EDGEWORTH EXPANSION FOR INDEPENENT BOUNDED INTEGER VALUED RANDOM VARIABLES.

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Abstract. We obtain asymptotic expansions for local probabilities of partial sums for uniformly bounded independent but not necessarily identically distributed integer-valued random variables. The expansions involve products of polynomials and trigonometric polynomials. Our results do not require any additional assumptions. As an application of our expansions we find necessary and sufficient conditions for the classical Edgeworth expansion. It turns out that there are two possible obstructions for the validity of the Edgeworth expansion of order r. First, the distance between the distribution of the underlying partial sums modulo some $h \in \mathbb{N}$ and the uniform distribution could fail to be $o(\sigma_N^{1-r})$, where σ_N is the standard deviation of the partial sum. Second, this distribution could have the required closeness but this closeness is unstable, in the sense that it could be destroyed by removing finitely many terms. In the first case, the expansion of order r fails. In the second case it may or may not hold depending on the behavior of the derivatives of the characteristic functions of the summands whose removal causes the break-up of the uniform distribution. We also show that a quantitative version of the classical Prokhorov condition (for the strong local central limit theorem) is sufficient for Edgeworth expansions, and moreover this condition is, in some sense, optimal.

Contents

1. Introduction.

Let $X_1, X_2, ...$ be a uniformly bounded sequence of independent integer-valued random variables. Set $S_N = X_1 + X_2 + ... + X_N$, $V_N = V(S_N) = \mathrm{Var}(S_N)$ and $\sigma_N = \sqrt{V_N}$. Assume also that $V_N \to \infty$ as $N \to \infty$. Then the central limit theorem (CLT) holds true, namely the distribution of $(S_N - \mathbb{E}(S_N))/\sigma_N$ converges to the standard normal distribution as $N \to \infty$.

Recall that the local central limit theorem (LLT) states that, uniformly in k we have

$$\mathbb{P}(S_N = k) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-(k-\mathbb{E}(S_N))^2/2V_N} + o(\sigma_N^{-1}).$$

This theorem is also a classical result, and it has origins in De Moivre-Laplace theorem. The stable local central limit theorem (SLLT) states that the LLT holds true for any integer-valued square integrable independent sequence $X_1', X_2', ...$ which differs from $X_1, X_2, ...$ by a finite number of elements. We recall a classical result due to Prokhorov.

Theorem 1.1. [32] *The SLLT holds iff for each integer $h > 1$,*

$$\sum_n \mathbb{P}(X_n \neq m_n \ mod\ h) = \infty \tag{1.1}$$

where $m_n = m_n(h)$ is the most likely residue of X_n modulo h.

We refer the readers' to [33, 36] for extensions of this result to the case when X_n's are not necessarily bounded (for instance, the result holds true when $\sup_n \|X_n\|_{L^3} < \infty$). Related results for local convergence to more general limit laws are discussed in [6, 25].

The above result provides a necessary and sufficient condition for the SLLT. It turns out that the difference between LLT and SLLT is not that big.

Proposition 1.2. *Suppose S_N obeys LLT. Then for each integer $h \geq 2$ at least one of the following conditions occur:*

either (a) $\sum_n \mathbb{P}(X_n \neq m_n(h) \ mod \ h) = \infty$.

or (b) $\exists j_1, j_2, \dots, j_k$ with $k < h$ such that $\sum_{s=1}^{k} X_{j_s}$ mod h is uniformly distributed. In that case for all $N \geq \max(j_1, \dots, j_k)$ we have that S_N mod h is uniformly distributed.

Since we could not find this result in the literature we include the proof in Section 6.

Next, we provide necessary and sufficient conditions for the regular LLT. We need an additional notation. Let $K = \sup_n \|X_n\|_{L^\infty}$. Call t *resonant* if $t = \frac{2\pi l}{m}$ with $0 < m \leq 2K$ and $0 \leq l < m$.

Theorem 1.3. *The following conditions are equivalent:*

(a) S_N satisfies LLT;

(b) For each $\xi \in \mathbb{R} \setminus \mathbb{Z}$, $\lim_{N\to\infty} \mathbb{E}\left(e^{2\pi i \xi S_N}\right) = 0$;

(c) For each non-zero resonant point ξ, $\lim_{N\to\infty} \mathbb{E}\left(e^{2\pi i \xi S_N}\right) = 0$;

(d) For each integer h the distribution of S_N mod h converges to uniform.

The proof of this result is also given in Section 6. We refer the readers to [7, 10] for related results in more general settings.

The local limit theorem deals with approximation of $P(S_N = k)$ up to an error term of order $o(\sigma_N^{-1})$. Given $r \geq 1$, the Edgeworth expansion of order r holds true if there are polynomials $P_{b,N}$, whose coefficients are uniformly bounded in N and their degrees do no depend on N, so that uniformly in $k \in \mathbb{Z}$ we have that

$$\mathbb{P}(S_N = k) = \sum_{b=1}^{r} \frac{P_{b,N}(k_N)}{\sigma_N^b} \mathfrak{g}(k_N) + o(\sigma_N^{-r}) \tag{1.2}$$

where $k_N = (k - \mathbb{E}(S_N))/\sigma_N$ and $\mathfrak{g}(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$. In Section 5 we will show, in particular, that Edgeworth expansions of any order r are unique up to terms of order $o(\sigma_N^{-r})$, and so the case $r = 1$ coincides with the LLT. Edgeworth expansions for discrete (lattice-valued) random variables have been studied in literature for iid random variables [18, Theorem 4.5.4] [30, Chapter VII], (see also [12, Theorem 5]), homogeneous Markov chains [28, Theorems 2-4], decomposable statistics [22], or dynamical systems [15] with good spectral properties such as expanding maps. Papers [2, 16] discuss the rate of convergence in the LLT. Results for non-lattice variables were obtained in [13, 1, 5, 3] (which considered random vectors) and [15] (see also [17] for corresponding results for random expanding dynamical systems).

In this paper we obtain analogues of Theorems 1.1 and 1.3 for higher order Edgeworth expansions for independent but not identically distributed integer-valued uniformly bounded random variables. We begin with the following result.

Theorem 1.4. *Let $K = \sup_j \|X_j\|_{L^\infty}$. For each $r \in \mathbb{N}$ there is a constant $R{=}R(r, K)$ such that the Edgeworth expansion of order r holds if*

$$M_N := \min_{2 \le h \le 2K} \sum_{n=1}^{N} \mathbb{P}(X_n \neq m_n(h) \ mod\ h) \ge R \ln V_N.$$

In particular, S_N obeys Edgeworth expansions of all orders if

$$\lim_{N \to \infty} \frac{M_N}{\ln V_N} = \infty.$$

The number $R(r, K)$ can be chosen according to Remark 3.6. This theorem is a quantitative version of Prokhorov's Theorem 1.1. We observe that logarithmic in V_N growth of various non-perioidicity characteristics of individual summands are often used in the theory of local limit theorems (see e.g. [20, 21, 23]). We will see from the examples of Section 10 that this result is close to optimal. However, to justify the optimality we need to understand the conditions necessary for the validity of the Edgeworth expansion.

Theorem 1.5. *For any $r \ge 1$, the Edgeworth expansion of order r holds if and only if for any nonzero resonant point t and $0 \le \ell < r$ we have*

$$\bar{\Phi}_N^{(\ell)}(t) = o\left(\sigma_N^{\ell+1-r}\right).$$

where $\bar{\Phi}_N(x) = \mathbb{E}[e^{ix(S_N - \mathbb{E}(S_N))}]$ and $\bar{\Phi}_N^{(\ell)}(\cdot)$ is its ℓ-th derivative.

This result generalizes Theorem 1.3, however in contrast with that theorem, in the case $r > 1$ we also need to take into account the behavior of the derivatives of the characteristic function at nonzero resonant points. The values of the characteristic function at the resonant points $2\pi l/m$ have clear probabilistic meaning. Namely, they control the rate equidistribution modulo m (see part (d) of Theorem 1.3 or Lemma 6.2). Unfortunately, the probabilistic meaning of the derivatives is less clear, so it is desirable to characterize the validity of the Edgeworth expansions of orders higher than 1 without considering the derivatives. Example 10.2 shows that this is impossible without additional assumptions. Some of the reasonable additional conditions are presented below.

We start with the expansion of order 2.

Theorem 1.6. *Suppose S_N obeys the SLLT. Then the following are equivalent:*

(a) Edgeworth expansion of order 2 holds;

(b) $|\Phi_N(t)| = o(\sigma_N^{-1})$ for each nonzero resonant point t;

(c) For each $h \le 2K$ the distribution of S_N mod h is $o(\sigma_N^{-1})$ close to uniform.

Corollary 7.3 provides an extension of Theorem 1.6 for expansions of an arbitrary order r under an additional assumption that $\varphi := \min\limits_{t\in\mathcal{R}} \inf\limits_{n} |\phi_n(t)| > 0$, where $\mathcal{R}$ is the set of all nonzero resonant points. The latter condition implies in particular, that for each ℓ there is a uniform lower bound on the distance between the distribution of $X_{n_1} + X_{n_2} + \cdots + X_{n_\ell} \bmod m$ and the uniform distribution, when $\{n_1, n_2, \dots, n_\ell\} \in \mathbb{N}^\ell$ and $m \ge 2$.

Next we discuss an analogue of Theorem 1.1 for expansions of order higher than 2. It requires a stronger condition which uses an additional notation. Given $j_1, j_2, \dots, j_s$ with $j_l \in [1, N]$ we write

$$S_{N;j_1,j_2,\dots,j_s} = S_N - \sum_{l=1}^{s} X_{j_l}.$$

Thus $S_{N;j_1,j_2,\dots,j_s}$ is a partial sum of our sequence with s terms removed. We will say that $\{X_n\}$ *admits an Edgeworth expansion of order r in a superstable way* (which will be denoted by $\{X_n\} \in EeSs(r)$) if for each $\bar{s}$ and each sequence $j_1^N, j_2^N, \dots, j_{s_N}^N$ with $s_N \le \bar{s}$ there are polynomials $P_{b,N}$ whose coefficients are $O(1)$ in N and their degrees do not depend on N so that uniformly in $k \in \mathbb{Z}$ we have that

$$\mathbb{P}(S_{N;j_1^N,j_2^N,\dots,j_{s_N}^N} = k) = \sum_{b=1}^{r} \frac{P_{b,N}(k_N)}{\sigma_N^b} \mathfrak{g}(k_N) + o(\sigma_N^{-r}) \tag{1.3}$$

and the estimates in $O(1)$ and $o(\sigma_N^{-r})$ are uniform in the choice of the tuples $j_1^N, \dots, j_{s_N}^N$. That is, by removing a finite number of terms we can not destroy the validity of the Edgeworth expansion (even though the coefficients of the underlying polynomials will of course depend on the choice of the removed terms). Let $\Phi_{N;j_1,j_2,\dots,j_s}(t)$ be the characteristic function of $S_{N;j_1,j_2,\dots,j_s}$.

Remark 1.7. We note that in contrast with SLLT, in the definition of the superstrong Edgeworth expansion one is only allowed to remove old terms, but not to add new ones. This difference in the definition is not essential, since adding terms with sufficiently many moments (in particular, adding bounded terms) does not destroy the validity of the Edgeworth expansion. See the proof of Theorem 1.8 (i) or the second part of Example 10.1, starting with equation (10.2), for details.

Theorem 1.8. *(1) $S_N \in EeSs(1)$ (that is, S_N satisfies the LLT in a superstable way) if and if it satisfies the SLLT.*

(2) For arbitrary $r \geq 1$ the following conditions are equivalent:

(a) $\{X_n\} \in EeSs(r)$;

(b) For each $j_1^N, j_2^N, \dots, j_{s_N}^N$ and each nonzero resonant point t we have $\Phi_{N;j_1^N,j_2^N,\dots,j_{s_N}^N}(t) = o(\sigma_N^{1-r})$;

(c) For each $j_1^N, j_2^N, \dots, j_{s_N}^N$, and each $h \leq 2K$ the distribution of $S_{N;j_1^N,j_2^N,\dots,j_{s_N}^N}$ mod h is $o(\sigma_N^{1-r})$ close to uniform.

To prove the above results we will show that for any order r, we can always approximate $\mathbb{P}(S_N = k)$ up to an error $o(\sigma_N^{-r})$ provided that instead of polynomials we use products of regular and the trigonometric polynomials. Those products allow us to take into account possible oscillatory behavior of $P(S_N = k)$ when k belongs to different residues mod h, where h is denominator of a *resonant frequency.* When $M_N \geq RV_N$ for R large enough, the new expansion coincides with the usual Edgeworth expansions. We thus derive that the condition $M_N \geq R \ln V_N$ is in a certain sense optimal.

2. Main result

Let $X_1, X_2, \dots$ be a sequence of independent integer-valued random variables. For each $N \in \mathbb{N}$ we set $S_N = \sum_{n=1}^{N} X_n$ and $V_N = \mathrm{Var}(S_N)$. We assume in this paper that $\lim_{N\to\infty} V_N = \infty$ and that there is a constant K such that

$$\sup_n \|X_n\|_{L^\infty} \leq K.$$

Denote $\sigma_N = \sqrt{V_N}$. For each positive integer m, let $q_n(m)$ be the second largest among

$$\sum_{l \equiv j \bmod m} \mathbb{P}(X_n = l) = \mathbb{P}(X_n \equiv j \bmod m),\ j = 1, 2, ..., m$$

and $j_n(m)$ be the corresponding residue class. Set

$$M_N(m) = \sum_{n=1}^{N} q_n(m) \quad \text{and} \quad M_N = \min_m M_N(m).$$

Theorem 2.1. *There $\exists J = J(K) < \infty$ and polynomials $P_{a,b,N}$, where $a \in 0, \ldots, J-1$, $b \in \mathbb{N}$, with degrees depending only on b but not on a, K or on any other characteristic of $\{X_n\}$, such that the coefficients of $P_{a,b,N}$ are uniformly bounded in N, and, for any $r \geq 1$ uniformly in $k \in \mathbb{Z}$ we have*

$$\mathbb{P}(S_N = k) - \sum_{a=0}^{J-1} \sum_{b=1}^{r} \frac{P_{a,b,N}((k - a_N)/\sigma_N)}{\sigma_N^b} \mathfrak{g}((k-a_N)/\sigma_N) e^{2\pi i a k/J} = o(\sigma_N^{-r})$$

where $a_N = \mathbb{E}(S_N)$ and $\mathfrak{g}(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$.

