha/2)}e^{-P(\log g)}$. A more serious drawback of this approach is the fact that higher differentiability (e.g. analyticity: see §3.1) of the original dynamics-weight pair is lost via symbolic dynamics, where only a metric space structure is available. See §4 for techniques which go a long way in overcoming this limitation.

2.2. *Axiom A flows.* Let X be a metric space. The *unweighted zeta function of a flow* $\Phi^t : X \to X$ with at most countably many closed orbits is defined by

$$\zeta^*(s) = \prod_{\tau \text{ primitive periodic orbit}} (1 - e^{-s\cdot\ell(\tau)})^{-1}, \tag{2.14}$$

where $\ell(\tau)$ is the *primitive length* of the closed orbit τ, i.e. the smallest $t_0 > 0$ such that $\Phi^{t_0}(x_0) = x_0$ for any point x_0 on the orbit. We use the terminology primitive periodic orbit to emphasize that each closed orbit is counted once in the Euler product expression (2.14) (in (2.2), a fixed point of f^n also appears as a fixed point of $f^{m\cdot n}$ for all $m \geq 1$).

In order to study the analytic properties of the zeta function (2.14), or more generally its weighted analogue $\zeta_G^*(s)$, where $G : X \to \mathbb{C}$ is bounded, say, and $-s\cdot\ell(\tau)$ is replaced in (2.14) by

$$\int_0^{\ell(\tau)} (G(\Phi^t(x_0)) - s)\, dt. \tag{2.15}$$

we shall use the Bowen and Ruelle (1975) approach to the ergodic theory of Axiom A flows. Just like Ratner's (1969) original approach in dimension three, it uses Markov sections and is based on the following symbolic model. Let Φ^t be a flow obtained by suspending the subshift σ of §2.1 under a positive return time $r \in \mathcal{F}_\theta^+$, i.e. set $\Sigma_A^r = \{(x,t) \mid x \in \Sigma_A, 0 \leq t \leq r(x)\}/\sim$ with $(x, r(x)) \sim (\sigma(x), 0)$, and define $\Phi^t : \Sigma_A^r \to \Sigma_A^r$ by $\Phi^t(x,u) = (x, u+t)$, if $0 \leq u + t < r(x)$, extending to other values of t with the equivalence $\sim$. Formally the unweighted zeta function (2.14) of Φ^t can be rewritten (using the notation (2.2) for the one-parameter weight $g(x) = e^{-sr(x)}$)

$$\zeta^*(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x\in \mathrm{Fix}\,\sigma^n} \exp\left(-s \sum_{k=0}^{n-1} r(\sigma^k x)\right) = \zeta_{\exp(-sr)}(1). \tag{2.16}$$

(The function $\zeta_{e^{-sr}}(z)$ which appears here is one of the many examples of two-variable zeta functions. See (5.10) later, and e.g. Parry and Pollicott (1990) for more details.)

In the case of the suspension of σ under the constant return time $r \equiv 1$, we get $\zeta^*(s) = 1/\operatorname{Det}(1 - e^{-s}\cdot A)$, so that the zeta function is not rational, but meromorphic in the whole complex plane; its poles are the countably many points s for which 1 is an eigenvalue of the finite matrix $e^{-s} \cdot A$. In particular, if s is a pole then $s + 2ki\pi$ is a pole for any integer k, so that there are countably many poles on the boundary of the half-plane of convergence. Note that this suspended flow is clearly not *topologically weak-mixing*, i.e. there exist a non-constant continuous function φ and $\beta \in \mathbb{R}$ with $\varphi \circ \Phi^t = e^{i\beta t}\varphi$ for all t. (See later in this subsection for more on the relationship between mixing properties of the flow and poles of the zeta function.)

For a general $r \in \mathcal{F}_\theta^+$, we consider the map $g \mapsto \zeta_g(1)$ as a complex-valued function on the Banach space $\mathcal{F}_\theta^+$. Using the notion of meromorphic functions between Banach spaces (see e.g. Dunford and Schwartz (1957)), Theorem 2.2 and (2.16) imply that $\zeta^*(s)$ is analytic and non-zero in the set of s such that $P(-\Re s \cdot r) < 0$, and admits a meromorphic extension to the set $P(-\Re s \cdot r) < \log \theta^{-1}$. Since $r > 0$, the map $t \mapsto P(-t \cdot r)$ is monotone decreasing and these sets are half-planes. Using Abramov's theorem on the entropy of a suspension, Bowen and Ruelle (1975) have shown that the topological pressure of σ satisfies $P(-h^*_{\text{top}} \cdot r) = 0$, where h^*_{top} is the topological entropy of the suspension of σ under r.

THEOREM 2.7. (Symbolic suspended flow zeta function) (Pollicott 1986, Ruelle 1987b, Haydn 1990b) *The unweighted zeta function $\zeta^*(s)$ of the suspension of a subshift of finite type under $r \in \mathcal{F}_\theta^+$ is analytic and non-zero in the half-plane $\Re s > h^*_{\text{top}}$, and admits a meromorphic extension to the set $\Re s > \delta$, with poles whenever $\mathcal{L}_{\exp(-sr)}$ has 1 as an eigenvalue, where $\delta < h^*_{\text{top}}$ is the unique real number such that $P(-\delta \cdot r) = \log \theta^{-1}$.*

See Parry and Pollicott (1990, ch. 6, 7, 9, 10) for other formulations, and for statements on weighted zeta functions of suspensions of shifts. The zeta function $\zeta^*(s)$ *may* have a non-polar singularity (Gallavotti 1976, Pollicott 1986) arbitrarily close to the bound δ in Theorem 2.7 (the constructions are inspired from the Fisher (1967) droplet model, see also §5.2). Theorem 2.7 may be reformulated for Axiom A flows using an appropriate counting procedure (see Parry and Pollicott (1990, ch. 9 and Appendix III) for details).

THEOREM 2.8. (Axiom A flow zeta function) *Let Φ be a C^1 Axiom A flow on a transitive basic set, with topological entropy h^*_{top}, and contraction coefficient $\theta = e^{-\gamma} < 1$. The zeta function $\zeta^*(s)$ is analytic and non-zero in the half-plane $\Re s > h^*_{\text{top}}$, and has a meromorphic extension to the half-plane $\Re s > h^*_{\text{top}} - (\gamma/2)$, with poles only when some $\mathcal{L}_{\exp(-sr_i)}$ acting on a suitable space has 1 as an eigenvalue, where $r_i : \Sigma_{A_i} \to \mathbb{R}_+$ is the return time arising from a Markov section. If Φ is topologically weak-mixing, there exists an open neighbourhood of the half-plane $\Re s \geq h^*_{\text{top}}$ in which the only singularity of $\zeta^*(s)$ is a simple pole at $s = h^*_{\text{top}}$.*

The following result is a consequence of Theorem 2.8 and Tauberian theorems.

THEOREM 2.9. (Prime orbit theorem) (Parry and Pollicott 1983) *Let Φ be a C^1 topologically weak-mixing Axiom A flow with topological entropy h^*_{top}. Then, if $\Pi(t)$ denotes the number of primitive periodic orbits τ such that $\exp(h^*_{\text{top}} \cdot \ell(\tau)) \leq t$, we have*

$$\Pi(t) \sim \frac{t}{\log t} \quad \textit{when } t \to \infty, \quad \textit{i.e. } \lim_{t\to\infty} \left| \frac{\Pi(t)}{(t/\log(t))} \right| = 1. \tag{2.17}$$

Another application of Theorem 2.8 is the proof of the regularity of the metric entropy of an Anosov flow when the flow is varied (Katok *et al* (1989); see Contreras (1992) for a more precise result, using a thermodynamic formalism but no zeta functions).

Just as in Theorem 2.6 for the discrete-time case, it is possible to relate the analytic properties of the Fourier transform of the (continuous-time) correlation functions associated to equilibrium states for a suspension of a subshift of finite type and Hölder continuous observables with the poles of the correspondingly weighted zeta function (Pollicott 1985, Ruelle 1987b, Haydn 1990a). Again, the case when the weight G is related to the unstable Jacobian of the flow is of special interest since it corresponds to the physical SRB measure (Bowen and Ruelle (1975), Bowen (1975, ch. 4)). In particular, a necessary condition to guarantee (via Paley–Wiener theorems) exponential decay of correlations is the existence of a vertical pole-free strip $P^*(\Re G) - \delta < \Re z \le P^*(\Re G)$, with the exception of the simple pole at $z = P^*(\Re G)$, where $P^*(\Re G)$ is the topological pressure of $\Re G$ with respect to the flow (see Bowen and Ruelle (1975)). However, Theorem 2.7, or its weighted analogues, do *not* exclude accumulation of poles along the vertical $\Re s = P^*(\Re G)$ for a weak-mixing flow. In fact, it is possible to construct examples of weak-mixing Axiom A flows with correlation functions (for equilibrium states of Hölder potentials) decaying arbitrarily slowly (Ruelle 1983, Pollicott 1984). It has been known for some time (Moore 1987, Ratner 1987, Collet *et al* 1984) that the correlation function decays exponentially in the case of geodesic flows on manifolds of *constant negative curvature*. The question of whether Anosov flows, or just geodesic flows on surfaces of non-constant negative curvature, have exponentially decaying correlation functions remained open for a long time. Recently, Chernov (1995) obtained, by using Markov approximations, a *subexponential* decay property ($C_{\varphi,\psi}(t) \le K_{\varphi,\psi} e^{-\beta\sqrt{t}}$ with $\beta > 0$) for the correlation function associated with the SRB measure of Anosov flows satisfying a uniform non-integrability condition (which basically implies that the stable and unstable foliations are Lipschitz) on three-manifolds. See Liverani (1996) for a conceptualized extension of Chernov's approach to higher dimensions, which explicitly uses stochastic perturbations of the flow. (The approaches of Chernov and Liverani do not seem to have connections with dynamical zeta functions.) More recently, Dolgopyat (1996a) proved *exponential decay of correlations* for the SRB measure and Hölder observables, in the case of $C^{2+\epsilon}$ weak-mixing Anosov flows on compact manifolds, with C^1 stable and unstable foliations (this smoothness requirement, which is satisfied in particular by geodesic flows in negative curvature, replaces in some sense Chernov's uniform non-integrability assumption). His result is based on a refined study of the spectral radius of operators $\mathcal{L}_{ge^{-sr}}$ from the above-mentioned approach of Pollicott and Ruelle. The proof also shows that the corresponding weighted zeta function is analytic in a half-plane $\Re s > P^*(\Re G)$ with the exception of the simple pole at $s = P^*(\Re G)$. Dolgopyat (1996a, 1996b) also showed that correlation functions associated to equilibrium states coming from Hölder weights, for Hölder observables, decay rapidly in the sense of Schwartz for C^∞ weak-mixing Anosov flows on compact manifolds (without assuming smoothness of the stable and unstable foliations), and for more general Axiom A flows under additional assumptions. In this case, it follows from his proof that the relevant weighted zeta function is pole-free in a domain $\{|\Re z - P^*(\Re G)| \le |\Im z|^{-\xi}, z \ne P^*(\Re G)\}$ (for some $\xi > 0$).

3. *Smooth expanding dynamics and flat traces*

We now turn our attention to the smooth, locally expanding situation: we fix $\gamma > 1$ and $r \ge 2$ or $r = \omega$, and consider pairs (f, g), where $f : M \to M$ is a C^r and γ-expanding transformation of a compact manifold, i.e. for any $x \in M$ and any $v \in T_x M$ we have $\|Df_x v\| \ge \gamma \|v\|$ (such maps are automatically topologically mixing because they are factors of full shifts), and $g : M \to \mathbb{C}$ is a C^r weight. We associate a transfer operator to (f, g) via (2.3). (Again, one particularly interesting weight is $g = 1/|\operatorname{Det} Df|$.) In fact, many results in this section hold in the more general setting where the finitely many contracting local inverse branches of a map f are replaced by a finite, countable or even uncountable (in this case the sum in (2.3) should be replaced by an integral) family of contractions f_i defined on M, or subsets of M (see Ruelle (1990) and Fried (1995a)), paired with weights g_i (which can be replaced by vector bundle maps). For the sake of simplicity, we restrict this study, however, to the dynamical situation (f, g) (see §6 for a discussion where it is important to allow more flexibility).

We shall see that, although the zeta function (2.2) still describes part of the discrete spectrum of $\mathcal{L}_g$, a better generalized Fredholm determinant is obtained by replacing the counting trace with a *flat trace*

$$\operatorname{Tr}^\flat \mathcal{L}_g = \sum_{x \in \operatorname{Fix} f} \frac{g(x)}{|\operatorname{Det}(1 - Df_x^{-1}(x))|}, \tag{3.1}$$

(where f_x^{-1} is the local inverse branch of f such that $f_x^{-1}(x) = x$) so that

$$\operatorname{Tr}^\flat \mathcal{L}_g^n = \sum_{x \in \operatorname{Fix} f^n} \frac{\prod_{k=0}^{n-1} g(f^k x)}{|\operatorname{Det}(1 - Df_x^{-n}(x))|}. \tag{3.2}$$

For the reader's convenience, we reproduce from the survey Baladi (1995a, §3.1) a heuristic argument motivating the denominator in (3.1)–(3.2).

First, observe that $\mathcal{L}_g$ can be written as an operator with a (highly non-smooth) kernel:

$$\begin{aligned} \mathcal{L}_g \varphi(x) &= \int_M \delta(fy - x) \cdot g(y) \cdot |\operatorname{Det} Df(y)| \cdot \varphi(y)\, dy \\ &= \sum_i \int_M \delta(y - f_i(x)) \cdot g(y) \cdot \varphi(y)\, dy, \end{aligned} \tag{3.3}$$

where $\delta(\cdot)$ is the Dirac delta, and the f_i are the finitely many contracting inverse branches of f (here, we neglect the problem of overcounting of periodic points on boundaries discussed above Theorem 2.4). Forgetting that the Dirac delta is not a continuous function, we apply classical Fredholm theory (Riesz and Sz.-Nagy 1955) to compute formal traces, and find the same expression as in (3.1):

$$\text{`Tr'}\, \mathcal{L}_g = \sum_i \int_M \delta(x - f_i x) \cdot g(x)\, dx = \sum_{x \in \operatorname{Fix} f} \frac{g(x)}{|\operatorname{Det}(1 - Df_x^{-1}(x))|}, \tag{3.4}$$

where the determinant in the denominator of (3.4) follows from the change of variable formula. We may regularize the kernel of our transfer operator by convolving the dirac with smooth functions, producing trace-class operators whose traces converge to the flat trace (3.4). (Note, however, that this approach is *not* the one which has been implemented to obtain the results mentioned in this section, an exception being the strategy applied by Tangerman (1986), who used heat operators.) The 'damping' or 'flattening' convolution procedure just described explains the terminology 'flat', which also refers to the analogy with the work of Atiyah and Bott (1964, 1967, 1968).

The flat traces can be used to construct a *generalized Fredholm determinant* (flat determinant)

$$d_g^\flat(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{Tr}^\flat \mathcal{L}_g^n = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \operatorname{Fix} f^n} \frac{\prod_{k=0}^{n-1} g(f^k x)}{|\operatorname{Det}(I - Df_x^{-n}(x))|}. \tag{3.5}$$

In the special case when $g = 1/|\operatorname{Det} Df| < 1$, we have another expression for the flat determinant:

$$d^\flat_{1/|\operatorname{Det} Df|} = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \operatorname{Fix} f^n} \frac{1}{|\operatorname{Det}(Df^n(x) - I)|}. \tag{3.6}$$

The weighted dynamical zeta function (2.2) can formally be expressed as a finite product of auxiliary flat determinants: in dimension one, it is simple to check that $\zeta_g(z) = d^\flat_{(g/Df)}/d^\flat_g(z)$ (by definition $d^\flat_{(g/Df)}$ is the determinant associated with $\mathcal{L}^{(1)} = \mathcal{L}_{(g/Df)}$, which can be interpreted as the action of the pair (f, g) on 1-forms). In dimension $d \geq 2$ we may use the formula $\operatorname{Det}(1 - B) = \sum_{\ell=0}^{d} (-1)^\ell \operatorname{Tr}(\Lambda^\ell B)$ where $\Lambda^\ell B$ is the ℓth exterior product of the finite matrix B. (The corresponding operators $\mathcal{L}_g^{(\ell)}$ describe the action of (f, g) on ℓ-forms, see Ruelle (1976b).)

We shall see in §3.1 that the transfer operator acting on holomorphic functions is compact (even nuclear in the sense of Grothendieck (1955, 1956)) when the pair (f, g) is analytic, so that the flat trace is the 'true' trace and $d^\flat(z)$ is the Grothendieck–Fredholm determinant of $\mathcal{L}_g$. In particular, $d^\flat(z)$ is an entire function whose zeros in the plane are the inverses of (all) the eigenvalues of $\mathcal{L}_g$, whereas the poles of the dynamical zeta function $\zeta_g(z)$ coincide with the inverse eigenvalues only in a disc. (Particularly enlightening examples are those of the maps $w \mapsto w^2$ and $x \mapsto 2x (\operatorname{mod} 1)$, on the unit circle and interval, respectively, with weight $g = 1/2$.) In §3.2 we consider the case of finite differentiability, where the operator is only quasicompact (as in §2), but where the flat determinant again 'sees' more of the discrete spectrum than the zeta function.

3.1. *Analytic expanding systems.* Let $\gamma > 1$, M be a compact, connected, real analytic manifold and $f : M \to M$ be a real analytic, γ-expanding map. We consider a complex neighbourhood $\mathcal{U}$ of M, and we set $\mathcal{A}$ to be the Banach space of holomorphic functions on $\mathcal{U}$ with a continuous extension to the boundary. If $\mathcal{U}$ is not too big, f and g can be extended analytically to $\mathcal{U}$, preserving the γ-expanding property of f. We refer to Baladi (1995a, 4.1) for a heuristical explanation (in dimension one) of why $\mathcal{L}_g$ is a nuclear operator of order 0 in the sense of Grothendieck (1955) (see Mayer (1991a) for a very readable account of the mathematical argument). Before mentioning the main result, we present the idea of the proof of Mayer (1976) that the trace (3.1) is the sum of eigenvalues of $\mathcal{L}_g$ when $M = S^1$. For this, writing F_k, $k = 1, \ldots, d$, for the finitely many inverse branches of f (in particular, $\mathcal{L}_g = \sum_k \mathcal{L}_{g,k}$ with $\mathcal{L}_{g,k}\varphi = (g\cdot\varphi)\circ F_k$), and noting the unique fixed point of F_k by z_k, it suffices to show that the spectrum of each $\mathcal{L}_{g,k}$ acting on $\mathcal{A}$ coincides with the set of simple eigenvalues $\mathcal{E}_k = \{0, g(z_k)\cdot(DF_k(z_k))^\ell, \ell \geq 0\}$. Indeed, this would imply that

$$\mathrm{Tr}\,\mathcal{L}_g = \sum_{k=1}^{d} \mathrm{Tr}\,\mathcal{L}_{g,k} = \sum_{k=1}^{d}\sum_{\ell\geq 0} g(z_k)(DF_k(z_k))^\ell = \sum_{z_k\in \mathrm{Fix}\, f} \frac{g(z_k)}{|1 - Df_{z_k}^{-1}(z_k)|}. \tag{3.7}$$

We now show that the spectrum of each $\mathcal{L}_{g,k}$ is a subset of $\mathcal{E}_k$: the eigenvalue property means that $\mathcal{L}_{g,k}\varphi(z) = \lambda\varphi(z) = (g\cdot\varphi)(F_k(z))$ for all z. Specializing to $z = z_k$ gives $\lambda = g(z_k)$ if $\varphi(z_k) \neq 0$. If $\varphi(z_k) = 0$ but $D\varphi(z_k)\neq 0$, we find $\lambda = g(z_k)DF_k(z_k)$. The general case is $D^j\varphi(z_k) = 0$, $0\leq j < \ell$, $D^\ell\varphi(z_k)\neq 0$.

THEOREM 3.1. (Flat determinant for analytic expanding maps) (Ruelle 1976b, 1990, Fried 1986a) *Let $\gamma > 1$, M be a compact, connected, real analytic manifold, and let $f : M \to M$ be a real analytic, γ-expanding map. Let $g : M \to \mathbb{C}$ be real analytic. Then the function $d_g^\flat$ defined in (3.5) is entire of finite order, and its zeros are the inverses of the non-zero eigenvalues of the compact (in fact nuclear) operator $\mathcal{L}_g$ acting on the Banach space $\mathcal{A}$. The dynamical zeta function $\zeta_g(z)$ (2.2) can be written as a quotient of entire functions of finite order $\zeta_g(z) = \bar{d}(z)/\tilde{d}(z)$.*

A more general statement can be found in Fried (1995a). The convergence to zero of the kth eigenvalue of $\mathcal{L}_g$ is exponential in dimension one and subexponential otherwise (see Fried 1986a). One of the key ingredients of the proof of Theorem 3.1 is the Cauchy integral formula which allows one to write the transfer operator in (smooth) kernel form. Analogous results hold for analytic Anosov diffeomorphisms or flows, under a very strong assumption of *analyticity of the stable/unstable foliations* (Ruelle 1976b, Fried 1986a). This assumption is satisfied for geodesic flows on compact surfaces of *constant* negative curvature, and gives a dynamical proof that the Selberg zeta function is meromorphic in the whole complex plane. Besides applications to quantum chaos and the cohomology of Kleinian groups mentioned in the introduction, the analytic expanding flat determinants were used to study the spectrum of the Feigenbaum period-doubling operator (Vul *et al* 1984, Christiansen *et al* 1990, Eckmann and Epstein 1990, Jiang *et al* 1992). Mayer's (1990, 1991b) beautiful analysis of the thermodynamic formalism for the Gauss map (useful in studying the Selberg zeta function) contains a rare occurrence of a transfer operator which is not only trace class when acting on a Hilbert space, but is also self-adjoint.