Moreover, $P_{0,1,N} \equiv 1$, and given K, r, there exists $R = R(K, r)$ such that if $M_N \geq R \ln V_N$ then we can choose $P_{a,b,N} = 0$ for $a \neq 0$.

We refer the readers to (6.1) for more details on these expansions in the case $r = 1$, and to Section 6 for a discussion about the relations with local limit theorems. The resulting expansions in the case $r = 2$ are given in (9.1). We note that the constants $J(K)$ and $R(K, r)$ can be recovered from the proof of Theorem 2.1.

Remark 2.2. Since the coefficients of the polynomials $P_{a,b,N}$ are uniformly bounded, the terms corresponding to $b = r + 1$ are of order $O(\sigma_N^{-(r+1)})$ uniformly in k. Therefore, in the r-th order expansion we actually get that the error term is $O(\sigma_N^{-(r+1)})$.

Remark 2.3. In fact, the coefficients of the polynomials $P_{a,b,N}$ for $a > 0$ are bounded by a constant times $(1 + M_N^q) e^{-c_0 M_N}$, where $c_0 > 0$ depends only on K and $q \geq 0$ depends only on r and K. Therefore, these coefficient are small when M_N is large. When $M_N \geq R(r, K) \ln V_N$ these coefficients become of order $o(\sigma_N^{-r})$. Therefore, they only contribute to the error term, and so we can replace them by 0, as stated in Theorem 2.1.

Remark 2.4. As in the derivation of the classical Edgeworth expansion, the main idea of the proof of Theorem 2.1 is the stationary phase analysis of the characteristic function. However, in contrast with the

iid case there may be resonances other than 0 which contribute to the oscillatory terms in the expansion. Another interesting case where the classical Edgeworth analysis fails is the case of iid terms where the summands are non-arithmetic but take only finitely many values. It is shown in [8] that in that case, the leading correction to the Edgeworth expansion also comes from resonances. However, in the case studied in [8] the geometry of resonances is more complicated, so in contrast to our Theorem 2.1, [8] does not get the expansion of all orders.

3. Edgeworth expansions under quantitative Prokhorov condition.

In this section we prove Theorem 1.4. In the course of the proof we obtain the estimates of the characteristic function on intervals not containing resonant points which will also play an important role in the proof of Theorem 2.1. The proof of Theorem 2.1 will be completed in Section 4 where we analyze additional contribution coming from nonzero resonant points which appear in the case $M_N \leq R \ln \sigma_N$. Those contributions constitute the source of the trigonometric polynomials in the generalized Edgeworth expansions.

3.1. Characterstic function near 0.

Here we recall some facts about the behavior of the characteristic function near 0, which will be useful in the proofs of Theorems 1.4 and 2.1. The first result holds for general uniformly bounded sequences $\{X_n\}$ (which are not necessarily integer-valued).

Proposition 3.1. *Suppose that* $\lim_{N\to\infty} \sigma_N = \infty$, *where* $\sigma_N = \sqrt{V_N} = \sqrt{V(S_N)}$. *Then for* $k = 1, 2, 3, ...$ *there exists a sequence of polynomials* $(A_{k,N})_N$ *whose degree* d_k *depends only on* k *so that for any* $r \geq 1$ *there is* $\delta_r > 0$ *such that for all* $N \geq 1$ *and* $t \in [-\delta_r \sigma_N, \delta_r \sigma_N]$,

$$\mathbb{E}\left(e^{it(S_N - \mathbb{E}(S_N))/\sigma_N}\right) = e^{-t^2/2}\left(1 + \sum_{k=1}^{r} \frac{A_{k,N}(t)}{\sigma_N^k} + \frac{t^{r+1}}{\sigma_N^{r+1}} O(1)\right). \tag{3.1}$$

Moreover, the coefficients of $A_{k,N}$ *are algebraic combinations of moments of the* X_m*'s and they are uniformly bounded in* N. *Furthermore*

$$A_{1,N}(t) = -\frac{i}{6}\gamma_N t^3 \quad \text{and} \quad A_{2,N}(t) = \Lambda_4(\bar{S}_N)\sigma_N^{-2}\frac{t^4}{4!} - \frac{1}{36}\gamma_N^2 t^6 \tag{3.2}$$

where $\bar{S}_N = S_N - \mathbb{E}(S_N)$, $\gamma_N = \mathbb{E}[(\bar{S}_N)^3]/\sigma_N^2$ *and* $\Lambda_4(\bar{S}_N)$ *is the fourth comulant of* $\bar{S}_N$.

The proof is quite standard, so we just sketch the argument. The idea is to fix some $B_2 > B_1 > 0$, and to partition $\{1, ..., N\}$ into intervals $I_1, ..., I_{m_N}$ so that $B_1 \leq \mathrm{Var}(S_{I_l}) \leq B_2$ where for each l we set $S_{I_l} = \sum_{j \in I_l} X_i$. It is clear that m_N/σ_N^2 is bounded away from 0 and ∞ uniformly in N. Recall next that there are constants C_p, $p \geq 2$ so that for any $n \geq 1$ and $m \geq 0$ we have

$$\left\| \sum_{j=n}^{n+m} \big(X_j - \mathbb{E}(X_j)\big) \right\|_{L^p} \leq C_p \left(1 + \left\| \sum_{j=n}^{n+m} \big(X_j - \mathbb{E}(X_j)\big) \right\|_{L^2} \right). \tag{3.3}$$

This is a consequence of the multinomial theorem and some elementary estimates, and we refer the readers to either Lemma 2.7 in [10], or Theorem 6.17 in [24] for such a result in a much more general settings. Using the latter estimates we get that the L^p-norms of S_{I_l} are uniformly bounded in l. This reduces the problem to the case when the variance of X_n is uniformly bounded from below, and all the moments of $X_n - \mathbb{E}(X_n)$ are uniformly bounded. In this case, the proposition follows by considering the Taylor expansion of the function $\ln \mathbb{E}\big(e^{it(S_N - \mathbb{E}(S_N))/\sigma_N}\big) + \frac{1}{2}t^2$, see [13, §XVI.6].

Proposition 3.2. *Given a square integrable random variable X, let $\bar{X} = X - \mathbb{E}(X)$. Then for each $h \in \mathbb{R}$ we have*

$$\left| \mathbb{E}(e^{ih\bar{X}}) - 1 \right| \leq \frac{1}{2} h^2 V(X).$$

Proof. Set $\varphi(h) = \mathbb{E}(e^{ih\bar{X}})$. Then by the integral form of the second order Taylor reminder we have

$$\begin{aligned} |\varphi(h) - \varphi(0) - h\varphi'(0)| = |\varphi(h) - \varphi(0)| &= \left| \int_0^h (t-h)\varphi''(t)dt \right| \\ &\leq V(X) \int_0^{|h|} (|h| - t)dt = \frac{1}{2} h^2 V(X). \end{aligned}$$

□

3.2. **Non resonant intervals.** As in almost all the proofs of the LLT, the starting point in the proof of Theorem 1.4 (and Theorem 2.1) is that for $k, N \in \mathbb{N}$ we have

$$2\pi \mathbb{P}(S_N = k) = \int_0^{2\pi} e^{-itk} \mathbb{E}(e^{itS_N}) dt. \tag{3.4}$$

Denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let

$$\Phi_N(t) = \mathbb{E}(e^{itS_N}) = \prod_{n=1}^{N} \phi_n(t) \quad \text{where} \quad \phi_n(t) = \mathbb{E}(e^{itX_n}).$$

Divide $\mathbb{T}$ into intervals I_j of small size δ such that each interval contains at most one resonant point and this point is strictly inside I_j. We call an interval resonant if it contains a resonant point inside. Then

$$2\pi\mathbb{P}(S_N = k) = \sum_j \int_{I_j} e^{-itk}\mathbb{E}(e^{itS_N})dt. \tag{3.5}$$

We will consider the integrals appearing in the above sum individually.

Lemma 3.3. *There are constants $C, c > 0$ which depend only on δ and K so that for any non-resonant interval I_j and $N \geq 1$ we have*

$$\int_{I_j} |\Phi_N(t)|dt \leq Ce^{-cV_N}.$$

Proof. Let $\hat{q}_n, \bar{q}_n$ be the largest and the second largest values of $\mathbb{P}(X_n = j)$ and let $\hat{j}_n, \bar{j}_n$ be the corresponding values. Note that

$$\phi_n(t) = \hat{q}_n e^{it\hat{j}_n} + \bar{q}_n e^{it\bar{j}_n} + \sum_{l \neq \hat{j}_n, \bar{j}_n} \mathbb{P}(X_n = l)e^{itl}. \tag{3.6}$$

Since I_j is non resonant, the angle between $e^{it\hat{j}_n}$ and $e^{it\bar{j}_n}$ is uniformly bounded from below. Indeed if this was not the case we would have $t\bar{j}_n - t\hat{j}_n \approx 2\pi l_n$ for some $l_n \in \mathbb{Z}$. Then $t \approx \frac{2\pi l_n}{m_n}$ where $m_n = \bar{j}_n - \hat{j}_n$ contradicting the assumption that I_j is non-resonant. Accordingly $\exists c_1 > 0$ such that $\left|e^{it\hat{j}_n} + e^{it\bar{j}_n}\right| \leq 2 - c_1$. Therefore

$$\left|\hat{q}_n e^{it\hat{j}_n} + \bar{q}_n e^{it\bar{j}_n}\right| \leq (\hat{q}_n - \bar{q}_n) + \bar{q}_n \left|e^{it\hat{j}_n} + e^{it\bar{j}_n}\right| \leq \hat{q}_n + \bar{q}_n - 2c_1\bar{q}_n.$$

Plugging this into (3.6), we conclude that $|\phi_n(t)| \leq 1 - 2c_1\bar{q}_n$ for $t \in I_j$. Multiplying these estimates over n and using that $1 - x \leq e^{-x}$, $x > 0$, we get

$$|\Phi_N(t)| \leq e^{-2c_1 \sum_n \bar{q}_n}.$$

Since $V(X_n) \leq c_2\bar{q}_n$ for a suitable constant c_2 we can rewrite the preceding as

$$|\Phi_N(t)| \leq e^{-c_3 V_N},\ c_3 > 0. \tag{3.7}$$

Integrating over I_j we obtain the result. □

3.3. **Prokhorov estimates.** Next we consider the case where I_j contains a nonzero resonant point $t_j = \frac{2\pi l}{m}$.

Lemma 3.4. *There is a constant c_0 which depends only on K so that for any nonzero resonant point $t_j = 2\pi l/m$ we have*

$$\sup_{t \in I_j} |\mathbb{E}(e^{itS_N})| \leq e^{-c_0 M_N(m)}. \tag{3.8}$$

Thus, for any $r \geq 1$ there is a constant $R = R(r, K)$ such that if $M_N(m) \geq R \ln V_N$, then the integral $\int_{I_j} e^{-itk}\mathbb{E}(e^{itS_N})dt$ is $o(\sigma_N^{-r})$ uniformly in k, and so it only contributes to the error term.

Proof. The estimate (3.8) follows from the arguments in [33], but for readers' convenience we recall its proof. Let X be an integer-valued random variable so that $\|X\|_{L^\infty} \leq K$. Let $t_0 = 2\pi l/m$ be a nonzero resonant point, where $\gcd(l, m) = 1$. Let $t \in \mathbb{T}$ be so that

$$|t - t_0| \leq \delta, \tag{3.9}$$

where δ is a small positive number. Let $\phi_X(\cdot)$ denote the characteristic function of X. Since $x \leq e^{x-1}$ for any real x we have

$$|\phi_X(t)|^2 \leq e^{|\phi(t)|^2-1}.$$

Next, we have

$$|\phi_X(t)|^2 - 1 = \phi(t)\phi(-t) - 1 = \sum_{j=-2K}^{2K} \sum_s \tilde{P}_j \left[\cos(t_j) - 1\right]$$

where

$$\tilde{P}_j = \sum_s \mathbb{P}(X = s)\mathbb{P}(X = j + s).$$

Fix some $-2K \leq j \leq 2K$. We claim that if δ in (3.9) is small enough and $j \not\equiv 0 \bmod m$ then for each integer w we have $|t - 2\pi w/j| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$ which depends only on K. This follows from the fact that $-2K \leq j \leq 2K$ and that $2\pi w/j \neq t_0$ (and there is a finite number of resonant points). Therefore,

$$\cos(tj) - 1 \leq -\delta_0$$

for some $\delta_0 > 0$. On the other hand, if $j = km$ for some integer k then with $w = lk$ we have

$$\begin{aligned}\cos(tj) - 1 &= -2\sin^2(tj/2) = -2\sin^2\left((tj - 2\pi w)/2\right) \\ &= -2\sin^2\left(j(t - t_0)/2\right) \leq -\delta_1(t - t_0)^2\end{aligned}$$

for some $\delta_1 > 0$ (assuming that $|t - t_0|$ is small enough). We conclude that

$$|\phi_X(t)|^2 - 1 \leq -\delta_0 \sum_{j\in A} \tilde{P}_j - \delta_1(t - t_0)^2 \sum_{j\in B} \tilde{P}_j$$

where $A = A(X)$ is the set of j's between $-2K$ and $2K$ so that $j \not\equiv 0 \bmod m$ and $B = B(X)$ is its complement in $\mathbb{Z} \cap [-2K, 2K]$. Let s_0 be the most likely residue of X mod m and s_1 be the second most likely residue class. Since

$$\mathbb{P}(X \equiv s_0 \bmod m) \geq \frac{1}{m} \quad \text{and} \quad \mathbb{P}(X \equiv s_1 \bmod m) = q_m(X)$$

it follows that $\sum_{j\in A}\tilde{P}_j \geq \frac{q_m(X)}{m}$.