For rational maps f of the Riemann sphere, much stronger properties can be proved. *Rationality* was obtained by Hinkkanen (1994) for unweighted zeta functions of rational maps, and by Hatjispyros (1997) and Hatjispyros and Vivaldi (1995) for the zeta functions of Chebyshev polynomials weighted by $(f')^k$. Waddington (1997) studied zeta functions associated with preperiodic points of hyperbolic rational maps. The striking results of Eremenko *et al* (1994) and Levin (1994), Levin *et al* (1991, 1994) on hyperbolic rational maps, in particular for some quadratic polynomials, have been briefly presented in Baladi (1995a, b).

Keller (1989) combined the Grothendieck–Fredholm approach together with the Hofbauer (1986) Markov extension presented in §5 to study piecewise invertible maps.

3.2. *Differentiable expanding systems.* Assume now that M is a compact connected C^∞ manifold, that $f : M \to M$ is C^r for some $1 \le r \le \infty$ and γ-expanding for some $\gamma > 1$, and that $g : M \to \mathbb{C}$ is C^r. We consider the Banach space $\mathcal{C}^r(M)$ of C^r functions $\varphi : M \to \mathbb{C}$ endowed with a norm $\|\cdot\|_r = \sum_{j=0}^r \|D^j \cdot\|_\infty$. The transfer operator (2.3) is again only quasicompact, but higher differentiability gives a better upper bound for the essential spectral radius.

THEOREM 3.2. (Quasicompactness) (Ruelle 1989) *Let $r \ge 1$, $\gamma > 1$, and let M be a differentiable compact connected manifold. Let $f : M \to M$ be C^r and γ-expanding, and let $g : M \to \mathbb{C}$ be C^r.*

(1) *The spectral radius of $\mathcal{L}_g : \mathcal{C}^r(M) \to \mathcal{C}^r(M)$ is bounded above by e^P (where $P = P(\log|g|) \in \mathbb{R} \cup \{-\infty\}$, defined in (2.4), is the spectral radius of $\mathcal{L}_{|g|}$ acting on bounded functions). If g is non-negative, the spectral radius coincides with e^P. If g is positive, e^P is a simple eigenvalue with a positive eigenfunction ψ_0 and the rest of the spectrum lies in a subset of a disc of radius strictly smaller than e^P.*

(2) *The essential spectral radius of $\mathcal{L}_g$ acting on $\mathcal{C}^r(M)$ is bounded above by e^P/γ^r.*

The first result in a differentiable, non-analytic setting was obtained by Tangerman (1986) who considered the C^∞ case and used a 'heat kernel' approach. The key bound used to obtain Theorem 3.2(2) is the following 'differentiable' version of (2.6) (which also appears in Tangerman's work): there exists $C > 0$ so that

$$\|\mathcal{L}_g^n \varphi\|_r \le C \sum_{j=0}^{r} \frac{\|D^j \varphi\|_\infty}{\gamma^{nj}}, \quad \forall \varphi \in \mathcal{C}^r(M), \forall n \in \mathbb{Z}^+. \tag{3.8}$$

The bound (3.8) is again proved by a combination of the chain rule and the (classical) Leibniz formula (the case $r = 1$ is essentially the same as (2.6), the reader is invited to check the case $r = 2$ as an exercise), see e.g. Fried (1995a, Lemma 1). To bound the essential spectral radius one then considers the sequence of operators $\mathcal{L}_g^n \Pi_n$, where Π_n is a finite rank projection constructed from local Taylor approximations of functions in $\mathcal{C}^r(M)$. (Contrary to the claim in Baladi (1995a, Proposition 3.1(2)), it is not known whether all complex numbers with modulus smaller than the essential spectral radius of $\mathcal{L}_g$ are eigenvalues.)

Exact formulas (as opposed to upper bounds) exist for the essential spectral radius in various settings: Collet and Isola (1991) obtained a formula for the one-dimensional case (see also Baladi *et al* (1996) for Hölder and Zygmund functions), Campbell and Latushkin (1997) have an expression of the essential spectral radius as a Lyapunov exponent, and Holschneider (1996) applied wavelet techniques to obtain the value of the essential spectral radius for transfer operators acting on a variety of functional Banach spaces (Besov, Triebel, Zygmund).

THEOREM 3.3. (Flat determinant) (Ruelle 1990) *Let $r \geq 1$. Let $f : M \to M$ be C^r and γ-expanding, and let g be C^r. The generalized Fredholm determinant $d_g^\flat(z)$ associated to (f, g) by (3.5) is analytic in the disc of radius $e^{-P}\gamma^r$ (with $P = P(\log|g|)$ defined by (2.4)), where its zeros are exactly the inverses of the eigenvalues of $\mathcal{L}_g : \mathcal{C}^r(M) \to \mathcal{C}^r(M)$ of modulus strictly larger than e^P/γ^r.*

Fried proved more general versions of Theorems 3.2 and 3.3, using in particular (Fried (1995a, §5)) a partition of unity to bypass the Manning-type overcounting argument used by Ruelle (1989). Fried (1995a, §4) also obtained control of the asymptotics of the eigenvalues.

We end this section with a very brief sketch of the proof of Theorem 3.3, when $M = S^1$, and for a strictly positive weight g. Assume (without restricting further generality) that $P = 0$ and that the operator $\mathcal{L}_g$ is normalized, i.e. preserves the constant function 1. Fixing some $\Theta > \gamma^{-r}$, Theorem 3.2(2) says that the spectrum of $\mathcal{L}_g : \mathcal{C}^r(M) \to \mathcal{C}^r(M)$ outside of the disc of radius Θ consists of $K(\Theta) < \infty$ eigenvalues λ_i, of finite algebraic multiplicity m_i, and we have the following *spectral decomposition*

$$\mathcal{L}_g^n \varphi = \sum_{i=1}^{K(\Theta)} \lambda_i^n (\psi_i L_i^n \psi_i^* \varphi) + \mathcal{R}\mathcal{L}_g^n \varphi, \quad \forall \varphi \in \mathcal{C}^r(M), \ \forall n \in \mathbb{Z}^+, \tag{3.9}$$

where each L_i is a m_i-dimensional matrix in Jordan form, each ψ_i is a row vector of elements of a basis of a generalized eigenspace in $\mathcal{C}^r(M)$ for λ_i, and each ψ_i^* is a column of vectors forming a basis of the generalized eigenspace for $\mathcal{L}_g^*$ and λ_i. Finally, there exists $C > 0$ so that $\|\mathcal{L}_g^n \mathcal{R}\| \leq C\Theta^n$ for all $n \in \mathbb{Z}^+$. In (3.9) we have decomposed $\mathcal{L}_g^n$ into a finite rank operator $\mathcal{M}_n(\Theta)$ (the sum over i), the trace of which is trivially equal to $\sum_{i=1}^{K(\Theta)} m_i \lambda_i^n$, and an exponentially decaying correction $\mathcal{R}\mathcal{L}_g^n$.

Consider now a Markov partition for the circle map f, fix some $n \in \mathbb{Z}^+$, and write $\mathcal{Z}_n$ for the nth refinement of the partition under the dynamics (we neglect the boundary problems which are in fact quite troublesome, especially in higher dimensions). Write χ_η for the characteristic function of $\eta \in \mathcal{Z}_n$, and choose a point x_η in each η, taking it to be a fixed point of f^n if it is possible. A crucial consequence of the Markov property is the dichotomy

$$(\mathcal{L}_g^n \chi_\eta)(x_\eta) = \begin{cases} g^{(n)}(x_\eta) & \text{if } f^n x_\eta = x_\eta, \\ 0 & \text{otherwise,} \end{cases} \tag{3.10}$$

where we introduced the notation $g^{(n)}(x) = \prod_{k=0}^{n-1} g(f^k(x))$. For $0 \leq q \leq r-1$ we set $e_{q,\eta}(x) = (x - x_\eta)^q \cdot \chi_\eta$, and $e_{q,\eta}^*(\varphi) = (1/q!) D^q(\varphi)(x_\eta)$. We may then rewrite the left-hand side of (3.10) as $e_{0,\eta}^*(\mathcal{L}_g^n e_{0,\eta})$, and yet another application of the chain rule and the Leibniz formula (most terms cancel in the process) shows that for all $0 \leq q \leq r-1$

$$e_{q,\eta}^*(\mathcal{L}_g^n e_{q,\eta}) = \frac{1}{q!} D^q(\mathcal{L}_g^n e_{q,\eta})(x_\eta) = \begin{cases} (Df_{x_\eta}^{-n}(x_\eta))^q \cdot g^{(n)}(x_\eta) & \text{if } f^n x_\eta = x_\eta, \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Observe now that by using (3.11), we may rewrite the flat trace (3.2) as

$$\begin{aligned}\operatorname{Tr}^\flat \mathcal{L}_g^n &= \sum_{x\in \mathrm{Fix}\, f^n} \frac{g^{(n)}(x)}{1-D(f_x^{-n}(x))}\\ &= \sum_{x\in \mathrm{Fix}\, f^n} \sum_{q=0}^{r-1} g^{(n)}(x)(D(f_x^{-n}(x))^q + E_n\\ &= \sum_{\eta\in\mathcal{Z}_n}\sum_{q=0}^{r-1}\frac{1}{q!}D^q(\mathcal{L}_g^n e_{q,\eta})(x_\eta) + E_n, \end{aligned} \tag{3.12}$$

where $|E_n| \le C/\gamma^{rn}$ is the remainder from a geometric series. Using the spectral decomposition (3.9) to expand the terms in the double sum in (3.12), we find by adding and subtracting the trace of $\mathcal{M}_n(\Theta)$ (which can also be written $\sum_{i=1}^{K(\Theta)} \lambda_i^n (L_i^n \psi_i^*)^* \psi_i$) that

$$\begin{aligned}&\sum_{\eta\in\mathcal{Z}_n}\sum_{q=0}^{r-1}\frac{1}{q!}D^q(\mathcal{L}_g^n e_{q,\eta})(x_\eta)\\ &\quad= \operatorname{Tr}\mathcal{M}_n(\Theta) + \left[\sum_{\eta\in\mathcal{Z}_n}\sum_{q=0}^{r-1}\frac{1}{q!}D^q(\mathcal{M}_n(\Theta)e_{q,\eta})(x_\eta) - \operatorname{Tr}\mathcal{M}_n(\Theta)\right]\\ &\qquad+\sum_{\eta\in\mathcal{Z}_n}\sum_{q=0}^{r-1}\frac{1}{q!}D^q\mathcal{R}\mathcal{L}_g^n(\Theta)e_{q,\eta}(x_\eta)\\ &\quad= \sum_{i=1}^{K(\Theta)} m_i\lambda_i^n + d_n^{(1)} + d_n^{(2)}, \end{aligned} \tag{3.13}$$

with

$$d_n^{(1)} = \sum_{i=1}^{K(\Theta)} \lambda_i^n (L_i^n\psi_i^*)^*\left(\sum_{\eta\in\mathcal{Z}_n}\sum_{q=0}^{r-1}\frac{1}{q!}D^q(\psi_i)(x_\eta)\cdot e_{q,\eta} - \psi_i\right),$$

and

$$d_n^{(2)} = \sum_{\eta\in\mathcal{Z}_n}\sum_{q=0}^{r-1}\frac{1}{q!}D^q\mathcal{R}\mathcal{L}_g^n(\Theta)e_{q,\eta}(x_\eta).$$

Since $\sum_{\eta\in\mathcal{Z}_n}\sum_{q=0}^{r-1}(1/q!)D^q(\psi_i)(x_\eta)\cdot e_{q,\eta}$ is just the sum of the local-order-r Taylor approximations of ψ_i (on intervals η of lengths of the order γ^{-n}), it is relatively straightforward to prove that $|d_1^{(n)}| \le C\gamma^{-rn}$. The bound $|d_2^{(n)}| \le C\Theta^n$ is more involved, and uses the information we have on the decay of $\mathcal{R}\mathcal{L}_g^n$ together with a telescoping argument due to Haydn (1990b).

Combining theorems from Ruelle (1990) and techniques developed in Baladi and Young (1993), the results in this section (and §5.1) have been extended to *random* settings, especially in the small noise situation. In Baladi (1997), annealed transfer operators and *annealed random dynamical zeta functions* $\zeta^{(a)}(z)$ (or Fredholm determinants) are defined by averaging over all possible closed random orbits $f_{\vec\omega}^{(n)}(x) = f_{\omega_{n-1}}\circ\cdots\circ f_{\omega_1}\circ f_{\omega_0}(x)$

$$\zeta^{(a)}(z) = \exp\sum_{n=1}^{\infty}\frac{z^n}{n}\int\sum_{f_{\vec\omega}^{(n)}(x)=x}\prod_{k=0}^{n-1} g_{\omega_k}(f_{\vec\omega}^{(k)}(x))p(d\omega_0)\dots p(d\omega_{n-1}). \tag{3.14}$$

It seems an interesting and non-trivial question to determine conditions ensuring that the *quenched random dynamical zeta function*

$$\zeta^{(q)}_{\vec\omega}(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{f^{(n)}_{\vec\omega}(x)=x} \prod_{k=0}^{n-1} g_{\omega_k}(f^{(k)}_{\vec\omega}(x)) \tag{3.15}$$

(or determinant) has poles (respectively zeros) which (for almost all $\vec\omega$) describe the Lyapunov spectrum of the corresponding ergodic product of random transfer operators $\mathcal{L}_{\omega_i}$ on $C^r(M)$. See Bogenschütz (1997) for recent results on the Lyapunov spectrum of such random operator cocycles.

4. *Smooth hyperbolic dynamics and flat traces*

The fact that the stable and unstable foliations of Axiom A or Anosov dynamical systems are usually only Hölder continuous, even for analytic diffeomorphisms or flows, is a major obstruction to the proof that the corresponding zeta functions admit meromorphic extensions to large domains: an alternative to the (at most Lipschitz) symbolic approach described in §2 is to construct an expanding system by projecting along stable manifolds, but this system will only be as smooth as the foliation. A dual description of this difficulty is the observation that it is not obvious to construct a space of functions (or distributions) on the manifold for which a transfer operator associated to the full hyperbolic dynamics f (as in (1.3)) is 'smoothness improving', i.e. reduces the higher-order part of the norm in the sense of (2.6) or (3.8). (See Liverani (1995) for such a construction.) In very vague terms, the distributions should be smooth along unstable manifolds but 'dual to smooth' along stable ones. A major breakthrough was obtained in the early 1990s by Rugh (1992, 1995, 1996a) who proved that the flat determinant is an entire function for analytic hyperbolic diffeomorphisms on surfaces (with an analogous statement for flows on three-dimensional manifolds). Fried (1995b) then gave a more conceptual and more general analysis, extending the results to higher dimensions. (Both Rugh and Fried's approaches involve an application of the Grothendieck theory as in Ruelle (1976b), and a combinatorial part based on Markov partitions using versions of Manning's (1971) counting argument.) Kitaev (1995a) then considered the technically much more difficult case of finite differentiability.

We now state the simplest possible version of the main results of Rugh, Fried, and Kitaev. We first define a continuous-time version of the *flat generalized Fredholm determinant*: for $\Phi^t : M \to M$ a flow with at most countably many periodic orbits and $g : M \to \mathbb{C}$ bounded, let

$$d^{\flat *}_g(s) = \exp - \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\tau \text{ primitive periodic orbit}} \frac{\exp(\int_0^{n\ell(\tau)} g(\Phi^t(x_0(\tau)) - s\, dt))}{|\operatorname{Det}(I - (DP_\tau^{-1})^n(\tau))|}, \tag{4.1}$$

where $\ell(\tau)$ is the primitive length of τ, $x_0(\tau)$ is an arbitrary point of τ, and DP_τ is the linearized Poincaré map of Φ for τ. The determinant (4.1) is neither a (Ruelle) dynamical zeta function like (2.14) nor exactly a Selberg zeta function, but something 'in between' (Fried (1995b, p. 179) uses the terminology 'correlation zeta function').

THEOREM 4.1. (Hyperbolic analytic determinant) *Let M be a compact connected analytic manifold, $g : M \to \mathbb{C}$ an analytic function, and $f : M \to M$ an analytic Axiom A diffeomorphism or $\Phi^t : M \to M$ an analytic Axiom A flow.*

(1) (Rugh 1992, 1996a) *The flat generalized Fredholm determinant $d_g^\flat(z)$ associated to (f, g) by (3.5) is an entire function if M is two-dimensional. The flat generalized Fredholm determinant $d_g^{\flat *}(s)$ associated to (Φ, g) by (4.1) is an entire function if M is three-dimensional.*

(2) (Fried 1995b) *The flat generalized Fredholm determinants $d_g^\flat(z)$ and $d_g^{\flat *}(s)$ defined by (3.5) and (4.1) extend to meromorphic functions in $\mathbb{C}$ (in any dimension).*

The flat determinants (3.5) and (4.1) are expected (Rugh 1995) to describe the correlation spectra, in particular to have zeros in bijection with the SRB correlation spectrum for the weight $g = 1/|\operatorname{Det} Df|$ (in the discrete time case), but this still requires some investigation. Our normalization of the flat determinants is consistent with that in §§2 and 3, but differs from the one used by Fried and Rugh, where $|\operatorname{Det}(Df^n(x) - I)|$ instead of $|\operatorname{Det}(I - Df^{-n}(x))|$ (and analogously for flows) appears in the denominator, and where the correlation spectrum of the SRB measure should correspond to $g \equiv 1$.

We now comment briefly on Theorem 4.1, limiting our discussion to the case of discrete-time dynamics. In both approaches, the transfer operator $\varphi \mapsto (g\varphi) \circ f^{-1}$ is not analyzed globally. In order to obtain tractable (local) nuclear operators, the manifold and dynamics are broken down into local pieces using Markov sections. A Manning-type argument is used to put the pieces together again: this is the reason why one only gets a meromorphic and not an entire function in Theorem 4.1(2). Rugh (1996a) conjectured that both flat determinants are actually entire functions in any dimension, i.e. that all 'poles' are artefacts from the trick to suppress boundary overcounting. (Rugh obtains the analyticity of the determinants in $\mathbb{C}$ in low dimension, by showing that all possible 'poles' are removable singularities.) Theorem 4.2 below proves Rugh's conjecture for C^∞ Anosov maps and C^∞ weights.

Rugh's (1992, 1996a) key idea was to write a two-dimensional (complexified, local) hyperbolic analytic diffeomorphism $(z_1', z_2') = f(z_1, z_2)$ on $D_1 \times D_2$ as

$$f(z_1, \psi_s(z_1, z_2')) = (\psi_u(z_1, z_2'), z_2'), \tag{4.2}$$

where both *pinning coordinates* ψ_u, ψ_s are analytic contractions (D_1 and D_2 are close to the stable, respectively unstable, direction). The transfer operator $\mathcal{L}_{(g/|\operatorname{Det} Df|)}(\varphi) = (\varphi \cdot g/|\operatorname{Det} Df|) \circ f^{-1}$ can then be written using a Cauchy integral

$$\mathcal{L}_{(g/|\operatorname{Det} Df|)}\varphi(z_1', z_2') = \int_{\partial D_1}\int_{\partial D_2} \frac{dz_1}{2i\pi}\frac{dz_2}{2i\pi}\frac{\epsilon_f \partial_1\psi_u(z_1, z_2')g(z_1, z_2)}{z_2 - \psi_s(z_1, z_2')}\frac{\varphi(z_1, z_2)}{z_1' - \psi_u(z_1, z_2')}, \tag{4.3}$$

where $\epsilon_f \in \{-1, 1\}$ is a well-chosen sign. The operator $\mathcal{L}_g$ can then be proved to be nuclear (Grothendieck 1955) when acting on the tensor product of functions holomorphic in D_1 with functions analytic *outside* of D_2, and its trace (in the ordinary sense) can be evaluated by Cauchy integration

$$\operatorname{Tr}\mathcal{L}_g = \frac{g(z_*)}{|\operatorname{Det}(I - Df^{-1}(z_*))|} = \operatorname{Tr}^\flat \mathcal{L}_g, \tag{4.4}$$

where z_* is the unique fixed point of f in $D_1 \times D_2$. Fried (1995b) extended the above procedure to a much more general setting. He introduced the notion of the *cross map* $C = (c_1, c_2) : D_1 \times D_2' \to D_1' \times D_2$ of a (local) hyperbolic map $f : D_1 \times D_2 \to D_1' \times D_2'$ (or more generally, of a hyperbolic correspondence), which in the two-dimensional analytic setting is given by $C(z_1, z_2') = (\psi_u(z_1, z_2'), \psi_s(z_1, z_2'))$, and in general satisfies

$$f(z_1, c_2(z_1, z_2')) = (c_1(z_1, z_2'), z_2'). \tag{4.5}$$

(The order of the stable and unstable directions is not the same in the papers of Rugh and Fried; we have adopted Rugh's choice.) The cross map C is in some sense a (contracting) *partial inverse* of f. Under suitable assumptions, Fried then associates a transfer operator to a complexified (local) hyperbolic map defined by its cross map by considering the *partial adjoint* of $\tilde{C}$, the action of the complexification of C on volume forms in the second variable (which involves the partial Jacobian of C in the second variable). (We skip completely Fried's beautiful analysis of the Banach function spaces.) A functoriality property analogous to the naturality of Rugh's kernel is proved, and the trace of the transfer operator is shown to satisfy a formula similar to (4.4). For this, Fried uses a fixed-point formula due to Atiyah and Bott (1964) instead of the Cauchy formula applied by Rugh.