Combining this with the trivial bound $\sum_{j\in B}\tilde{P}_j \geq \mathbb{P}^2(X\equiv s_0) \geq \frac{1}{m^2}$ we obtain

$$|\phi_X(t)| \leq \exp - \left[\frac{1}{2}\left(\frac{\delta_0 q_m(X)}{m} + \frac{\delta_1(t-t_0)^2}{m^2}\right)\right].$$

Applying the above with $t_0 = t_j$ and $X = X_n$, $1 \leq n \leq N$ we get that

$$|\Phi_N(t)| \leq e^{-c_0 M_N(m) - \bar{c}_0 N(t-t_j)^2} \leq e^{-c_0 M_N(m)} \tag{3.10}$$

where c_0 is some constant. □

Remark 3.5. Using the first inequality in (3.10) and arguing as in [33, page 264], we can deduce that there are positive constants C, c_1, c_2 such that

$$\int_{I_j} |\mathbb{E}(e^{itS_N})|dt \leq C\left(e^{-c_1\sigma_N} + \frac{e^{-c_2 M_N(m)}}{\sigma_N}\right). \tag{3.11}$$

This estimate plays an important role in the proof of the SLLT in [33], but for our purposes a weaker bound (3.8) is enough. Note also that in order to prove (3.8) we could have just used the trivial inequality $\cos(t_j) - 1 \leq 0$ when $j \equiv 0 \bmod m$, but we have decided to present this part from [33] in full.

Remark 3.6. Let $d_{\mathcal{R}}$ be the minimal distance between two different resonant points. Then, when $\delta < 2d_{\mathcal{R}}$, we can take $\delta_0 = 1 - \cos(d_{\mathcal{R}})$ in the proof of Lemma 3.4. Therefore, we can take $c_0 = \frac{1-\cos(d_{\mathcal{R}})}{4K}$ in (3.8). Hence Lemma 3.4 holds with $R(r,K) = \frac{r+1}{2c_0}$.

3.4. **Proof of Theorem 1.4.** Fix some $r \geq 1$. Lemmas 3.3 and 3.4 show that if $M_N \geq R(r,K)\ln V_N$, then all the integrals in the right hand side of (3.5) are of order $o(\sigma_N^{-r})$, except for the one corresponding to the resonant point $t_j = 0$. That is, for any $\delta > 0$ small enough, uniformly in k we have

$$2\pi\mathbb{P}(S_N = k) = \int_{-\delta}^{\delta} e^{-ihk}\Phi_N(h)dh + o(\sigma_N^{-r}).$$

In order to complete the proof of Theorem 1.4, we need to expand the above integral. Making a change of variables $h \to h/\sigma_N$ and using Proposition 3.1, we conclude that if δ is small enough then

$$\int_{-\delta}^{\delta} e^{-ihk}\Phi_N(h)dh =$$

$$\sigma_N^{-1}\int_{-\delta\sigma_N}^{\delta\sigma_N} e^{-ihk_N}e^{-h^2/2}\left(1+\sum_{u=1}^{r}\frac{A_{u,N}(h)}{\sigma_N^k}+\frac{h^{r+1}}{\sigma_N^{r+1}}O(1)\right)dh$$

where $k_N = (k-\mathbb{E}(S_N))/\sigma_N$. Since the coefficients of the polynomials $A_{u,N}$ are uniformly bounded in N, we can just replace the above integral with the corresponding integral over all $\mathbb{R}$ (i.e. replace $\pm\delta\sigma_N$ with $\pm\infty$). Now the Edgeworth expansions are achieved using that for any nonnegative integer q we have that $(it)^q e^{-t^2/2}$ is the Fourier transform of the q-th derivative of $\mathbf{n}(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ and that for any real a,

$$\int_{-\infty}^{\infty} e^{-iat}\widehat{\mathbf{n}^{(q)}}(t)dt = \mathbf{n}^{(q)}(a) = \frac{1}{\sqrt{2\pi}}(-1)^q H_q(a)e^{-a^2/2} \tag{3.12}$$

where $H_q(a)$ is the q-th Hermite polynomial.

4. Generalized Edgeworth expansions: Proof of Theorem Theorem 2.1

4.1. **Contributions of resonant intervals.** Let $r \geq 1$. As in the proof of Theorem 1.4, our starting point is the equality

$$2\pi\mathbb{P}(S_N = k) = \int_0^{2\pi} e^{-itk}\mathbb{E}(e^{itS_N})dt = \sum_j \int_{I_j} e^{-itk}\mathbb{E}(e^{itS_N})dt \tag{4.1}$$

which holds for any $k \in \mathbb{N}$. We will consider the integrals appearing in the above sum individually. By Lemma 3.3 the integrals over non-resonant intervals are of order $o(\sigma_N^{-r})$, and so they can be disregarded. Moreover, in §3.4 we have expanded the integral over the resonant interval containing 0. Now we will see that in the case $M_N < R(r,K)\ln V_N$ the contribution of nonzero resonant points need not be negligible.

Let $t_j = \frac{2\pi l}{m}$ be a nonzero resonant point so that $M_N(m) < R(r,K)\ln V_N$ and let I_j be the resonant interval containing it. Theorem 2.1 will follow from an appropriate expansion of the integral

$$\int_{I_j} e^{-itk}\mathbb{E}(e^{itS_N})dt.$$

We need the following simple result, which for readers' convenience is formulated as a lemma.

Lemma 4.1. *There exists $\bar{\varepsilon} > 0$ so that for each $n \geq 1$ with $q_n(m) \leq \bar{\varepsilon}$ we have $|\phi_n(t_j)| \geq \frac{1}{2}$. In fact, we can take $\bar{\varepsilon} = \frac{1}{4m}$.*

Proof. Recall that $t_j = 2\pi l/m$. The lemma follows since for any random variable X we have $|\mathbb{E}(e^{it_jX})| =$

$$\left| e^{it_j s(m,X)} - \sum_{u \not\equiv s(m,X) \bmod m} \left(e^{it_j s(m,X)} - e^{it_j u}\right) P(X \equiv u \bmod m) \right|$$

$$\geq 1 - 2mq(m,X)$$

where $s(m,X)$ is the most likely value of X mod m and $q(m,X)$ is the second largest value among $P(X \equiv u \bmod m)$, $u = 0,1,2,...,m-1$. Therefore, we can take $\bar{\varepsilon} = \frac{1}{4m}$. □

Next, set $\bar{\varepsilon} = \frac{1}{8K}$ and let $N_0 = N_0(N, t_j, \bar{\varepsilon})$ be the number of all n's between 1 to N so that $q_n(m) > \bar{\varepsilon}$. Then $N_0 \leq \frac{R \ln V_N}{\bar{\varepsilon}}$ because $M_N(m) \leq R \ln V_N$. By permuting the indexes $n = 1, 2, ..., N$ if necessary we can assume that $q_n(m)$ is non increasing. Let N_0 be the largest number such that $q_{N_0} \geq \bar{\varepsilon}$. Decompose

$$\Phi_N(t) = \Phi_{N_0}(t)\Phi_{N_0,N}(t) \tag{4.2}$$

where $\Phi_{N_0,N}(t) = \prod_{n=N_0+1}^{N} \phi_n(t)$.

Lemma 4.2. *If the length δ of I_j is small enough then for any $t = t_j + h \in I_j$ and $N \geq 1$ we have*

$$\Phi_{N_0,N}(t) = \Phi_{N_0,N}(t_j)\Phi_{N_0,N}(h)\Psi_{N_0,N}(h)$$

where

$$\Psi_{N_0,N}(h) = \exp\left[O(M_N(m)) \sum_{u=1}^{\infty} (O(1))^u h^u \right].$$

Proof. Denote

$$\mu_n = \mathbb{E}(X_n), \quad \bar{X}_n = X_n - \mu_n, \quad \bar{\phi}_n(t) = \mathbb{E}(e^{it\bar{X}_n}).$$

Let $j_n(m)$ be the most likely residue mod m for X_n. Decompose

$$\bar{X}_n = s_n + Y_n + Z_n$$

where $Z_n \in m\mathbb{Z}$, $s_n = j_n(m) - \mu_n$, so that $\mathbb{P}(Y_n \neq 0) \leq mq_n(m)$. Then for $t = t_j + h$,

$$\bar{\phi}_n(t) = e^{it_j s_n} \mathbb{E}\left(e^{it_j Y_n} e^{ih\bar{X}_n}\right) = \bar{\phi}_n(t_j)\psi_n(h) \tag{4.3}$$

where

$$\psi_n(h) = \left(1 + \frac{ih\mathbb{E}(e^{it_j Y}\bar{X}_n) - \frac{h^2}{2}\mathbb{E}(e^{it_j Y}(\bar{X}_n)^2) + \dots}{\mathbb{E}(e^{it_j Y_n})}\right).$$

Next, using that for any $x \in (-1,1)$ we have

$$1+x = e^{\ln(1+x)} = e^{x-x^2/2+x^3/3-\dots}$$

we obtain that for h close enough to 0,

(4.4)

$$\psi_n(h) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{1}{\mathbb{E}(e^{it_jY_n})}\sum_{q=1}^{\infty}\frac{(ih)^q}{q!}\mathbb{E}(e^{it_jY_n}(\bar{X}_n)^q)\right)^k\right)$$

$$= \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sum_{1\le j_1,\dots,j_k}\frac{1}{(\mathbb{E}(e^{it_jY_n}))^k}\prod_{r=1}^{k}\frac{(ih)^{j_r}}{j_r!}\mathbb{E}(e^{it_jY_n}(\bar{X}_n)^{j_r})\right)$$

$$= \exp\left(\sum_{u=1}^{\infty}\left(\sum_{k=1}^{u}\frac{(-1)^{k+1}}{k}\sum_{j_1+\dots+j_k=u}\prod_{r=1}^{k}\frac{\mathbb{E}(e^{it_jY_n}(\bar{X}_n)^{j_r})}{\mathbb{E}(e^{it_jY_n})j_r!}\right)(ih)^u\right).$$

Observe next that

$$\mathbb{E}[e^{it_jY_n}(\bar{X}_n)^{j_r}] = \mathbb{E}\left[(e^{it_jY_n}-1)\big((\bar{X}_n)^{j_r}-\mathbb{E}[(\bar{X}_n)^{j_r}]\big)\right]+\mathbb{E}[(\bar{X}_n)^{j_r}]\mathbb{E}(e^{it_jY_n})$$

and so with $C = 2K$, we have

$$\frac{\mathbb{E}[e^{it_jY_n}(\bar{X}_n)^{j_r}]}{\mathbb{E}(e^{it_jY_n})} = O(q_n(m))O(C^{j_r}) + \mathbb{E}[(\bar{X}_n)^{j_r}].$$

Plugging this into (4.4) and using that for h small enough,

$$\exp\left[\sum_{u=1}^{\infty}\left(\sum_{k=1}^{u}\frac{(-1)^{k+1}}{k}\sum_{j_1+\dots+j_k=u}\prod_{r=1}^{k}\frac{\mathbb{E}(\bar{X}_n^{j_r})}{j_r!}\right)(ih)^u\right] = \mathbb{E}\left(e^{ih\bar{X}_n}\right)$$

we conclude that

$$\psi_n(h) = \mathbb{E}(e^{ih\bar{X}_n})\exp\left[\sum_{u=1}^{\infty}(O(1))^uO(q_n(m))h^u\right].$$

Therefore,

$$\Phi_{N_0,N}(t) = \Phi_{N_0,N}(t_j)\Phi_{N_0,N}(h)\Psi_{N_0,N}(h)$$

where $\Psi_{N_0,N}(h) = \exp\left[O(M_N(m))\sum_{u=1}^{\infty}(O(1))^uh^u\right]$. □

Remark 4.3. We will see in §4.2 that the coefficients of the polynomials appearing in Theorem 2.1 depend on the coefficients of the power

series $\Psi_{N_0,N}(h)$ (see, in particular, (4.22)). The first term in this series is $ih\sum_{n=N_0+1}^{N} a_{n,j}$, where

$$a_{n,j}=\frac{\mathbb{E}[(e^{it_jY_n}-1)\bar{X}_n]}{\mathbb{E}(e^{it_jY_n})}=\frac{\mathbb{E}(e^{it_jX_n}\bar{X}_n)}{\mathbb{E}(e^{it_jX_n})} \tag{4.5}$$

while the second term is $\frac{h^2}{2}\sum_{n=N_0+1}^{N} b_{n,j}$, where

$$\begin{aligned} b_{n,j}&=\frac{\mathbb{E}[(e^{it_jY_n}-1)\bar{X}_n]^2}{\mathbb{E}(e^{it_jY_n})^2}-\frac{\mathbb{E}[(e^{it_jY_n}-1)((\bar{X}_n)^2-V(X_n))]}{\mathbb{E}(e^{it_jY_n})}\\ &=a_{n,j}^2-\frac{\mathbb{E}\big(e^{it_jX_n}(\bar{X}_n)^2\big)}{\mathbb{E}(e^{it_jX_n})}. \end{aligned} \tag{4.6}$$

In Section 9 we will use (4.5) to compute the coefficients of the polynomials from Theorem 2.1 in the case $r=2$, and (4.6) is one of the main ingredients for the computation in the case $r=3$ (which will not be explicitly discussed in this manuscript).

The next step in the proof of Theorem 2.1 is the following.