Theorem 4.1 can be used to study the dynamical zeta function (2.2), respectively (2.14) and (2.15), as explained after (3.5). As shown by Fried (1986a, b, 1988, 1995b) (see also Moscovici and Stanton (1991), Sánchez-Morgado (1996)), inspired by observations of Milnor, and Ray and Singer, one can sometimes express the *Ray–Singer or Reidemeister torsion* of an orthogonal (acyclic) representation $\alpha : \Pi_1(M) \to Gl(m, \mathbb{C})$ of a manifold M in terms of a special value of the dynamical zeta function (for the geodesic flow)

$$R(z) = \prod_{\tau \in \mathcal{P}} \mathrm{Det}(I - e^{-z\ell(\tau)} \alpha(\tau))$$

or the *torsion* dynamical zeta function

$$Z_\alpha(z) = \exp - \sum_{\tau \in \mathcal{P}} \sum_{n=1}^{\infty} \frac{1}{n} e^{-zn\ell(\tau)} \epsilon_\tau \operatorname{Tr} \alpha(\tau)$$

(where $\mathcal{P}$ is the set of primitive closed orbits and $\epsilon_\gamma \in \{\pm 1\}$). The possible availability of this topological information gives a motivation for extending the domain of analyticity of these zeta functions.

We mention here the Fredholm determinant for semiclassical quantization introduced by Cvitanović *et al* (1993), the rigorous foundation of which is expected to be found in Theorem 4.1 and its extensions.

We now move to finitely differentiable systems.

THEOREM 4.2. (Differentiable hyperbolic determinant) (Kitaev 1995a) *Let M be a compact, connected C^∞ manifold, $r \geq 1$, $f : M \to M$ a C^r Anosov diffeomorphism with contraction constant $\theta < 1$, and $g : M \to \mathbb{C}$ a C^r function. Define*

$$\widehat{R}_g = \lim_{n \to \infty} \sup_{x \in M} \left| \prod_{k=0}^{n-1} g(f^k(x)) \right|^{1/n}. \tag{4.6}$$

Then the flat generalized Fredholm determinant $d_g^\flat(z)$ associated to (f, g) by (3.5) is an analytic function in the disc of radius $\theta^{-r/2}\widehat{R}_g^{-1}$. In particular, if f and g are C^∞ then the flat determinant is an entire function.

Note the analogy between the exponent $r/2$ in Theorem 4.2 (where comparison with Theorem 3.3 indicates a loss of one-half of the regularity 'because' of the co-existence of contraction and expansion) and the exponent $\alpha/2$ in the two sided α-Hölder case of Theorems 2.5 and 2.6 (see Lemma 2.3). The results announced in Kitaev (1995a) actually apply to *mixed transfer operators* constructed by summing over a family of transfer operators associated to differentiable systems, all hyperbolic with respect to the same cone field; also, the lower bound given there for the radius of convergence of the flat determinant is more precise than the rough one given in Theorem 4.2. The argument involves replacing the global operator by a *regular operator*, i.e. a sum of local operators (analogous to Rugh's (1992) rectangle maps and Fried's (1995a) system of hyperbolic correspondences), showing, however, that the determinant is unchanged in the process. This uses a partition of unity, a tool not available in the analytic setting of Theorem 4.1. Kitaev then analyses the local transfer operators by replacing them with *ϵ-perturbative* operators which are ϵ-close to operators associated with linear dynamics and constant weights, controlling the errors. Iterates of the ϵ-perturbative operators, restricted to suitable finite-dimensional subspaces of generalized functions, give rise to finite-dimensional matrices, whose traces approximate the flat traces of these iterates. (No pre-built machinery is used here.)

5. *Countable state dynamics in dimension one*

In this section and the next we restrict our study to one-dimensional maps and weights, but consider situations which allow for (countable) 'grammars', as opposed to the finite Markov symbolic dynamics which were used more or less explicitly in the expanding or hyperbolic cases of §§2 to 4. It will often be convenient to work with Banach spaces of functions admitting discontinuities, in general functions of bounded variation; allowing singularities, one also gives up the flat determinants, and reverts to the counting zeta function (2.2). The one-dimensional setting has also been a testing ground for extending the theory of §§2 to 4 to a non-uniformly hyperbolic situation (see §5.1, where the phenomenology does not change essentially, in particular operators still have gaps and correlation functions still decay exponentially), or even allowing neutral periodic orbits (§5.2), where the situation changes drastically.

5.1. *Uniformly and non-uniformly hyperbolic maps.* Let I be a compact interval, say $[0, 1]$, and consider a continuous map $f : I \to I$ for which there exists a finite (the extension to countable is possible under some technical assumptions) partition $0 = a_0 < a_1 < \cdots < a_N = 1$ into intervals such that $f|_{[a_i,a_{i+1}]}$ is strictly monotone. Recall that the variation of a function $\varphi : I \to \mathbb{C}$ is defined to be

$$\mathrm{var}_I\, \varphi = \sup\left\{ \sum_i |\varphi(x_i) - \varphi(x_{i-1})| \mid \{x_i\} \text{ finite ordered subset of } I \right\}. \tag{5.1}$$

The variation enjoys a rather nice change of variable formula since $\mathrm{var}_J\, \varphi \circ h = \mathrm{var}_{h(J)}\, \varphi$ (for $J \subset I$ an interval and $h : J \to h(J)$ a homeomorphism). It satisfies a Leibniz inequality

$$\mathrm{var}(\varphi\psi) \le \mathrm{var}\, \varphi \sup |\psi| + \sup |\varphi| \,\mathrm{var}\, \psi. \tag{5.2}$$

A more annoying bound is

$$\mathrm{var}_I(\chi_J \psi) \le \mathrm{var}_J\, \psi + 2 \sup_J |\psi|, \tag{5.3}$$

(where χ_J is the characteristic function of the interval $J \subset I$). The space $\mathcal{B}$ of functions $\varphi : I \to \mathbb{C}$ of bounded variation is endowed with the Banach norm $\|\varphi\| = \sup |\varphi| + \mathrm{var}\, \varphi$.

For $g : I \to \mathbb{C}$ of bounded variation, one defines the transfer operator $\mathcal{L}_g$ associated to (f, g) by (2.3). Although $\mathcal{L}_g$ usually does not preserve the Banach space of continuous functions (an exception is when the partition $\mathcal{Z}$ of I into intervals of monotonicity of f satisfies a Markov property), it is not difficult to check that $\mathcal{L}_g$ is a bounded operator when acting on $\mathcal{B}$. Following results of Hofbauer and Keller (1982, 1984), an analogue of Theorems 2.1 and 2.2 was proved.

THEOREM 5.1. (Quasicompactness and zeta functions) (Baladi and Keller 1990) *Let $f : I \to I$ be a piecewise monotone map and let $g : I \to \mathbb{C}$ be a continuous map of bounded variation.*

(1) *The spectral radius of $\mathcal{L}_g : \mathcal{B} \to \mathcal{B}$ is not larger than $e^{P(\log|g|)}$ and coincides with $e^{P(\log|g|)}$ if $g > 0$. The essential spectral radius of $\mathcal{L}_g$ is equal to*

$$\widehat{R}_g := \limsup_{n\to\infty} \sup_{x\in I} \left| \prod_{k=0}^{n-1} g(f^k(x)) \right|^{1/n}. \tag{5.4}$$

(2) *Assume that the partition $\mathcal{Z}$ into intervals of monotonicity of f is generating (i.e. the maximal length of the intervals of monotonicity of f^n tends to zero when $n \to \infty$). Then the dynamical zeta function $\zeta_g(z)$ defined by (2.2) is analytic in the disc of radius $e^{-P(\log|g|)}$ and admits a meromorphic extension to the disc of radius $\widehat{R}_g^{-1}$, where its poles are exactly the inverses of the eigenvalues of $\mathcal{L}_g$ outside of the disc of radius $\widehat{R}_g$ (the order of the pole coincides with the algebraic multiplicity of the eigenvalue).*

Theorem 5.1 is mainly interesting when $\widehat{R}_g < e^{P(\log|g|)}$. If f is piecewise C^1, we get a strict inequality for the natural weight $g = 1/|f'|$ (for which a fixed point of the transfer operator corresponds to an absolutely continuous invariant measure for f, since the dual $\mathcal{L}_g^*$ preserves the Lebesgue measure) if $\sup g < 1$, i.e. if the map is *piecewise expanding*.

To prove the upper bound for the essential spectral radius one considers the sequence of finite-rank operators $\mathcal{L}_g^n \Pi_n$, where Π_n is a projection to functions constant on the intervals of monotonicity of f^n, using the basic properties of the variation semi-norm mentioned above. The lower bound had been obtained by Keller (1984). The proof of the result concerning zeta functions is very similar to the proof of Theorem 2.2 if the partition into intervals of monotonicity is Markov. In the general case, a Markov extension due to Hofbauer (1986) is used: the tower map $\hat{f} : \widehat{I} \to \widehat{I}$ (with $\pi : \widehat{I} \to I$, such that $\pi \circ \hat{f} = f \circ \pi$) is a piecewise monotone map defined on a countable family of intervals, possessing a countable Markov partition with good combinatorics at infinity. One proves the desired result for the lifted objects $\widehat{\mathcal{L}}_{\hat{g}}$ and $\hat{\zeta}_{\hat{g}}$ and then pushes them back downstairs.

An elegant alternative proof of Theorem 5.1 was obtained by Ruelle (1994) who viewed $f : I \to I$ as embedded in the full shift with d symbols (the weight function in the extended space still has bounded variation). The continuity assumption of g in Theorem 5.1 can be suppressed (Ruelle 1994, Baladi 1995b).

We mention now results of Keller and Nowicki (1992), Young (1992), and Ruelle (1993), which apply in particular to some smooth unimodal interval maps f with $g = 1/|f'|$, where the function $1/|f'|$ is unbounded (in particular not in $\mathcal{B}$) so that Theorem 5.1 does not apply.

We first introduce some exponents measuring the hyperbolicity of a piecewise monotone interval map $f : I \to I$. The *cylinder decay exponent* is defined by

$$\lambda_\eta := \inf_n \inf_{\eta \in \mathcal{Z}_n} |\eta|^{-1/n}. \tag{5.5}$$

Assuming that f is C^1, we introduce the *hyperbolicity exponent of periodic orbits*

$$\lambda_{\text{per}} := \inf_n \inf_{x \in \text{Fix}\, f^n} |(f^n)'(x)|^{1/n}. \tag{5.6}$$

Assuming further that zero is the only critical point, the *Collet–Eckmann exponent* (exponential of the Lyapunov exponent of the critical value) is given by

$$\lambda_{\text{CE}} := \liminf_{n \to \infty} |(f^n)'(f(0))|^{1/n}. \tag{5.7}$$

If f has negative Schwarzian derivative then $\lambda_\eta > 1$ if and only if $\lambda_{\text{per}} > 1$ if and only if $\lambda_{\text{CE}} > 1$ (Nowicki and Sands (1996)). Once more we restrict our study to the simplest cases (more general S-unimodal maps can be considered, as well as different versions of the weight g).

THEOREM 5.2. *Let $f : [-1, 1] \to [-1, 1]$ be a quadratic map $f(x) = a - x^2$, and set $g = 1/|f'|$.*

(1) (Keller and Nowicki 1992) *Assume that $\lambda_{\text{CE}} > 1$. The weighted dynamical zeta function $\zeta_g(z)$ defined by (2.2) is meromorphic and non-zero in the disc of radius $\Theta := \max\{\sqrt{\lambda_{\text{CE}}}, \sqrt{\lambda_{\text{per}}}, \lambda_\eta\}$, where its poles coincide with the inverses of the eigenvalues of a transfer operator associated with a tower extension $\hat{f} : \hat{I} \to \hat{I}$ of f. In particular, $\zeta_g(z)$ is analytic in the open unit disc, and if f is topologically mixing its only singularity on the closed disc is a simple pole at $z = 1$.*

(2) (Ruelle 1993) *In fact, $\zeta_g(z)$ extends to a meromorphic, non-vanishing function in the disc of radius $\lambda_{\text{per}} \geq \Theta$.*

The Collet–Eckmanns condition $\lambda_{\text{CE}} > 1$ was proved to imply exponential decay of correlations (for the unique absolutely continuous invariant measure and observables of bounded variation) for non-flat topologically mixing unimodal maps with negative Schwarzian derivative by Keller and Nowicki (1992), under some weak technical assumptions, using the spectral properties of the (Markov tower extension) transfer operator mentioned in Theorem 5.2(1). Young (1992) independently proved exponential decay of correlations, using a Benedicks–Carleson-type approach to construct a slightly different (non-Markov) tower extension. A much more general tower construction has been recently developed by Young (1996): this new tower (which has been used, in particular, to show exponential decay of correlations for the SRB measure of 'good' Hénon maps, Benedicks and Young (1996)) *does* satisfy a Markov property, so it could therefore perhaps be indicated that we study zeta functions of more complicated, higher-dimensional, non-uniformly hyperbolic systems. See §6 for alternatives.

Nowicki and Sands (1996) recently proved in the context of topologically mixing S-unimodal maps that the Collet–Eckmann condition $\lambda_{\mathrm{CE}} > 1$ is in fact *equivalent* to the property of exponential decay of correlations for a unique absolutely continuous invariant measure and observables of bounded variation. It is tempting to conjecture in the same context that the zeta function $\zeta_{1/|f'|}(z)$ admits a meromorphic extension to a disc of radius greater than 1, with a simple pole at $z = 1$ as the only singularity, *if and only if* $\lambda_{\mathrm{CE}} > 1$ (the 'if' direction follows from the results we stated). The modulus of the first singularity not equal to 1 of $\zeta_{1/|f'|}(z)$ seems to be a rather natural hyperbolicity exponent of the map. When this exponent is equal to 1, one could try to study the nature of the singularity on the unit circle. (See §5.2 for branch cuts.) The question of equivalence between the presence of a gap in the singularities of a weighted zeta function or weighted determinant, and the existence of a unique SRB measure satisfying exponential decay of correlations for Hölder observables, in the setting of Hénon maps is much more challenging, since zeta functions of Hénon maps are basically unexplored mathematically (see §6 for more comments).

We end with a few words about Ruelle's (1993) elegant proof of Theorem 5.2(2), based on an application of the *Bochner tube theorem* (see e.g. Bochner and Martin (1948)) which says that any function $F(u, s)$ which is holomorphic in two 'tubes'

$$T_i = \{(u, s) \in \mathbb{C}^2 \mid (\Re u, \Re s) \in K_i \subset \mathbb{R}^2\}, \quad i = 1, 2 \tag{5.8}$$

(where K_1, K_2 are two open domains of $\mathbb{R}^2$ with $K_1 \cap K_2 \neq \emptyset$), admits a holomorphic extension to the tube

$$T = \{(u, s) \in \mathbb{C}^2 \mid (\Re u, \Re s) \in K\}, \tag{5.9}$$

where $K \subset \mathbb{R}^2$ is the convex hull of $K_1 \cup K_2$. To apply the tube theorem, set $h_s(x) = h(x, s) = |f'(x)|^s$ for $\Re s > 0$, and continue $h(x, s)$ analytically. Introduce an auxiliary zeta function

$$d(z, s) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}\, f^n} \prod_{k=0}^{n-1} h_s(f^k x), \tag{5.10}$$

noting that $\zeta_{1/|f'|}(z) = 1/d(z, -1)$. Since the cardinality of Fix f^n is at most 2^n, the function $d(z, s)$ is holomorphic in the tube

$$T_1 := \{(u, s) = (\log z, s) \mid 2|z|\lambda_{\mathrm{per}}^{\Re s} = 2e^{\Re u}\lambda_{\mathrm{per}}^{\Re s} < 1\}. \tag{5.11}$$

Applying Theorem 5.1 to (f, h_s), and noting that $\widehat{R}_{|f'|^s} = (\widehat{R}_{|f'|})^{\Re s}$, we find that

$d(z,s) = 1/\zeta_{h_s}(z)$ is holomorphic in a second tube:

$$T_2 := \{(u,s) = (\log z, s) \mid |z|(\widehat{R}_{|f'|})^{\Re s} = e^{\Re u}, (\widehat{R}_{|f'|})^{\Re s} < 1\}. \tag{5.12}$$

The reader is invited to find and draw the bases K_1, K_2 of the tubes T_1, T_2, and (using the inequality $\widehat{R}_{|f'|} \geq \lambda_{\mathrm{per}}$) to verify that the convex hull of $K_1 \cup K_2$ contains the set $\{(u,s) \mid \Re u + \Re s \log \lambda_{\mathrm{per}} < 0\}$. The Bochner theorem yields that $d(z,s)$ is holomorphic for $|z|\lambda_{\mathrm{per}}^{\Re s} < 1$, which for $\Re s = -1$ gives the announced condition $|z| < \lambda_{\mathrm{per}}$.

Ruelle (1993) obtained a meromorphic extension of the zeta function, with no relation to spectral properties of a transfer operator or exponential decay of correlations. Pollicott (1995) was later able to extract information on decay of correlations from this analytic completion approach, under additional conditions.

5.2. *Parabolic maps and intermittency.* All the discrete-time results mentioned so far were for systems admitting enough hyperbolicity to guarantee exponential decay of correlations, proved by showing that an appropriate transfer operator acting on a well-chosen Banach space had a spectral gap (and the zeta function or generalized determinant a corresponding meromorphic extension). Situations where the spectral radius and essential spectral radius of the transfer operator coincide, with 'abnormal' (e.g. power-law) decay of correlations have been studied in statistical mechanics (see, in particular, Fisher (1967), and the rich literature on phase transitions). In dynamics, numerical experiments indicate (see e.g. Cvitanović *et al* (1997), Dahlqvist (1995, 1996)) that branch cuts in the zeta functions and 'phase transitions' should be expected when neutral periodic orbits are present in an otherwise hyperbolic system, but there are still few mathematical results. In fact the presence of a single fixed point with a zero Lyapunov exponent suffices to destroy the usual hyperbolic picture, in particular it may happen that there is no SBR measure, see e.g. the two-dimensional 'almost Anosov' model of Hu and Young (1995).

The term 'intermittency' was used by Pomeau and Manneville (1980) to describe a general class of dissipative dynamical systems at the boundary of the transition to turbulence. Here, we only mention two recent studies in dimension one (for systems with strong Markov properties). The first one, due to Isola, who applies inducing techniques, is concerned with differentiable maps, while the second, due to Rugh, requires analyticity in order to use the Grothendieck–Fredholm theory. Before discussing the two studies, we mention an example, studied by Gaspard and Wang (see Wang (1989)), where the zeta function of a linearized map can be computed explicitly:

$$f(x) = \begin{cases} \dfrac{x}{1-x} & x \leq 1/2, \\ 2x-1 & x > 1/2. \end{cases} \tag{5.13}$$

Indeed, one can associate to f a piecewise linear map $\hat{f}$ (with countably many pieces) with an explicitly computable zeta function:

$$\hat{\zeta}_{1/|\hat{f}'|}(z) = \frac{z}{(1-z)^2 \log(1/1-z)},$$

having a logarithmic branch point at $z = 1$.