Lemma 4.4. *For $t=t_j+h\in I_j$ we can decompose*

$$\Phi_{N_0}(t)=\Phi_{N_0}(t_j+h)=\sum_{l=0}^{L}\frac{\Phi_{N_0}^{(l)}(t_j)}{l!}h^l+O\left((h\ln V_N)^{L+1}\right). \tag{4.7}$$

Proof. The lemma follows from the observation that the derivatives of Φ_{N_0} satisfy $|\Phi_{N_0}^{(k)}(t)|\le O(N_0^k)\le(C\ln V_N)^k$. □

4.2. **Completing the proof.** Recall (4.1) and consider a resonant interval I_j which does not contain 0 such that $M_N(m)\le R\ln\sigma_N$. Set $U_j=[-u_j,v_j]=I_j-t_j$. Let N_0 be as described below Lemma 4.1. Denote

$$S_{N_0,N}=S_N-S_{N_0},\ S_0=0, \tag{4.8}$$

$$V_{N_0,N}=\mathrm{Var}(S_N-S_{N_0})=V_N-V_{N_0}\quad\text{and}\quad\sigma_{N_0,N}=\sqrt{V_{N_0,N}}.$$

Then

$$V_{N_0,N}=V_N+O(\ln V_N)=V_N(1+o(1)). \tag{4.9}$$

Denote $h_{N_0,N}=h/\sigma_{N_0,N}$. By (3.1), if $|h_{N,N_0}|$ is small enough then

$$\mathbb{E}(e^{ih_{N_0,N}S_{N_0,N}})= \tag{4.10}$$

$$e^{ih_{N_0,N}\mathbb{E}(S_{N_0,N})}e^{-h^2/2}\left(1+\sum_{k=1}^{r}\frac{A_{k,N_0,N}(h)}{\sigma_{N_0,N}^k}+\frac{h^{r+1}}{\sigma_{N_0,N}^{r+1}}O(1)\right)$$

where $A_{k,N_0,N}$ are polynomials with bounded coefficients (the degree of $A_{k,N_0,N}$ depends only on k). Let us now evaluate $\int_{I_j} e^{-itk}\mathbb{E}(e^{itS_N})dt$. By Lemma 4.2,

$$\int_{I_j} e^{-itk}\Phi_N(t)dt = \tag{4.11}$$

$$e^{-it_jk}\Phi_{N_0,N}(t_j)\int_{U_j} e^{-ihk}\Phi_{N_0}(t_j+h)\Phi_{N_0,N}(h)\Psi_{N_0,N}(h)\,dh.$$

Therefore, it is enough to expand the integral on the RHS of (4.11). Fix a large positive integer L and plug (4.7) into (4.11). Note that for N is large enough, h_0 small enough and $|h|\le h_0$, Proposition 3.2 and (4.9) show that there exist positive constants c_0, c such that

$$|\Phi_{N_0,N}(h)| = |\mathbb{E}(e^{ihS_{N_0,N}})| \le e^{-c_0(V_N-V_{N_0})h^2} \le e^{-cV_Nh^2}. \tag{4.12}$$

Thus, the contribution coming from the term $O\left((h\ln V_N)^{L+1}\right)$ in the right hand side of (4.7) is at most of order

$$V_N^{R\delta}(\ln V_n)^{L+1}\int_{-\infty}^{\infty} h^{L+1}e^{-cV_Nh^2}dh$$

where δ is the diameter of I_j. Changing variables $x=\sigma_N h$, where $\sigma_N = \sqrt{V_N}$ we get that the latter term is of order $(\ln V_n)^{L+1}\sigma_N^{-(L+1-2R\delta)}$ and so when L is large enough we get that this term is $o(\sigma_N^{-r-1})$ (alternatively, we can take $L=r$ and δ to be sufficiently small). This means that it is enough to expand each integral of the form

$$\int_{U_j} e^{-ihk}h^l\Phi_{N_0,N}(h)\Psi_{N_0,N}(h)dh \tag{4.13}$$

where $l=0,1,...,L$ (after changing variables the above integral is divided by $\sigma_{N_0,N}^{l+1}$). Next, Lemma 4.2 shows that for any $d\in\mathbb{Z}$ we have

$$\Psi_{N_0,N}(h) = 1+\sum_{u=1}^{d} C_{w,N}h^u + h^{d+1}O(1+M_N(m)^{d+1})|V_N|^{O(|h|)}, \tag{4.14}$$

where $C_{w,N}=C_{w,N,t_j}$ are $O(M_N^u(m)) = O((\ln V_N)^u)$. Note that, with $a_{n,j}$ and $b_{n,j}$ defined in Remark 4.3, we have

$$C_{1,N} = i\sum_{n=N_0+1}^{N} a_{n,j} \tag{4.15}$$

and

$$C_{2,N} = \frac{1}{2}\sum_{n=N_0+1}^{N} b_{n,j} - \frac{1}{2}\left(\sum_{n=N_0+1}^{N} a_{n,j}\right)^2.$$

Take d large enough and plug (4.14) into (4.13). Using (4.12), we get again that the contribution of the term

$$h^{d+1}O(1+M_N(m)^{d+1})|V_N|^{O(|h|)}h^l\Phi_{N_0,N}(h)$$

to the above integral is $o(\sigma_N^{-r})$. Thus, it is enough to expand each term of the form

$$\int_{U_j} e^{-ihk}h^q\Phi_{N_0,N}(h)dh$$

where $0 \le q \le L+d$. Using (4.10) and making the change of variables $h \to h/\sigma_{N_0,N}$ it is enough to expand

(4.16)

$$\int_{-\infty}^{\infty} e^{-ih(k-\mathbb{E}[S_{N_0,N}])/\sigma_{N_0,N}}h^q e^{-h^2/2}\left(1+\sum_{w=1}^{r}\frac{A_{w,N_0,N}(h)}{\sigma_{N_0,N}^w}+\frac{h^{r+1}}{\sigma_{N_0,N}^{r+1}}O(1)\right)\frac{dh}{\sigma_{N_0,N}}.$$

This is achieved by using that $(it)^q e^{-t^2/2}$ is the Fourier transform of the q-th derivative of $\mathbf{n}(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ and that for any real a,

$$\int_{-\infty}^{\infty} e^{-iat}\widehat{\mathbf{n}^{(q)}}(t)dt = \mathbf{n}^{(q)}(a) = \frac{1}{\sqrt{2\pi}}(-1)^q H_q(a)e^{-a^2/2} \tag{4.17}$$

where $H_q(a)$ is the q-th Hermite polynomial.

Note that in the above expansion we get polynomials in the variable $k_{N_0,N} = \dfrac{k-\mathbb{E}[S_N - S_{N_0}]}{\sigma_{N,N_0}}$, not in the variable $k_N = \frac{k-\mathbb{E}(S_N)}{\sigma_N}$. Since $k_{N_0,N} = k_N\alpha_{N_0,N} + O(\ln \sigma_N/\sigma_N)$, where $\alpha_{N_0,N} = \sigma_N/\sigma_{N_0,N} = O(1)$, the binomial theorem shows that such polynomials can be rewritten as polynomials in the variable k_N whose coefficients are uniformly bounded in N. We also remark that in the above expansions we get the exponential terms

$$e^{-\frac{(k-a_{N_0,N})^2}{2(V_N-V_{N_0})}} \quad \text{where} \quad a_{N_0,N} = \mathbb{E}[S_N - S_{N_0}]$$

and not $e^{-(k-a_N)^2/2V_N}$ (as claimed in Theorem 2.1). In order to address this fix some $\varepsilon < 1/2$. Note that for $|k-a_{N_0,N}| \ge V_N^{\frac{1}{2}+\varepsilon}$ we have

$$e^{-\frac{(k-a_{N_0,N})^2}{2(V_N-V_{N_0})}} = o(e^{-cV_N^{2\varepsilon}}) \quad \text{and} \quad e^{-\frac{(k-a_{N_0,N})^2}{2V_N}} = o(e^{-cV_N^{2\varepsilon}}) \text{ for some } c>0.$$

Since both terms are $o(\sigma_N^{-s})$ for any s, it is enough to explain how to replace $e^{-\frac{(k-a_{N_0,N})^2}{2(V_N-V_{N_0})}}$ with $e^{-\frac{(k-a_N)^2}{2V_N}}$ when $|k-a_{N_0,N}| \le V_N^{\frac{1}{2}+\varepsilon}$ (in which case $|k-a_N| = O(V_N^{\frac{1}{2}+\varepsilon})$). For such k's we can write

$$\exp\left[-\frac{(k-a_{N_0,N})^2}{2(V_N-V_{N_0})}\right] = \tag{4.18}$$

$$\exp\left[-\frac{(k-a_{N_0,N})^2}{2V_N}\right] \exp\left[-\frac{(k-a_{N_0,N})^2 V_{N_0}}{2V_N(V_N-V_{N_0})}\right].$$

Since $\frac{(k-a_{N_0,N})^2 V_{N_0}}{2V_N(V_N-V_{N_0})} = O\left(V_N^{-(1-3\varepsilon)}\right)$, for any d_1 we have

$$\exp\left[-\frac{(k-a_{N_0,N})^2 V_{N_0}}{2V_N(V_N-V_{N_0})}\right] = \tag{4.19}$$

$$\sum_{j=0}^{d_1} \frac{V_{N_0}^j}{2^j (V_N-V_{N_0})^j j!} \left(\frac{(k-a_{N_0,N})^2}{\sigma_N^2}\right)^j + O(V_N^{-(d_1+1)(2-3\varepsilon)}).$$

Note that (using the binomial formula) the first term on the above right hand side is a polynomial of the variable $(k-a_N)/\sigma_N$ whose coefficients are uniformly bounded in N.

Next we analyze the first factor in the RHS of (4.18). As before, it is enough to consider k's such that $|k-a_N| \le V_N^{\frac{1}{2}+\varepsilon}$ for a sufficiently small ε. We have

$$\exp\left[-\frac{(k-a_{N,N_0})^2}{2V_N}\right] = \tag{4.20}$$

$$\exp\left[-\frac{(k-a_N)^2}{2V_N}\right] \exp\left[-\frac{2(k-a_N)a_{N_0}+a_{N_0}^2}{2V_N}\right].$$

Note that $\frac{(k-a_N)a_{N_0}+a_{N_0}^2}{2V_N} = k_N\beta_{N_0,N} + \theta_{N_0,N}$, where

$$\beta_{N_0,N} = \frac{a_{N_0}}{2\sigma_N} = O\left(\frac{\ln\sigma_N}{\sigma_N}\right) \quad \text{and} \quad \theta_{N_0,N} = \frac{a_{N_0}^2}{2V_N} = O\left(\frac{\ln^2\sigma_N}{V_N}\right).$$

Approximating $e^{\frac{(k-a_N)a_{N_0}+a_{N_0}^2}{2V_N}}$ by a polynomial of a sufficiently large degree d_2 in the variable $\frac{(k-a_N)a_{N_0}+a_{N_0}^2}{2V_N}$ completes the proof of existence of polynomials $P_{a,b,N}$ claimed in the theorem (the Taylor reminder in the last approximation is of order $O\left(V_N^{-d_2(\frac{1}{2}-\varepsilon)}\right)$, so we can take $d_2 = 4(r+1)$ assuming that ε is small enough).

Finally, let us show that the coefficients of the polynomials $P_{a,b,N}$ constructed above are uniformly bounded in N. In fact, we will show

that for each nonzero resonant point $t_j = 2\pi l/m$, the coefficients of the polynomials coming from integration over I_j are of order

$$O\left((1 + M_N^{q_0}(m))e^{-M_N(m)}\right),$$

where $q_0 = q_0(r)$ depends only on r.

Observe that the additional contribution to the coefficients of the polynomials coming from the transition between the variables k_N and $k_{N_0,N}$ is uniformly bounded in N. Hence we only need to show that the coefficients of the (original) polynomials in the variable $k_{N_0,N}$ are uniformly bounded in N. The possible largeness of these coefficient can only come from the terms C_{u,N,t_j}, for $u = 0, 1, 2, ..., d$ which are of order $M_N^u(m)$, respectively. However, the corresponding terms are multiplied by terms of the form $\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(\ell)}(t_j)$ for certain ℓ's which are uniformly bounded in N (see also (4.22)). We conclude that there are constants $W_j \in \mathbb{N}$ and $a_j \in \mathbb{N}$ which depend only on t_j and r so that the coefficients of the resulting polynomials are composed of a sum of at most W_j terms of order $(M_N(m))^{a_j}\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(\ell)}(t_j)$, where $\ell \le E(r)$ for some $E(r) \in \mathbb{N}$. Next, we have

$$\Phi_{N_0}^{(\ell)}(t_j)\Phi_{N_0,N}(t_j) = \tag{4.21}$$

$$\sum_{\substack{n_1,\dots,n_k \le N_0;\\ \ell_1+\dots+\ell_k=\ell}} \gamma_{\ell_1,\dots,\ell_k} \left(\prod_{q=1}^{k} \phi_{n_q}^{(\ell_q)}(t_j)\right) \left[\prod_{n\le N,\, n\neq n_k} \phi_n(t_j)\right]$$

where $\gamma_{\ell_1,\dots,\ell_k}$ are bounded coefficients of combinatorial nature. Using (3.8) we see that for each $n_1, \dots, n_k$ the product in the square brackets is at most $Ce^{-c_0 M_N(m)+O(1)}$ for some $C, c_0 > 0$. Hence

$$|\Phi_{N_0}^{(\ell)}(t_j)\Phi_{N_0,N}(t_j)| \le \hat{C} N_0^{\ell}\, e^{-c_0 M_N(m)},\ \hat{C} > 0.$$

Now, observe that the definition of N_0 gives $M_N(m) \ge \varepsilon_0 N_0$, $\varepsilon_0 > 0$. Therefore $|\Phi_{N_0}^{(\ell)}(t_j)\Phi_{N_0,N}(t_j)| \le C_0 M_N^{\ell}(m)e^{-c_0 M_N(m)}$, and so each one of the above coefficients is of order $M_N^{\ell'}(m)e^{-c_0 M_N(m)}$ for some ℓ' which does not depend on N. □

Remark 4.5. The transition between the variables $k_{N_0,N}$ and k_N changes the monomials of the polynomials $P_{a,b,N}$, $a \neq 0$ coming from integration over I_j, for $t_j \neq 0$ into monomials of the form $\frac{c_N a_{N_0}^{j_1}\sigma_{N_0}^{j_2}k_N^{j_3}}{\sigma_N^u}$ for some bounded sequence (c_N), $j_1, j_2, j_3 \ge 0$ and $u \in \mathbb{N}$. As we have explained, the coefficients of these monomials are uniformly bounded. Still, it seems more natural to consider such monomials as part of the polynomial $P_{a,b+u,N}$. In this case we still get polynomials with bounded

coefficients since a_{N_0} and σ_{N_0} are both $O(N_0)$, $N_0 = O(M_N(m))$ and c_N contains a term of the form $\Phi^{(\ell)}_{N_0}(t_j)\Phi_{N_0,N}(t_j)$.