The first set-up is as follows. Let $f : [0, 1] \to [0, 1]$ be such that there exists $q \in]0, 1[$ with $f|_{[0,q[}, f|_{[q,1]}$ strictly monotone increasing and C^1 with Hölder derivative, and $f([0, q[) = [0, 1[$, $f([q, 1]) = [0, 1]$ (in particular, $f(0) = 0$), and both inverse branches Lipschitz continuous. Assume that there exists $\gamma > 1$ such that $f'(x) > \gamma$ for $x \in [q, 1]$. Assume also that $f'(0) = 1$ and $f'(x) > 1$ for $x \in]0, q[$, and that there are constants $a > 0$, $s > 0$ so that

$$f(x) = x + ax^{1+s}(1 + u(x)) \text{ for } x \mapsto 0_+, \tag{5.14}$$

with $u(0) = 0$, and $u'(x) = \mathcal{O}(x^{t-1})$ for $x \mapsto 0_+$ for some $t > 0$. It is known (Thaler 1980) that such a map f admits a unique invariant σ-finite absolutely continuous measure ν (which is finite if and only if $s < 1$). Set $c_0 = 1$, $c_n = f_1^{-1}(c_{n-1})$. The countably many intervals on which the piecewise expanding (and piecewise surjective) induced map $A_n = [c_{n-1}, c_n]$ is monotonic have finite ν-measure. We may thus introduce

$$D(z) = \frac{1}{(1 - z)\sum_{n=0}^{\infty} \nu(A_{n+1})z^n}. \tag{5.15}$$

(It is intuitively clear that the lengths of the intervals A_n, on which the time to return to the good region is n, should play a key role in the properties of the zeta function, and more generally the statistical properties of f.) Isola (1996) proves that the coefficients a_n in the power expansion $D(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy

$$a_n \sim_{n\to\infty} \begin{cases} (1 + \mathcal{O}(1)n^{1-1/s})/\nu([0, 1]) & \text{if } 0 < s < 1, \\ \mathcal{O}(1)/\log n & \text{if } s = 1, \\ \mathcal{O}(1)n^{-1+1/s} & \text{if } s > 1. \end{cases} \tag{5.16}$$

He then uses the asymptotics (5.16) to study the analytic properties of $D(z)$. A main result announced in Isola (1996) is that the zeta function $\zeta_g(z)$ defined by (2.2) for the weight $g = 1/|f'|$ has a non-polar singularity at $z = 1$ and can be written

$$\zeta_g(z) = \frac{D(z)L(z)}{1 - z}, \tag{5.17}$$

where $L(z)$ is analytic in $|z| < 1$ and extends to a continuous function on $|z| \leq 1$, with $L(1) \neq 0$.

Isola's argument is based on a study of a family of transfer operators (more precisely, an operator-valued power series), associated to the induced piecewise expanding map which can be modelled by a countable full shift, and uses, in particular, results from Prellberg (1991). Isola has also studied the decay of correlations for the unique absolutely continuous measure when $0 < s < 1$, and other statistical properties (such as the scaling rate of test functions with finite average) when $s \geq 1$. We refer the reader to Lopes (1993) and Yuri (1995, 1996) for related works.

We introduce the analytic setting of the second result. Let $\Delta \subset \mathbb{C} \setminus \{0\}$ be a simply connected open domain containing an open sector $S_{R,\phi} = \{re^{i\xi}, -\phi \leq \xi \leq \phi, 0 < r < R\}$ of angle $\phi > \pi/2$. Let $f_i : \Delta \to \Delta$, $i = 1, 2$, be two injective analytic maps with continuous extensions to $\overline{\Delta}$. Assume that $\overline{f_2\Delta} \subset \Delta$ (i.e. f_2 is a contraction). Assume also that $\overline{f_1\Delta} \subset \Delta \cup \{0\}$ and that there are constants $a > 0$, $\epsilon > 0$ so that for $z \in \Delta$

$$f_1(z) = z - az^2 + \mathcal{O}(|z|^{2+\epsilon}). \tag{5.18}$$

Set $g_i = f_i'$ and define a transfer operator acting on analytic functions $\varphi : \Delta \to \mathbb{C}$ by

$$\mathcal{M}\varphi(z) = \varphi(f_1 z)g_1(z) + \varphi(f_2 z)g_2(z). \tag{5.19}$$

Write $\Xi_*^n = \{1, 2\}^n \setminus (1, \ldots, 1)$, and for any n-tuple $(i_1, \ldots, i_n) \in \Xi_*^n$, let $z_{i_1 \ldots i_n}$ be the (necessarily unique) fixed point of the composition $f_{i_1} \circ \cdots \circ f_{i_n}$ in $\overline{\Delta}$. (Note that the indifferent fixed point $z = 0$ does not appear.) Write $f^{(n)}_{i_1 \ldots i_n}(z_{i_1 \ldots i_n})$ for the derivative of $f_{i_1} \circ \cdots \circ f_{i_n}$ at $z_{i_1 \ldots i_n}$. Finally, define the generalized Fredholm determinant of (f_i, g_i) by

$$d(\lambda) = \exp - \sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \sum_{(i_1 \ldots i_n) \in \Xi_*^n} \frac{f^{(n)}_{i_1 \ldots i_n}(z_{i_1 \ldots i_n})}{1 - f^{(n)}_{i_1 \ldots i_n}(z_{i_1 \ldots i_n})}. \tag{5.20}$$

THEOREM 5.3. (Fatou coordinates) (Rugh 1996b) *Let* f_i, g_i *(*$i = 1, 2$*) and* $\mathcal{M}$ *be as above. Then there exists a Banach space* $\mathcal{H}$ *of functions defined on an open domain* U *containing the compact maximal invariant set of the pair* (f_1, f_2) *(except for* 0*), such that:*

(1) *the spectral radius of* $\mathcal{M}$ *is equal to* 1*; the spectrum decomposes into* $[0, 1] \cup \sigma_p$*;*
(2) *the points in* σ_p *are eigenvalues of finite multiplicity that can only accumulate at* 1 *and* 0*;*
(3) *the determinant* $d(\lambda)$ *(5.20) is holomorphic in* $\overline{\mathbb{C}} - [0, 1]$*, where its zeros are exactly the eigenvalues of* $\mathcal{M}$ *acting on* $\mathcal{H}$ *(the order of the zero coincides with the multiplicity of the eigenvalue). The function* $d(\lambda)$ *can be analytically extended from each side of* $[0, 1]$ *to an open neighbourhood of* $]0, 1[$.

We refer to Rugh (1996b) for the general statement, and a description of the abstract space $\mathcal{H}$, which is obtained by pulling back a Banach space of holomorphic functions via the Fatou coordinate conjugating f_1 to the translation $T(w) = w + 1$. The key insight is that the transfer operator in the Fatou coordinates is conjugated to the translation operator T, which can be written, when acting on functions Ψ expressible as Laplace transforms $\int_0^\infty \psi(t)e^{-wt}\,dt$, as

$$T\Psi(w) = \int_0^\infty e^{-t}\psi(t)e^{-wt}\,dt. \tag{5.21}$$

However, (5.21) is basically an explicit spectral decomposition of T showing, in particular, that its spectrum is $[0, 1]$. (See Contucci and Knauf (1997) for analogous results on the spectrum of the transfer operator of Farey type maps.)

Theorem 5.3 can be applied to an analytic two-branched interval map f whenever its local inverse branches satisfy the conditions on f_1, f_2. (In particular, such f do not admit finite absolutely continuous measures. One can nevertheless ask whether the discrete spectrum of $\mathcal{M}$ can be reinterpreted in terms of scaling rates.) The generalized determinant (5.20) is then just $d(\lambda) = d^\flat_{1/f'}(\lambda^{-1})/(1 - \lambda^{-1})$ with $d^\flat$ as defined by (3.5).

For other results on complex maps, we refer in particular to the extensive study of 'jump transformations' (inducing) associated to parabolic maps of Aaronson *et al* (1993), and the article of Denker *et al* (1996) on the transfer operators for rational transformations (where the subexponential approach to equilibrium is proved, see Haydn (1996) for an exponential control of the supremum norm of Hölder observables). See also Smirnov's (1996) spectral analysis of the transfer operator associated to polynomial Julia sets in the Riemann sphere, acting on Sobolev spaces.

6. *Kneading operators and sharp traces*

In addition to the approach described in §5, which was closely related to that used in §§2 to 4, another strategy based on the powerful *kneading theory* of Milnor–Thurston is available, for the moment in one real or complex dimension, and will be described next.

The dynamical zeta functions and generalized Fredholm determinants we have seen up to now involved sums over periodic points, requiring in particular the set of periodic points to be at most countable. In their pioneering paper on one-dimensional dynamics, Milnor and Thurston (1988) associated to any piecewise monotone interval map $f : I \to I$ (with finitely many, say N, monotonicity intervals) a *negative zeta function*

$$\zeta^-(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} 2\# \operatorname{Fix}^- f^n, \tag{6.1}$$

where the set of *negative fixed points* of f^n is

$$\operatorname{Fix}^- f^n = \{x \in I \mid f^n x = x,\, f^n \text{ monotone decreasing in a neighbourhood of } x\}.$$

The important fact, of course, is that $\operatorname{Fix}^- f^n$, is a finite set. The principle 'what goes down must go up' explains why it is natural to double the negative periodic points (if f is piecewise expanding it is not very difficult to show that $\zeta^-(z)$ is just the usual unweighted dynamical zeta function (1.1) of f, up to a simple polynomial factor).

Milnor and Thurston (1988) proved (using a homotopy argument) the surprising equation

$$\zeta^-(z) \operatorname{Det}(1 - D(z)) = 1, \tag{6.2}$$

where $D(z)$ is the *kneading matrix*, a finite $(N+1) \times (N+1)$ matrix, with coefficients power series associated to the itineraries of the turning points. These power series (the *kneading invariants*) embody a rather complete description of the map f, and the one-dimensional kneading theory is by now extremely well developed (see e.g. de Melo and van Strien (1993)). (Milnor and Thurston's version of (6.2) involves a trivial polynomial correction due to the fact that they worked with an $(N-1) \times (N-1)$ matrix, see Baladi and Ruelle (1994).)

One can rewrite $\zeta^-(z)$ as a *Lefschetz zeta function*, $\zeta^L(z)$ where all periodic points x are counted, but with a weight $L(x) \in \{0, -1, 1\}$ (cancellations may occur, in particular, in homtervals). This second formulation (Baladi and Ruelle 1994) makes it easier to define a weighted negative (Lefschetz) dynamical zeta function, especially if the weight is locally constant. Formula (6.2) was extended to weighted and 'non-functional' (where the local inverse branches of a given f are replaced by an arbitrary family of local homeomorphisms) situations in a series of papers (Baladi and Ruelle 1994, Baladi 1995c, Ruelle 1996a, Baladi and Ruelle 1996) where the 'usual' relationship between the poles of the zeta function and the inverse eigenvalues of a transfer operator acting on functions of bounded variation (or C^r with rth derivative of bounded variation, Ruelle (1996b)) was established. (See also Mori (1990, 1992).) Again we limit ourselves to simplest statements, referring also to the review in Baladi (1995b) for an outline.