Remark 4.6. As can be seen from the proof, the resulting expansions might contain terms corresponding to σ_N^{-s} for $s > r$. Such terms can be disregarded. For $\frac{|k-a_N|}{\sigma_N} \le V_N^\varepsilon$ this follows because the coefficients of our exapansions are $O(1)$ and for $\frac{|k-a_N|}{\sigma_N} \ge V_N^\varepsilon$ this follows from (4.12). In practice, some of the polynomials $P_{a,b,N}$ with $b \le r$ might have coefficients which are $o(\sigma_N^{b-r})$ (e.g. when $b+u > r$ in the last remark) so they also can be disregarded. The question when the terms $P_{a,b,N}$ may be disregarded is in the heart of the proof of Theorem 1.5 given in the next section.

4.3. **A summary.** The proofs of Proposition 1.2, Theorem 1.3 and Theorem 1.5 will be based on careful analysis of the formulas of the polynomials from Theorem 2.1. For this purpose, it will be helpful to summarize the main conclusions from the proof of Theorem 2.1. Let $r \ge 1$ and $t_j = 2\pi l/m$ be a nonzero resonant point. Then the arguments in the proof of Theorem 2.1 yield that the contribution to the expansion coming from t_j is

$$\mathbf{C}_j(k) := \tag{4.22}$$

$$e^{-it_jk}\Phi_{N_0,N}(t_j)\sum_{s\le r-1}\left(\sum_{u+l=s}\frac{\Phi^{(l)}_{N_0}(t_j)C_{u,N}}{l!}\right)\int_{U_j}e^{-ihk}h^s\Phi_{N_0,N}(h)dh$$

where $U_j = I_j - t_j$, $C_{u,N}$ are given by (4.14) and $C_{0,N} = 1$. When $t_j = 0$ then it is sufficient to consider only $s = 0$, $N_0 = 0$ and the contribution is just the integral

$$\int_{-\delta}^{\delta} e^{-ihk}\Phi_N(h)dh$$

where δ is small enough. As in (4.16), changing variables we can replace the integral corresponding to h^s with

$$\sigma_{N_0,N}^{-s-1}\int_{-\infty}^{\infty} e^{-ih(k-\mathbb{E}[S_{N_0,N}])/\sigma_{N_0,N}}h^s e^{-h^2/2} \tag{4.23}$$

$$\times\left(1+\sum_{w=1}^{r}\frac{A_{w,N_0,N}(h)}{\sigma_{N_0,N}^w}+\frac{h^{r+1}}{\sigma_{N_0,N}^{r+1}}O(1)\right)dh.$$

After that was established, the proof was completed using (4.17) and some estimates whose whose purpose was to make the transition between the variables $k_{N_0,N}$ and k_N.

5. Uniqueness of trigonometric expansions.

In several proofs we will need the following result.

Lemma 5.1. *Let $r \geq 1$ and $d \geq 0$. Set $\mathcal{R}_0 = \mathcal{R} \cup \{0\}$ where $\mathcal{R}$ is the set of nonzero resonant points. For any $t_j \in \mathcal{R}_0$, let $A_{0,N}(t_j)$,...,$A_{d,N}(t_j)$ be sequences so that, uniformly in k such that $k_N = \frac{k - \mathbb{E}(S_N)}{\sigma_N} = O(1)$ we have*

$$\sum_{t_j \in \mathcal{R}_0} e^{-it_j k} \left(\sum_{m=0}^{d} k_N^m A_{m,N}(t_j) \right) = o(\sigma_N^{-r}).$$

Then for all m and t_j

$$A_{m,N}(t_j) = o(\sigma_N^{-r}). \tag{5.1}$$

In particular the polynomials from the definition of the (generalized) Edgeworth expansions are unique up to terms of order $o(\sigma_N^{-r})$.

Proof. The proof is by induction on d. Let us first set $d = 0$. Then, for any $k \in \mathbb{N}$ we have

$$\sum_{t_j \in \mathcal{R}_0} e^{-it_j k} A_{0,N}(t_j) = o(\sigma_N^{-r}). \tag{5.2}$$

Let T be the number of nonzero resonant points, and let us relabel them as $\{x_1, ..., x_T\}$. Consider the vector

$$\mathfrak{A}_N = (A_{0,N}(0), A_{0,N}(x_1), ..., A_{0,N}(x_T)).$$

Let $\mathcal{V}$ be the transpose of the Vandermonde matrix of the distinct numbers $\alpha_j = e^{-ix_j}$, $j = 0, 1, 2, ..., T$ where $x_0 := 0$. Then $\mathcal{V}$ is invertible and by considering $k = 0, 1, 2, ..., T$ in (5.2) we see that (5.2) holds true if and only if

$$\mathfrak{A}_N = \mathcal{V}^{-1} o(\sigma_N^{-r}) = o(\sigma_N^{-r}).$$

Alternatively, let Q be the least common multiple of the denominators of $t_j \in \mathcal{R}$. Let $a_N(p) = A_{0,N}(2\pi p/Q)$ if $2\pi p/Q$ is a resonant point and 0 otherwise. Then for $m = 0, 1, ..., Q-1$ we have

$$\hat{a}_N(m) := \sum_{p=0}^{Q-1} a_N(p) e^{-2\pi pm/Q} = o(\sigma_N^{-r}).$$

Therefore, by the inversion formula of the discrete Fourier transform,

$$a_N(p) = Q^{-1} \sum_{m=0}^{Q-1} \hat{a}_N(m) e^{2\pi imp/Q} = o(\sigma_N^{-r}).$$

Assume now that the theorem is true for some $d \geq 0$ and any sequences functions $A_{0,N}(t_j), ..., A_{d,N}(t_j)$. Let $A_{0,N}(t_j), ..., A_{d+1,N}(t_j)$ be

sequences so that uniformly in k such that $k_N := \dfrac{k - \mathbb{E}(S_N)}{\sigma_N} = O(1)$ we have

$$\sum_{t_j \in \mathcal{R}_0} e^{-it_j k} \left(\sum_{m=0}^{d+1} k_N^m A_{m,N}(t_j) \right) = o(\sigma_N^{-r}). \tag{5.3}$$

Let us replace k with $k' = k + [\sigma_N]Q$, where Q is the least common multiply of all the denominators of the nonzero t_j's. Then $e^{-it_j k} = e^{-it_j k'}$. Thus,

$$\sum_{t_j \in \mathcal{R}_0} e^{-it_j k} \left(\sum_{m=0}^{d+1} (k_N'^m - k_N^m) A_{m,N}(t_j) \right) = o(\sigma_N^{-r}).$$

Set $L_N = [\sigma_N]Q/\sigma_N \approx Q$. Then the LHS above equals

$$L_N \sum_{t_j \in \mathcal{R}_0} e^{-it_j k} \left(\sum_{s=0}^{d} k_N^s \mathcal{A}_{s,N}(t_j) \right)$$

where

$$\mathcal{A}_{s,N}(t_j) = \sum_{m=s+1}^{d+1} A_{m,N}(t_j) L_N^{m-s-1}.$$

By the induction hypothesis we get that

$$\mathcal{A}_{s,N}(t_j) = o(\sigma_N^{-r})$$

for any $s = 0, 1, ..., d$. In particular

$$\mathcal{A}_{d,N}(t_j) = A_{d+1,N}(t_j) = o(\sigma_N^{-r}).$$

Substituting this into (5.3) we can disregard the last term $A_{d+1,N}(t_j)$. Using the induction hypothesis with $A_{0,N}(t_j), A_{1,N}(t_j), ..., A_{d,N}(t_j)$ we obtain (5.1). □

6. First order expansions

In this section we will consider the case $r = 1$. By (4.22) and (4.23), we see that the contribution coming from the integral over I_j is

$$\sigma_{N_0,N}^{-1} e^{-it_j k} \Phi_N(t_j) \sqrt{2\pi} e^{-k_{N_0,N}^2/2} + o(\sigma_N^{-1})$$

where $k_{N_0,N} = (k - \mathbb{E}(S_{N_0,N}))/\sigma_{N_0,N}$. Now, using the arguments at the end of the proof of Theorem 2.1 when $r = 1$ we can just replace $e^{-k_{N_0,N}^2/2}$ with $e^{-(k-\mathbb{E}(S_N))^2/2V_N}$ (since it is enough to consider the case

when $k_{N_0,N}$ and $k_{0,N}$ are of order V_N^{ε}). Therefore, taking into account that $\sigma_{N_0,N}^{-1} - \sigma_N^{-1} = O(\sigma_N^{-2} N_0)$ we get that

$$\sqrt{2}\pi \mathbb{P}(S_N = k) = \tag{6.1}$$
$$\left(1 + \sum_{t_j \in \mathcal{R}} e^{-it_j k} \Phi_N(t_j)\right) \sigma_N^{-1} e^{-(k-\mathbb{E}[S_N])^2/2V_N} + o(\sigma_N^{-1}).$$

Here $\mathcal{R}$ is the set of all nonzero resonant points $t_j = 2\pi l_j/m_j$. Indeed the contribution of the resonant points satisfying $M_N(m_j) \leq R(r,K) \ln V_N$ is analyzed in §4.2. The contribution of the other nonzero resonant points t is $o(\sigma_N^{-1})$ due to (3.8) in Section 3. In particular, (3.8) implies that $\Phi_N(t) = o(\sigma_N^{-1})$ so adding the points with $M_N(m_j) \geq R(r,K) \ln V_N$ only changes the sum in the RHS of (6.1) by $o(\sigma_N^{-1})$.

Corollary 6.1. *The local limit theorem holds if and only if* $\max_{t \in R} |\Phi_N(t)| = o(1)$.

Proof. It follows from (6.1) that the LLT holds true if and only if for any k we have

$$\sum_{t_j \in \mathcal{R}} e^{-it_j k} \Phi_N(t_j) = o(1).$$

Now, the corollary follows from Lemma 5.1. □

Before proving Theorem 1.3 we recall a standard fact which will also be useful in the proofs of Theorems 1.6 and 1.8.

Lemma 6.2. *Let $\{\mu_N\}$ be a sequence of measures probability measures on $\mathbb{Z}/m\mathbb{Z}$ and $\{\gamma_N\}$ be a positive sequence. Then $\mu_N(a) = \frac{1}{m} + O(\gamma_N)$ for all $a \in \mathbb{Z}/m\mathbb{Z}$ if and only iff $\hat{\mu}_N(b) = O(\gamma_N)$ for all $b \in (\mathbb{Z}/m\mathbb{Z}) \setminus \{0\}$ where $\hat{\mu}$ is the Fourier transform of μ.*

Proof. If $\mu_N(a) = \frac{1}{m} + O(\gamma_n)$ then

$$\hat{\mu}_N(b) = \sum_{a=0}^{m-1} \mu_N(a) e^{2\pi i ab/m} = \sum_{a=0}^{m-1} \frac{1}{m} e^{2\pi i ab/m} + O(\gamma_N) = O(\gamma_N).$$

Next $\hat{\mu}_N(0) = 1$ since μ_N are probabilities. Hence if $\hat{\mu}_N(b) = O(\gamma_N)$ for all $b \in (\mathbb{Z}/m\mathbb{Z}) \setminus \{0\}$ then

$$\mu_N(a) = \frac{1}{m} \sum_{b=0}^{m-1} \hat{\mu}_N(b) e^{-2\pi i ba/m} = \frac{1}{m} \left[1 + \sum_{b=1}^{m-1} \hat{\mu}_N(b) e^{-2\pi i ba/m}\right] = \frac{1}{m} + O(\gamma_N)$$

as claimed. □

Proof of Theorem 1.3. The equivalence of conditions (b) and (c) comes from the fact that for non-resonant points the characteristic function decays faster than any power of σ_N (see (3.7)).

The equivalence of (a) and (c) is due to Corollary 6.1. Finally, the equivalence between (c) and (d) comes from Lemma 6.2. □

Remark 6.3. Theorem 1.3 can also be deduced from [7, Corollary 1.4]. Indeed the corollary says that either the LLT holds or there is an integer $h \in (0, 2K)$ and a bounded sequence $\{a_N\}$ such that the limit

$$\mathbf{p}(j) = \lim_{N\to\infty} \mathbb{P}(S_N - a_N = j \bmod h)$$

exists and moreover if $k - a_n \equiv j \bmod h$ then

$$\sigma_N \mathbb{P}(S_N = k) = \mathbf{p}(j) h \mathfrak{g}\left(\frac{k - \mathbb{E}(S_N)}{\sigma_N}\right) + o\left(\sigma_N^{-1}\right).$$

Thus in the second case the LLT holds iff $\mathbf{p}(j) = \frac{1}{h}$ for all j which is equivalent to S_N being asymptotically uniformly distributed mod h and also to the Fourier transform of $\mathbf{p}(j)$ regarded as the measure on $\mathbb{Z}/(h\mathbb{Z})$ being the δ measure at 0. Thus the conditions (a), (c) and (d) of the theorem are equivalent. Also by the results of [7, Section 2] (see also [§3.3.2][10]) if $\mathbb{E}\left(e^{i\xi S_N}\right)$ does not converge to 0 for some non zero ξ then $\left(\frac{2\pi}{\xi}\right)\mathbb{Z} \bigcap 2\pi\mathbb{Z}$ is a lattice in $\mathbb{R}$ which implies that ξ is resonant, so condition (b) of the theorem is also equivalent to the other conditions.