Let $I \subset \mathbb{R}$ be a compact interval, and Ω be a finite set of indices. For each $\omega \in \Omega$, let $I_\omega \subset I$ be an open interval, $f_\omega : I_\omega \to f_\omega(I_\omega)$ a homeomorphism (setting also $\epsilon_\omega = 1$

if f_ω preserves orientation, $\epsilon_\omega = -1$ otherwise). Finally, let $g_\omega : I \to \mathbb{C}$ be of bounded variation (in particular, dg_ω is a complex measure), continuous, and supported in I_ω. We define a transfer operator acting on the space $\mathcal{B}$ of functions $\varphi : I \to \mathbb{C}$ of bounded variation (or just on bounded functions) by

$$\mathcal{L}\varphi = \sum_{\omega\in\Omega} (\varphi \circ f_\omega) g_\omega. \tag{6.3}$$

We also introduce a formal dual of $\mathcal{L}$

$$\widehat{\mathcal{L}}\varphi = \sum_{\omega\in\Omega} \epsilon_\omega (\varphi \circ f_\omega^{-1})(g_\omega \circ f_\omega^{-1}). \tag{6.4}$$

(Note that if the f_ω are the local inverse branches of a piecewise monotone interval map f then $\widehat{\mathcal{L}}\varphi = (\varphi \circ f)(g \circ f)\epsilon_f$, where ϵ_f is the 'sign of the slope of f', ignoring boundary problems.) Finally, writing $\|\cdot\|_\infty$ for the operator norm of $\mathcal{L}$ or $\widehat{\mathcal{L}}$ acting on bounded functions, we define

$$R := \lim_{n\to\infty} (\|\mathcal{L}^n\|_\infty)^{1/n}, \quad \widehat{R} := \lim_{n\to\infty} (\|\widehat{\mathcal{L}}^n\|_\infty)^{1/n}. \tag{6.5}$$

Up to exchanging $\mathcal{L}$ and $\widehat{\mathcal{L}}$, we can assume to fix ideas that $\widehat{R} \le R$.

To state the result we shall use the sign function

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{6.6}$$

which has the property that $\frac{1}{2} d(\operatorname{sgn})$ is δ_0 the Dirac mass at zero. Define also the *sharp trace* of the data $(\Omega, I_\omega, f_\omega, g_\omega)$ by

$$\operatorname{Tr}^\# \mathcal{L} := \sum_{\omega\in\Omega} \int_{I_\omega} dg_\omega(x) \tfrac{1}{2} \operatorname{sgn}(f_\omega x - x), \tag{6.7}$$

(definition (6.7) clearly extends to the iterates $\mathcal{L}^n$). The sharp trace has the trace property

$$\operatorname{Tr}^\#(\mathcal{L}_1\mathcal{L}_2) = \operatorname{Tr}^\#(\mathcal{L}_2\mathcal{L}_1) \tag{6.8}$$

for any transfer operators $\mathcal{L}_1$, $\mathcal{L}_2$ of the form (6.3).

THEOREM 6.1. (Sharp traces and sharp determinants) *Assume that $\widehat{R} \le R$.*

(1) (Ruelle 1991, 1996a) *The spectral radius $\rho(\mathcal{L})$ of $\mathcal{L}$ acting on $\mathcal{B}$ satisfies $\widehat{R} \le \rho(\mathcal{L}) \le R$. If all functions g_ω are real and non-negative, then $\rho(\mathcal{L}) = R$, if, additionally, $\widehat{R} < R$ then R is an eigenvalue with non-negative eigenfunction.*

(2) (Baladi and Ruelle 1996) *The sharp determinant*

$$\operatorname{Det}^\#(1 - z\mathcal{L}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \operatorname{Tr}^\# \mathcal{L}^m. \tag{6.9}$$

defines a holomorphic function in the disc of radius $\widehat{R}^{-1}$ where its zeros are exactly the inverses of the eigenvalues of $\mathcal{L}$ of modulus at least $\widehat{R}$. The order of the zero coincides with the algebraic multiplicity of the eigenvalue.

An application of integration by parts and change of variables rule in an integral yields $\mathrm{Tr}^{\#}\,\widehat{\mathcal{L}} = -\,\mathrm{Tr}^{\#}\,\mathcal{L}$. We thus have the *functional equation*

$$\mathrm{Det}^{\#}(1 - z\widehat{\mathcal{L}}) = \frac{1}{\mathrm{Det}^{\#}(1 - z\mathcal{L})}. \tag{6.10}$$

Note also that one can rewrite $\mathrm{Det}^{\#}(1 - z\mathcal{L})$ as a (weighted) Lefschetz zeta function whenever the set of periodic orbits is countable (using $d(\mathrm{sgn}) = \delta_0$).

Theorem 6.1(1) improves and generalizes Theorem 5.1 above (we emphasize that Theorem 6.1 holds without any transversality assumption on the f_ω, which could, for example have uncountably many fixed points). We give a brief sketch of Ruelle's (1996b) elegant proof of the upper bound for the essential spectral radius, where the Leibniz formula appears in a more explicit avatar than (2.7). We start from the fact that the derivative gives a Banach space isomorphism between the space of functions of bounded variation, quotiented by functions vanishing except on an at most countable set, and the space of finite complex measures. We then consider the operator $\overline{\mathcal{L}} = d\mathcal{L}d^{-1}$ acting on finite measures and observe that the Leibniz rule produces a decomposition $\overline{\mathcal{L}} = \overline{\mathcal{L}}_1 + \overline{\mathcal{L}}_2$, where $\overline{\mathcal{L}}_1 = \sum_\omega g_\omega \epsilon_\omega (f_\omega^{-1})^*$ obviously has spectral radius bounded by $\widehat{R}$, and where $\overline{\mathcal{L}}_2 = \sum_\omega dg_\omega d^{-1}$ is compact because the integration operator d^{-1} is compact.

The proof of Theorem 6.1(2) is by regularization. Specifically, it uses a family of almost trace class operators, the *kneading operators* $\mathcal{D}(z)$ which are analogues of Milnor and Thurston's kneading matrix. These operators act on $L^2(d\mu)$ where the auxiliary measure μ is set to be $\mu = \sum_\omega |dg_\omega| + \sum_\omega |d(g_\omega \circ f_\omega^{-1})|$ (so as to guarantee the existence of the Radon–Nikodym derivatives $dg_\omega/d\mu$). They are defined by

$$\mathcal{D}(z) = z\mathcal{N}(1 - z\mathcal{L})^{-1}S\varphi, \tag{6.11}$$

for any z not in the spectrum of $\mathcal{L}$, where the (smoothness improving, compact) integration operator S is defined by

$$\mathcal{S}\varphi(x) = \int_I \tfrac{1}{2}\,\mathrm{sgn}(x - y)\varphi(y)\,d\mu(y), \tag{6.12}$$

and where the auxiliary operator $\mathcal{N}$ is given by

$$\mathcal{N}\varphi = \sum_{\omega\in\Omega} (\varphi \circ f_\omega)\frac{dg_\omega}{d\mu}. \tag{6.13}$$

Since $\mathcal{D}(z)\varphi(x) = \int \mathcal{D}_{xy}(z)\varphi(y)\,d\mu(y)$ has a bounded kernel, whenever $1/z$ is not in the spectrum of $\mathcal{L}$, it is a Hilbert–Schmidt operator in $L^2(d\mu)$ so that the regularized determinant $\mathrm{Det}_2(1+\mathcal{D}(z))$ of order two is well-defined (Simon 1979). We may thus set

$$\mathrm{Det}^*(1 + \mathcal{D}(z)) := \exp\left[\int \mathcal{D}_{xx}(z)\,d\mu(x)\right]\mathrm{Det}_2(1 + \mathcal{D}(z)). \tag{6.14}$$

The key identity in the proof of Theorem 6.1(2) is the following analogue of (6.2):

$$\mathrm{Det}^{\#}(1 - z\mathcal{L})\,\mathrm{Det}^*(1 + \mathcal{D}(z)) = 1. \tag{6.15}$$

The first proof of (6.15) in Baladi and Ruelle (1996) was by a series resummation argument (involving repeated use of integration by parts and change of variables). A more conceptual proof can be extracted from the method used in a one-dimensional complex partial analogue of Theorem 6.1 (Baladi *et al* 1995). We do not state the complex result for lack of space, mentioning only that $\mathrm{sgn}(x)/2$ is replaced by the function $\sigma(z) = 1/(\pi z)$, and derivation is replaced by $\bar\partial$ (using in particular $\bar\partial\sigma = \delta_0$), and that in the simplest cases we have the formula

$$\mathrm{Tr}^{\#}\,\mathcal{L} = \sum_{\omega\in\Omega}\ \sum_{x\in\mathrm{Fix}\, f_\omega(x)} \frac{g_\omega(x)}{1-\partial f_\omega(x)}$$

(note the absence of absolute value).

We sketch the conceptual proof of (6.15) now: after unifying the sharp trace (6.7) of transfer operators with the 'trace' of kernel operators $\mathrm{Tr}^*\,\mathcal{D}(z) := \int \mathcal{D}_{xx}\,d\mu(x)$, we check that the unified trace, noted Tr^*, satisfies (6.8). Hence, the usual determinant formulas hold for

$$\mathrm{Det}^*(1-\mathcal{M}) = \exp - \sum_n \frac{1}{n}\,\mathrm{Tr}^*\,\mathcal{M}^n .$$

The argument is then quite literally a three-line proof:

$$\begin{aligned}\mathrm{Det}^*(1+\mathcal{D}(z))\,\mathrm{Det}^{\#}(1-z\mathcal{L}) &= \mathrm{Det}^*(1+z\mathcal{N}(1-z\mathcal{L})^{-1}S)\,\mathrm{Det}^*(1-z\mathcal{L})\\ &= \mathrm{Det}^*(1+zS\mathcal{N}(1-z\mathcal{L})^{-1})\,\mathrm{Det}^*(1-z\mathcal{L})\\ &= \mathrm{Det}^*(1+z(S\mathcal{N}-\mathcal{L})) = 1, \qquad (6.16)\end{aligned}$$

where we used $\mathrm{Tr}^{\#}\,\mathcal{L} = \mathrm{Tr}^*(S\mathcal{N})$, and more generally $\mathrm{Tr}^*(S\mathcal{N}-\mathcal{L})^n = 0$ for $n \geq 1$, to get the last equality.

A kneading approach to dynamical zeta functions in higher dimensions is still lacking. For Hénon-like (or more generally once-folding) maps, it can be hoped that the *pruning-front* approach of Cvitanović *et al* (1988), which is in the process of being made rigorous (de Carvalho 1996), will lead to a two-dimensional kneading theory which could include a kneading operator analysis of naturally weighted sharp zeta functions (see also Ishii (1997) for a kneading theory of the Lozi map). The pruning front conjecture is supported by many very interesting numerical studies (see e.g. Bäcker and Dullin (1997), Hansen (1993), the book by Cvitanović *et al* (1997) and references therein). General ideas to define sharp traces in higher dimensions have been advanced by Kitaev (1995b) and could perhaps apply to differentiable dynamical systems without any topological assumptions (in particular, admitting countably or uncountably many fixed points).

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