Proof of Proposition 1.2. Let S_N satisfy LLT. Fix $m \in \mathbb{N}$ and suppose that $\sum_n q_n(m) < \infty$. Let s_n be the most likely residue of X_n mod m. Then for $t = \frac{2\pi l}{m}$ we have

$$\phi_n(t) = e^{its_n} - \sum_{j \not\equiv s_n \bmod m} \mathbb{P}(X_n \equiv j \bmod \mathrm{m}) \left(e^{its_n} - e^{itj}\right),$$

so that $1 \geq |\phi_n(t)| \geq 1 - 2mq_n(m)$. It follows that for each $\varepsilon > 0$ there is $N(\varepsilon)$ such that $\left|\prod_{n=N(\varepsilon)+1}^{\infty} \phi_n(t)\right| > 1 - \varepsilon$. Applying this for $\varepsilon = \frac{1}{2}$ we have

$$\frac{1}{2} \leq \liminf_{N\to\infty} \left|\Phi_{N(1/2),N}(t)\right| \neq 0. \tag{6.2}$$

On the other hand the LLT implies that

$$\lim_{N\to\infty} \Phi_N(t) = 0. \tag{6.3}$$

Since $\Phi_N = \Phi_{N(1/2)}\Phi_{N(1/2),N}$, (6.2) and (6.3) imply that $\Phi_{N(1/2)}(t) = 0$. Since $\Phi_{N(1/2)}\left(\frac{2\pi l}{m}\right) = \prod_{n=1}^{N(1/2)} \phi_n\left(\frac{2\pi l}{m}\right)$ we conclude that there exists $n_l \le N(1/2)$ such that $\phi_{n_l}(\frac{2\pi l}{m}) = 0$. Hence $Y = X_{n_1} + X_{n_2} + \dots X_{n_{m-1}}$ satisfies $\mathbb{E}\left(e^{2\pi i(k/m)Y}\right) = 0$ for $k = 1, \dots m-1$. By Lemma 6.2 both Y and S_N for $N \ge N(1/2)$ are uniformly distributed. This proves the proposition. □

7. Characterizations of Edgeworth expansions of all orders.

7.1. **Derivatives of the non-perturbative factor.** Next we prove the following result.

Proposition 7.1. *Fix $r \ge 1$, and assume that $M_N \le R(r,K)\ln\sigma_N$ (possibly along a subsequence). Then Edgeworth expansions of order r hold true (i.e. (1.2) holds for such N's) iff for each $t_j \in \mathcal{R}$ and $0 \le \ell < r$ (along the underlying subsequence) we have*

$$\sigma_N^{r-\ell-1}\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(\ell)}(t_j) = o(1). \tag{7.1}$$

Proof. First, in view of (4.22) and (4.12), it is clear that the condition (7.1) is sufficient for expansions of order r.

Let us now prove that the condition (7.1) is necessary for the expansion of order r.

We will use induction on r. For $r = 1$ (see (6.1)) our expansions read

$$\mathbb{P}(S_N = k) = \sigma_N^{-1}e^{-k_N^2/2}\left[1 + \sum_{t_j\in\mathcal{R}} e^{-it_jk}\Phi_N(t_j)\right] + o(\sigma_N^{-1}).$$

Therefore if

$$\mathbb{P}(S_N = k) = \sigma_N^{-1}e^{-k_N^2/2}P_N(k_N) + o(\sigma_N^{-1})$$

for some polynomial P_N then Lemma 5.1 tells us that, in particular $\Phi_N(t_j) = o(1)$ for each $t_j \in \mathcal{R}$.

Let us assume now that the necessity part in Proposition 7.1 holds for $r' = r-1$ and prove that it holds for r. We will use the following lemma.

Lemma 7.2. *Assume that for some $t_j \in \mathcal{R}$,*

$$\sigma_N^{r-2-l}\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(l)}(t_j) = o(1),\ l = 0,1,...,r-2. \tag{7.2}$$

Then, up to an $o(\sigma_N^{-r})$ error term, the contribution of t_j to the generalized Edgeworh expansions of order r is

$$e^{-it_jk}e^{-k_N^2/2}\left(\frac{\Phi_{N_0}(t_j)}{\sigma_N}+\sum_{q=2}^{r}\frac{\mathscr{H}_{N,q}(k_N)}{\sigma_N^q}\right) \tag{7.3}$$

with

$$\mathscr{H}_{N,q}(x)=\mathscr{H}_{N,q}(x;t_j)= \tag{7.4}$$
$$\mathcal{H}_{N,q,1}(x)+\mathcal{H}_{N,q,2}(x)+\mathcal{H}_{N,q,3}(x)+\mathcal{H}_{N,q,4}(x)$$

where

$$\mathcal{H}_{N,q,1}(x)=\frac{(i)^{q-1}H_{q-1}(x)\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(q-1)}(t_j)}{(q-1)!},$$

$$\mathcal{H}_{N,q,2}(x)=\frac{(i)^{q-1}H_{q-1}(x)\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(q-2)}(t_j)C_{1,N,t_j}}{(q-2)!},$$

$$\mathcal{H}_{N,q,3}(x)=\frac{a_{N_0}(i)^{q-2}H'_{q-2}(x)\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(q-2)}(t_j)}{(q-2)!},$$

$$\mathcal{H}_{N,q,4}(x)=-\frac{xa_{N_0}(i)^{q-2}H_{q-2}(x)\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(q-2)}(t_j)}{(q-2)!},$$

and H_q are Hermite polynomials.

Here C_{1,N,t_j} is given by (4.15) when $M_N(m)\le R(K,r)\ln\sigma_N$, and $C_{1,N,t_j}=0$ when $M_N(m)>R(K,r)\ln\sigma_N$. (Note that in either case $C_{1,N,t_j}=O(M_N(m))=O(\ln\sigma_N)$).

As a consequence, when the Edgeworth expansions of order r hold true and (7.2) holds, then uniformly in k so that $k_N=O(1)$ we have

$$\frac{\Phi_N(t_j)}{\sigma_N}+\sum_{q=2}^{r}\frac{\mathscr{H}_{N,q}(k_N;t_j)}{\sigma_N^q}=o(\sigma_N^{-r}). \tag{7.5}$$

The proof of the lemma will be given in §7.2 after we finish the proof of Proposition 7.1.

By the induction hypothesis the condition (7.2) holds true. Let us prove now that for $\ell=0,1,2,...,r-1$ and $t_j\in\mathcal{R}$ we have

$$\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(\ell)}(t_j)=o(\sigma_N^{-r+1+\ell}).$$

Let us write

$$\frac{\Phi_N(t_j)}{\sigma_N}+\sum_{q=2}^{r}\frac{\mathscr{H}_{N,q}(k_N)}{\sigma_N^q}=\sum_{m=0}^{r-1}k_N^mA_{m,N}(t_j).$$

Applying Lemmas 5.1 and 7.2 we get that

$$A_{m,N}(t_j)=o(\sigma_N^{-r})$$

for each $0 \leq m \leq r-1$ and $t_j \in \mathcal{R}$.

Fix $t_j \in \mathcal{R}$. Using Lemma 7.2 and the fact that the Hermite polynomials H_u have the same parity as u and that their leading coefficient is 1 we have

$$A_{r-1,N}(t_j) = \sigma_N^{-r}(i)^{r-2}\Big(i\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-1)}(t_j)/(r-1) + \Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-2)}(t_j)(iC_{1,N} - a_{N_0})\Big) = o(\sigma_N^{-r}) \tag{7.6}$$

and

$$A_{r-2,N}(t_j) = \sigma_N^{-r+1}(i)^{r-3}\Big(i\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-2)}(t_j)/(r-2) + \Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-3)}(t_j)(iC_{1,N} - a_{N_0})\Big) = o(\sigma_N^{-r}). \tag{7.7}$$

Since $\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-3)}(t_j) = o(\sigma_N^{-1})$, (7.7) yields

$$\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-2)}(t_j) = o(\sigma_N^{-1}\ln\sigma_N).$$

Plugging this into (7.6) we get

$$\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-1)}(t_j) = o(1).$$

Therefore we can just disregard $\mathscr{H}_{r,N}(k_N;t_j)$ since its coefficients are of order $o(\sigma_N^{-r})$. Since the term $\mathcal{H}_{r,N}(k_N;t_j)$ no longer appears, repeating the above arguments with $r-1$ in place of r we have

$$A_{r-3,N}(t_j) = \sigma_N^{-r+2}(i)^{r-4}\Big(i\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-3)}(t_j)/(r-3) + \Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-4)}(t_j)(iC_{1,N} - a_{N_0})\Big) = o(\sigma_N^{-r}).$$

Since $\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-4)}(t_j) = o(\sigma_N^{-2})$, the above asymptotic equality yields that

$$\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-3)}(t_j) = o(\sigma_N^{-2}\ln\sigma_N).$$

Plugging this into (7.7) we get

$$\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(r-2)}(t_j) = o(\sigma_N^{-1}).$$

Hence, we can disregard also the term $\mathscr{H}_{r-1,N}(k_N;t_j)$. Proceeding this way we get that $\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(\ell)}(t_j) = o(\sigma_N^{\ell+1-r})$ for any $0 \leq \ell < r$. $\square$

Before proving Lemma 7.2, let us state the following result, which is a consequence of Proposition 7.1 and (3.8).

Corollary 7.3. *Suppose that for each nonzero resonant point t we have $\inf_n |\phi_n(t)| > 0$. Then for any r, the sequence S_N obeys Edgeworth expansions of order r if and only if $\Phi_N(t) = o(\sigma_N^{1-r})$ for each nonzero resonant point t.*

7.2. **Proof of Lemma 7.2.**

Proof. First, because of (7.2), for each $0 \le s \le r-1$, the terms indexed by $l < s-1$ in (4.22), are of order $o(\sigma_N^{-r})$ and so they can be disregarded. Therefore, we need only to consider the terms indexed by $l = s$ and $l = s-1$. For such l, using again (7.2) we can disregard all the terms in (4.23) indexed by $w \ge 1$, since the resulting terms are of order $o(\sigma_N^{-w-r} \ln \sigma_N) = o(\sigma_N^{-r})$. Now, since $\sigma_{N_0,N}^{-1} - \sigma_N^{-1} = O(V_{N_0}/\sigma_N^3)$ we can replace $\sigma_{N_0,N}$ with σ_N^{-1} in (4.22), as the remaining terms are of order $o(\sigma_N^{-r-1})$. Therefore, using (4.17) we get the following contributions from $t_j \in \mathcal{R}$,

$$e^{-it_jk}\, e^{-k_{N_0,N}^2/2} \left(\frac{\Phi_N(t_j)}{\sigma_N} + \sum_{q=2}^{r} \frac{\mathcal{H}_{N,q}(k_{N_0,N})}{\sigma_N^q} \right)$$

where $\mathcal{H}_{N,q}(x) = \mathcal{H}_{N,q,1}(x) + \mathcal{H}_{N,q,2}(x)$ and $\mathcal{H}_{N,q,j}, j = 1, 2$ are defined after (7.4). Note that when $x = O(1)$ and $q < r$,

$$\frac{\mathcal{H}_{N,q,1}(x)}{\sigma_N^q} = o(\sigma_N^{-r+1}) \quad \text{and} \quad \frac{\mathcal{H}_{N,q,2}(x)}{\sigma_N^q} = o(\sigma_N^{-r} \ln \sigma_N). \tag{7.8}$$

while when $q = r$,

$$\frac{\mathcal{H}_{N,r,1}(x)}{\sigma_N^r} = O(\sigma_N^{-r} \ln \sigma_N) \quad \text{and} \quad \frac{\mathcal{H}_{N,r,2}(x)}{\sigma_N^r} = o(\sigma_N^{-r} \ln \sigma_N). \tag{7.9}$$

Next

$$k_{N_0,N} = (1 + \rho_{N_0,N}) k_N + \frac{a_{N_0}}{\sigma_N} + \theta_{N_0,N}$$

where $\rho_{N_0,N} = \sigma_N/\sigma_{N_0,N} - 1 = O(\ln \sigma_N/\sigma_N^2)$ and

$$\theta_{N_0,N} = a_{N_0} \left(\frac{1}{\sigma_{N_0,N}} - \frac{1}{\sigma_N} \right) = O(\ln^2 \sigma_N/\sigma_N^3).$$

Hence, when $|k_{N_0,N}| \le \sigma_N^{\varepsilon}$ (and so $k_N = O(\sigma_N^{\varepsilon})$) for some $\varepsilon > 0$ small enough then for each $m \ge 1$ we have

$$k_{N_0,N}^m = k_N^m + m k_N^{m-1} a_{N_0}/\sigma_N + o(\sigma_N^{-1}).$$

Therefore, (7.8) and (7.9) show that upon replacing $H_{q-1}(k_{N_0,N})$ with $H_{q-1}(k_N)$ the only additional term is

$$\frac{a_{N_0}(i)^{q-1}H'_{q-1}(k_N)\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(q-1)}(t_j)}{(q-1)!\sigma_N^{q+1}}$$

for $q = 2, 3, ..., r-1$. We thus get that the contribution of t_j is

$$e^{-it_jk}\ e^{-k_{N_0,N}^2/2}\left(\frac{\Phi_N(t_j)}{\sigma_N}+\sum_{q=2}^{r}\frac{\mathcal{C}_{N,q}(k_N)}{\sigma_N^q}\right)$$

where

$$\mathcal{C}_{N,q}(x)=\mathcal{H}_{N,q}(x)+\frac{a_{N_0}(i)^{q-2}H'_{q-2}(x)\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(q-2)}(t_j)}{(q-2)!}.$$

Note that $\mathcal{C}_{N,2}(\cdot)=\mathcal{H}_{N,2}(\cdot)$. Finally, we can replace $e^{-k_{N_0,N}^2/2}$ with

$$(1-k_Na_{N_0}/\sigma_N)e^{-k_N^2/2}$$

since all other terms in the transition between $e^{-k_{N_0,N}^2/2}$ to $e^{-k_N^2/2}$ are of order $o(\sigma_N^{-1})$ (see (4.18) and (4.19)). The term $-k_Na_{N_0}/\sigma_N$ shifts the u-th order term to the $u+1$-th term, $u = 1, 2, ..., r-1$ multiplied by $-k_Na_{N_0}$. Next, relying on (7.8) and (7.9) we see that after multiplied by $k_Na_{N_0}/\sigma_N$, the second term $\mathcal{H}_{N,q,2}(k_N)$ from the definition of $\mathcal{H}_{N,q}(k_N)$ is of order $o(\sigma_N^{-r-1}\ln^2\sigma_N)\sigma_N^q$ and so this product can be disregarded. Similarly, we can ignore the additional contribution coming from multiplying the second term from the definition of $\mathcal{C}_{N,q}(k_N)$ by $-k_Na_{N_0}/\sigma_N$ (since this term is of order $o(\sigma_N^{-r}\ln\sigma_N)\sigma_N^q$). We conclude that, up to a term of order $o(\sigma_N^{-r})$, the total contribution of t_j is

$$e^{-it_jk}\ e^{-k_N^2/2}\left(\frac{\Phi_N(t_j)}{\sigma_N}+\sum_{q=2}^{r}\frac{\mathscr{H}_{N,q}(k_N;t_j)}{\sigma_N^q}\right)$$

where $\mathscr{H}_{N,q}(x;t_j)=\mathcal{C}_{N,q}(x)-\dfrac{xa_{N_0}(i)^{q-2}H_{q-2}(x)\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(q-2)}(t_j)}{(q-2)!}$

which completes the proof of (7.3).

Next we prove (7.5). On the one hand, by assumption we have Edgeworth expansions or order r, and, on the other hand, we have the expansions from Theorem 2.1. Therefore, the difference between the two must be $o(\sigma_N^{-r})$. Since the usual Edgeworth expansions contain no terms corresponding to nonzero resonant points, applying Lemma 5.1 and (7.3) we obtain (7.5). □

Note that the formulas of Lemma 7.2 together with already proven Proposition 7.1 give the following result.

Corollary 7.4. *Suppose that $\mathbb{E}(S_N)$ is bounded, S_N admits the Edgeworth expansion of order $r-1$, and, either*

(a) for some $\bar{\varepsilon} \le 1/(8K)$ we have $N_0 = N_0(N, t_j, \bar{\varepsilon}) = 0$ for each each nonzero resonant point t_j,

or (b) $\varphi := \min_{t\in\mathcal{R}} \inf_n |\phi_n(t)| > 0$.

Then

$$\sqrt{2\pi}\mathbb{P}(S_N = k)$$

$$= e^{-k_N^2/2}\left[\mathcal{E}_r(k_N) + \sum_{t_j\in\mathcal{R}}\left(\frac{\Phi_N(t_j)}{\sigma_N} + \frac{ik_N C_{1,N,t_j}\Phi_N(t_j)}{\sigma_N^2}\right)e^{-it_jk}\right] + o(\sigma_N^{-r})$$

where $\mathcal{E}_r(\cdot)$ is the Edgeworth polynomial of order r (i.e. the contribution of $t=0$), and we recall that

$$iC_{1,N,t_j} = -\sum_{n=1}^{N}\frac{\mathbb{E}(e^{it_jX_n}\bar{X}_n)}{\mathbb{E}(e^{it_jX_n})}.$$

Proof. Part (a) holds since under the assumption that $N_0 = 0$ all terms $\mathcal{H}_{N,q,j}$ in (7.4) except $\mathcal{H}_{N,2,2}$ vanish. Part (b) holds since in this case the argument proceeds similarly to the proof of Theorem 2.1 if we set $N_0 = 0$ for any t_j (since we only needed N_0 to obtain a positive lower bound on $|\phi_n(t_j)|$ for $t_j \in \mathcal{R}$ and $N_0 < n \le N$). □

Remark 7.5. Observe that $\sigma_N^{-1} \gg |C_{1,N,t_j}|\sigma_N^{-2}$, so if the conditions of the corollary are satisfied but $|\Phi_N(t_j)| \le c\sigma_N^{1-r}$ (possibly along a subsequence), then the leading correction to the Edgeworth expansion comes from

$$e^{-k_N^2/2}\sum_{t_j\in\mathcal{R}}\left(\frac{\Phi_N(t_j)}{\sigma_N}\right).$$

Thus Corollary 7.4 strengthens Corollary 7.3 by computing the leading correction to the Edgeworth expansion when the expansion does not hold.

7.3. **Proof of Theorem 1.5.** We will use the following.

Lemma 7.6. *Let t_j be a nonzero resonant point, $r > 1$ and suppose that $M_N \le R\ln\sigma_N$, $R = R(r,K)$ and that $|\mathbb{E}(S_N)| = O(\ln\sigma_N)$. Then (7.1) holds for all $0 \le \ell < r$ if and only if*

$$|\Phi_N^{(\ell)}(t_j)| = o\left(\sigma_N^{-r+\ell+1}\right) \tag{7.10}$$

for all $0 \le \ell < r$.

Proof. Let us first assume that (7.1) holds. Recall that

$$\Phi_N(t) = \Phi_{N_0}(t)\Phi_{N_0,N}(t) \tag{7.11}$$

with

$$\Phi_{N_0,N}(t) = \Phi_{N_0,N}(t_j)\Phi_{N_0,N}(h)\Psi_{N_0,N}(h) \tag{7.12}$$

where $t = t_j + h$ and

$$\Psi_{N_0,N}(h) = \exp\left[O(M_N(m))\sum_{u=1}^{\infty}(O(1))^u h^u\right]. \tag{7.13}$$

For $\ell = 0$ the result reduces to (7.11). For larger ℓ's we have

$$\Phi_N^{(\ell)}(t_j) = \Phi_{N_0,N}(t_j)\Phi_{N_0}^{(\ell)}(t_j) + \sum_{k=0}^{\ell-1}\binom{\ell}{k}\Phi_{N_0,N}^{(\ell-k)}(t_j)\Phi_{N_0}^{(k)}(t_j). \tag{7.14}$$

Fix some $k < \ell$. Then by (7.12),

$$\begin{aligned}\Phi_{N_0,N}^{(\ell-k)}(t_j) = \Phi_{N_0,N}(t_j)\sum_{u=0}^{\ell-k}\binom{\ell-k}{u}\Phi_{N_0,N}^{(u)}(0)\Psi_{N_0,N}^{(\ell-k-u)}(0)\\ = O(\ln^{\ell-k}\sigma_N)\Phi_{N_0,N}(t_j)\end{aligned}$$

where we have used that $\bar{S}_{N_0,N} = S_{N_0,N} - \mathbb{E}(S_{N_0,N})$ satisfies $|\mathbb{E}[(\bar{S}_{N_0,N})^q]| \le C_q\sigma_{N_0,N}^{2q}$, (see (3.3)). Therefore

$$\Phi_{N_0,N}^{(\ell-k)}(t_j)\Phi_{N_0}^{(k)}(t_j) = O(\ln^{\ell-k}\sigma_N)\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(k)}(t_j). \tag{7.15}$$

Finally, by (7.1) we have

$$\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(k)}(t_j) = o(\sigma_N^{k+1-r})$$

and so, since $k < \ell$,

$$\Phi_{N_0}^{(\ell-k)}(t_j)\Phi_{N_0}^{(k)}(t_j) = o(\sigma_N^{\ell+1-r}).$$

This completes the proof that (7.10) holds.

Next, suppose that (7.10) holds for each $0 \le \ell < r$. Let use prove by induction on ℓ that

$$|\Phi_{N_0,N}\Phi_{N_0}^{(\ell)}(t_j)| = o\left(\sigma_N^{-r+\ell+1}\right). \tag{7.16}$$

For $\ell = 1$ this follows from (7.14). Now take $\ell > 1$ and assume that (7.16) holds with k in place of ℓ for each $k < \ell$. Then by (7.14), (7.15) and the induction hypothesis we get that

$$\Phi_N^{(\ell)}(t_j) = \Phi_{N_0,N}(t_j)\Phi_{N_0}^{(\ell)}(t_j) + o(\sigma_N^{\ell+1-r}).$$

By assumption we have $\Phi_N^{(\ell)}(t_j) = o(\sigma_N^{\ell+1-r})$ and hence

$$\Phi_{N_0,N}(t_j)\Phi_{N_0}^{(\ell)}(t_j) = o(\sigma_N^{\ell+1-r})$$

as claimed. □

Theorem 1.5 in the case $M_N \le R \ln \sigma_N$ follows now by first replacing X_n with $X_n - c_n$, where (c_n) is a bounded sequence of integers so that $\mathbb{E}[S_N - C_N] = O(1)$, where

$$C_N = \sum_{j=1}^{n} c_j \tag{7.17}$$

(see Lemma 3.4 in [10]), and then applying Lemma 7.6 and Proposition 7.1.

It remains to consider the case when $M_N(m) \ge \bar{R} \ln \sigma_N$ where $\bar{R}$ is large enough. In that case, by Theorem 1.4, the Edgeworth expansion of order r hold true, and so, after the reduction to the case when $\mathbb{E}(S_N)$ is bounded, it is enough to show that $\Phi_N^{(\ell)}(t_j) = o\left(\sigma_N^{-r+\ell+1}\right)$ for all $0 \le \ell < r$. By the arguments of Lemma 7.6 it suffices to show that for each $0 \le \ell < r$ we have $\Phi_{N_0}^{(\ell)}(t_j)\Phi_{N_0,N}(t_j) = o(\sigma_N^{-r})$. To this end we write

$$\Phi_{N_0}^{(\ell)}(t_j)\Phi_{N_0,N}(t_j) = \sum_{\substack{n_1,\dots,n_k \le N_0;\\ \ell_1+\dots+\ell_k=\ell}} \gamma_{\ell_1,\dots,\ell_k} \left(\prod_{q=1}^{k} \phi_{n_q}^{(\ell_q)}(t_j)\right) \left[\prod_{n\le N,\, n\ne n_k} \phi_n(t_j)\right]$$

where $\gamma_{\ell_1,\dots,\ell_k}$ are bounded coefficients of combinatorial nature. Using (3.8) we see that for each $n_1, \dots, n_k$ the product in the square brackets is at most $Ce^{-c_0 M_N(m)+O(1)}$ for some $C, c_0 > 0$. Hence

$$|\Phi_{N_0}^{(\ell)}(t_j)\Phi_{N_0,N}(t_j)| \le \hat{C} N_0^{\ell}\, e^{-cM_N(m)}.$$

It remains to observe that the definition of N_0 gives $M_N(m) \ge \hat{\varepsilon} N_0$. Therefore $|\Phi_{N_0}^{(\ell)}(t_j)\Phi_{N_0,N}(t_j)| \le C^* M_N^{\ell}(m)\, e^{-cM_N(m)} = o(\sigma_N^{-r})$ provided that $M_N \ge \bar{R} \ln \sigma_N$ for $\bar{R}$ large enough. □

8. Edgeworth expansions and uniform distribution.

8.1. **Proof of Theorem 1.6.** In view of Proposition 7.1 with $r = 2$, it is enough to show that if $\Phi_N(t_j) = o(\sigma_N^{-1})$ then the SLLT implies that

$$|\Phi_{N_0,N}(t_j)\Phi'_{N_0}(t_j)| = o(1) \tag{8.1}$$

for any non-zero resonant point t_j (note that the equivalence of conditions (b) and (c) of the theorem follows from Lemma 6.2).

Denote $\Phi_{N;k}(t) = \prod_{l\neq k, l\leq N} \phi_l(t)$.

Let us first assume that $\phi_k(t_j) \neq 0$ for all $1 \leq k \leq N$. Then $\phi_k'(t_j)\Phi_{N;k}(t_j) = \phi_k'(t_j)\Phi_N(t_j)/\phi_k(t_j)$. Let $\varepsilon_N = \frac{\ln \sigma_N}{\sigma_N}$. If for all $1 \leq k \leq N_0$ we have $|\phi_k(t_j)| \geq \varepsilon_N$ then

$$\left|\Phi_{N_0,N}(t_j)\Phi_{N_0}'(t_j)\right| = \left|\sum_{k=1}^{N_0} \phi_k'(t_j)\Phi_{N;k}(t_j)\right| \leq |\Phi_N(t_j)| \sum_{k=1}^{N_0} |\phi_k'(t_j)/\phi_k(t_j)|$$

$$\leq C\varepsilon_N^{-1} N_0 |\Phi_N(t_j)| \leq C'\sigma_N |\Phi_N(t_j)| \to 0 \text{ as } N \to \infty$$

where we have used that $N_0 = O(\ln V_N)$. Next suppose there is at least one $1 \leq k \leq N_0$ such that $|\phi_k(t_j)| < \varepsilon_N$. Let us pick some $k = k_N$ with the latter property. Then for any $k \neq k_N$, $1 \leq k \leq N_0$ we have

$$|\phi_k'(t_j)\Phi_{N;k}(t_j)| \leq C|\phi_{k_N}(t_j)| < C\varepsilon_N.$$

Therefore,

$$\left|\sum_{k\neq k_N,\, 1\leq k\leq N_0} \phi_k'(t_j)\Phi_{N;k}(t_j)\right| \leq \frac{C' \ln^2 \sigma_N}{\sigma_N} = o(1).$$

It follows that

$$\Phi_{N_0,N}(t_j)\Phi_{N_0}'(t_j) = \Phi_{N;k_N}(t_j)\phi_{k_N}'(t_j) + o(1). \tag{8.2}$$

Next, in the case when $\phi_{k_0}(t_j) = 0$ for some $1 \leq k_0 \leq N_0$, then (8.2) clearly holds true with $k_N = k_0$ since all the other terms vanish.

In summary, either (8.1) holds or we have (8.2). In the later case, using (3.8) we obtain

$$\left|\mathbb{E}\left(e^{it_j S_{N;k_N}}\right)\right| \leq e^{-c_2 \sum_{s\neq k_N, 1\leq s\leq N} q_s(m)} = e^{-c_2 M_N(m) - q_{k_N}(m)} \tag{8.3}$$

where $S_{N;k} = S_N - X_k$, and $c_2 > 0$ depends only on K. Since the SLLT holds true, M_N converges to ∞ as $N \to \infty$. Taking into account that $0 \leq q_{k_N}(m) \leq 1$ we get that the left hand side of (8.3) converges to 0, proving (8.1). □

8.2. **Proof of Theorem 1.8.** We start with the proof of part (1). Assume that the LLT holds true in a superstable way. Let $X_1', X_2', ...$ be a square integrable integer-valued independent sequence which differs from $X_1, X_2, ...$ by a finite number of elements. Then there is $n_0 \in \mathbb{N}$ so that $X_n = X_n'$ for any $n > n_0$. Set $S_N' = \sum_{n=1}^{N} X_n'$, $Y = S_{n_0}'$ and

$Y_N = Y\mathbb{I}(|Y| < \sigma_N^{1/2+\varepsilon})$, where $\varepsilon > 0$ is a small constant. By the Markov inequality we have

$$\mathbb{P}(|Y| \geq \sigma_N^{1/2+\varepsilon}) = \mathbb{P}(|Y|^2 \geq \sigma_N^{1+2\varepsilon}) \leq \|Y\|_{L^2}^2 \sigma_N^{-1-2\varepsilon} = o(\sigma_N^{-1}).$$

Therefore, for any $k \in \mathbb{N}$ and $N > n_0$ we have

$$\begin{aligned}\mathbb{P}(S'_N = k) &= \mathbb{P}(S_{N;1,2,...,n_0} + Y_N = k) + o(\sigma_N^{-1}) \\ &= \mathbb{E}[\mathbb{P}(S_{N;1,2,...,n_0} = k - Y_N | X'_1, ..., X'_{n_0})] + o(\sigma_N^{-1}) \\ &= \mathbb{E}[P_{N:n_1,...,n_0}(k - Y_N)] + o(\sigma_N^{-1})\end{aligned}$$

where $P_{N:n_1,...,n_0}(s) = \mathbb{P}(S_{N;1,2,...,n_0} = s)$ for any $s \in \mathbb{Z}$. Since the LLT holds true in a superstable way, we have, uniformly in k and the realizations of $X'_1, ..., X'_{n_0}$ that

$$P_{N:n_1,...,n_0}(k - Y_N) = \frac{e^{-(k-Y_N-\mathbb{E}(S_N))^2/(2V_N)}}{\sqrt{2\pi}\sigma_N} + o(\sigma_N^{-1}).$$

Therefore,

$$\mathbb{P}(S'_N = k) = \tag{8.4}$$

$$\frac{e^{-(k-\mathbb{E}(S_N))^2/2V_N}}{\sqrt{2\pi}\sigma_N}\mathbb{E}\big(e^{-(k-\mathbb{E}(S_N))Y_N/V_N + Y_N^2/(2V_N)}\big) + o(\sigma_N^{-1}).$$

Next, since $|Y_N| \leq \sigma_N^{1/2+\varepsilon}$ we have that $\|Y_N^2/(2V_N)\|_{L^\infty} \leq \sigma_N^{2\varepsilon-1}$, and so when $\varepsilon < 1/2$ we have $\|Y_N^2/2V_N\|_{L^\infty} = o(1)$. Recall that $k_N = (k - \mathbb{E}(S_N))/\sigma_N$. Suppose first that $|k_N| \geq \sigma_N^{\varepsilon}$ with $\varepsilon < 1/4$.

Since

$$\big|(k - \mathbb{E}(S_N))Y_N/V_N\big| \leq |k_N|\sigma_N^{\varepsilon-\frac{1}{2}},$$

we get that the RHS of (8.4) is $o(\sigma_N^{-1})$ (uniformly in such k's).

On the other hand, if $|k_N| < \sigma_N^{\varepsilon}$ then

$$\mathbb{E}\big(e^{-(k-\mathbb{E}(S_N))Y_N/V_N + Y_N^2/2V_N}\big) = 1 + o(1)$$

(uniformly in that range of k's).

We conclude that, uniformly in k, we have

$$\mathbb{P}(S'_N = k) = \frac{e^{-(k-\mathbb{E}(S_N))^2/(2V_N)}}{\sqrt{2\pi}\sigma_N} + o(\sigma_N^{-1}).$$

Lastly, since $\sup_N |\mathbb{E}(S_N) - \mathbb{E}(S'_N)| < \infty$ and $\sup_N |\mathrm{Var}(S_N) - \mathrm{Var}(S'_N)| < \infty$,

$$\mathbb{P}(S'_N = k) = \frac{e^{-(k-\mathbb{E}(S'_N))^2/(2V'_N)}}{\sqrt{2\pi}\sigma'_N} + o(1/\sigma'_N)$$

where $V'_N = \mathrm{Var}(S'_N)$ and $\sigma'_N = \sqrt{V'_N}$.

Conversely, if the SLLT holds then $M_N(h) \to \infty$ for each $h \geq 2$. Now if t is a nonzero resonant point with denominator h then (3.8) gives

$$|\Phi_{N;j_1^N,j_2^N,...,j_{s_N}^N}(t)| \leq Ce^{-cM_N(h)+\bar{C}\bar{s}},\ C,\bar{C}>0$$

for any choice of $j_1^N,...,j_{s_N}^N$ and $\bar{s}$ with $s_N \leq \bar{s}$. Since the RHS tends to 0 as $N \to \infty$, $\{X_n\} \in EeSS(1)$ completing the proof of part (1).

For part (2) we only need to show that (a) is equivalent to (b) as the equivalence of (b) and (c) comes from Lemma 6.2. By replacing again X_n with $X_n - c_n$ it is enough to prove the equivalency in the case when $\mathbb{E}(S_N) = O(1)$. The proof that (a) and (b) are equivalent consists of two parts. The first part is the following statement whose proof is a straightforward adaptation of the proof of Theorem 1.5 and is therefore omitted.

Proposition 8.1. *$\{X_n\} \in SsEe(r)$ if and only if for each $\bar{s}$, each sequence $j_1^N, j_2^N, \ldots, j_{s_N}^N$ with $s_N \leq \bar{s}$, each $\ell < r$ and each $t \in \mathcal{R}$ we have*

$$\Phi^{(\ell)}_{N;j_1^N,j_2^N,...,j_{s_N}^N}(t) = o(\sigma_N^{\ell+1-r}). \tag{8.5}$$

Note that the above proposition shows that the condition $\Phi_{N;j_1^N,j_2^N,...,j_{s_N}^N}(t) = o(\sigma_N^{1-r})$ is necessary.

The second part of the argument is to show that if

$$\Phi_{N;j_1^N,j_2^N,...,j_{s_N}^N}(t) = o(\sigma_N^{1-r})$$

holds for every finite modification of S_N with $s_N \leq \bar{s} + \ell$ (uniformly) then (8.5) holds for every modifications with $s_N \leq \bar{s}$ so that the condition $\Phi_{N;j_1^N,j_2^N,...,j_{s_N}^N}(t) = o(\sigma_N^{1-r})$ is also sufficient.

To this end we introduce some notation. Fix a nonzero resonant point $t = \frac{2\pi l}{m}$. Let $\check{\Phi}_N$ be the characteristic function of the sum $\check{S}_N$ of all X_n's such that $1 \leq n \leq N$, $n \notin \{j_1^N, j_2^N, ..., j_{s_N}^N\}$ and $q_n(m) \geq \bar{\epsilon}$. Let $\check{N}$ be the number of terms in $\check{S}_N$. Denote $\tilde{S}_N = S_{N;j_1^N,j_2^N,...,j_{s_N}^N} - \check{S}_N$ and let $\tilde{\Phi}_N(t)$ be the characteristic function of $\tilde{S}_N$. Similarly to the proof of Theorem 1.5 it suffices to show that for each $\ell < r$

$$\left|\check{\Phi}_N^{(\ell)}\tilde{\Phi}_N(t)\right| = o(\sigma_N^{1+\ell-r})$$

and, moreover, we can assume that $M_N(m) \leq \bar{R}\ln\sigma_N$ so that $\check{N} = O(\ln\sigma_N)$. We have (cf. (4.21)) ,

$$\check{\Phi}_N^{(\ell)}\tilde{\Phi}_N(t) = \sum_{\substack{n_1,...,n_k;\\ \ell_1+\cdots+\ell_k=\ell}} \prod_{w=1}^{k} \gamma_{\ell_1,...,\ell_k}\phi_{n_w}^{(\ell_w)}(t_j) \prod_{n\notin\{n_1,n_2...,n_k,j_1^N,j_2^N...,j_{s_N}^N\}} \phi_n(t_j)$$

where the summation is over all tuples $n_1, n_2, \dots, n_k$ such that $q_{n_w}(m) \geq \bar{\epsilon}$. Note that the absolute value of each term in the above sum is bounded by $C|\Phi_{N;n_1,\dots,n_k,j_1\dots,j^N_{s_N}}(t_j)| = o(\sigma_N^{1-r})$. It follows that the whole sum is

$$o\left(\sigma_N^{1-r}\check{N}^\ell\right) = o\left(\sigma_N^{1-r}\ln^\ell \sigma_N\right)$$

completing the proof. □

Remark 8.2. Lemma 6.2 and Theorem 1.5 show that the convergence to uniform distribution on any factor $\mathbb{Z}/h\mathbb{Z}$ with the speed $o(\sigma_N^{1-r})$ is necessary for Edgeworth expansion of order r. This is quite intuitive. Indeed calling $\mathscr{E}_r$ the Edgeworth function of order r, (i.e. the contribution from zero), then it is a standard result from numerical integration (see, for instance, [9, Lemma A.2]) that for each $s \in \mathbb{N}$ and each $j \in \mathbb{Z}$

$$\sum_{k\in\mathbb{Z}} h\mathscr{E}_r\left(\frac{j+hk}{\sqrt{\sigma_N}}\right) = \int_{-\infty}^{\infty} \mathscr{E}_r(x)dx + o\left(\sigma_N^{-s}\right) = 1 + o\left(\sigma_N^{-s}\right)$$

where in the last inequality we have used that the non-constant Hermite polynomials have zero mean with respect to the standard normal law (since they are orthogonal to the constant functions). However, using this result to show that

$$\sum_{k\in\mathbb{Z}} \mathbb{P}(S_N = j + kh) = \frac{1}{h} + o\left(\sigma_N^{1-r}\right)$$

requires a good control on large values of k. While it appears possible to obtain such control using the large deviations theory it seems simpler to estimate the convergence rate towards uniform distribution from our generalized Edgeworth expansion.

9. Second order expansions

In this section we will compute the polynomials in the general expansions in the case $r = 2$.

First, let us introduce some notations which depend on a resonant point t_j. Let $t_j = 2\pi l_j/m_j$ be a nonzero resonant point such that $M_N(m_j) \leq R(2,K)\ln V_N$ where $R(2,K)$ is specified in Remark 3.6. Let $\check{\Phi}_{j,N}$ be the characteristic function of the sum $\check{S}_{j,N}$ of all X_n's such that $1 \leq n \leq N$ and $q_n(m_j) \geq \bar{\epsilon} = \frac{1}{8K}$. Note that $\check{S}_{j,N}$ was previously denoted by S_{N_0}. Let $\tilde{S}_{N,j} = S_N - \check{S}_{N,j}$ and denote by $\tilde{\Phi}_{N,j}$ its characteristic function. (In previous sections we denoted the same expression by $S_{N_0,N}$, but here we want to emphasize the dependence on t_j.) Let $\gamma_{N,j}$ be the ratio between the third moment of $\tilde{S}_{N,j} - \mathbb{E}(\tilde{S}_{N,j})$ and its variance. Recall that by (3.3) $|\gamma_{N,j}| \leq C$ for some C. Also,

let C_{1,N,t_j} be given by (4.15), with the indexes rearranged so that the n's with $q_n(m) \geq \bar{\varepsilon}$ are the first N_0 ones (C_{1,N,t_j} is at most of order $M_N(m) = O(\ln V_N)$). For the sake of convenience, when either $t_j = 0$ or $M_N(m_j) \geq R(2,K)\ln V_N$ we set $C_{1,N,t_j} = 0$, $\tilde{S}_{N,j} = S_N$ and $\check{S}_{N,j} = 0$. In this case $\tilde{\Phi}_{N,j} = \tilde{\Phi}_N$ and $\check{\Phi}_{N,j} \equiv 1$. Also denote $k_N = (k - \mathbb{E}(S_N))/\sigma_N$, $\bar{S}_N = S_N - \mathbb{E}(S_N)$, and $\gamma_N = \mathbb{E}(\bar{S}_N^3)/V_N$, ($\gamma_N$ is bounded).

Proposition 9.1. *Uniformly in k, we have*

$$\sqrt{2\pi}\mathbb{P}(S_N = k) = \Big(1 + \sum_{t_j \in \mathcal{R}} e^{-it_j k}\Phi_N(t_j)\Big)\sigma_N^{-1}e^{-k_N^2/2} \tag{9.1}$$
$$-\sigma_N^{-2}e^{-k_N^2/2}\left(\gamma_N k_N^3/6 + \sum_{t_j\in\mathcal{R}} e^{-it_jk}\tilde{\Phi}_{N,j}(t_j)P_{N,j}(k_N)\right)$$
$$+o(\sigma_N^{-2})$$

where

$$P_{N,j}(x) = \big(\check{\Phi}_{N,j}(t_j)(iC_{1,N,t_j} - \mathbb{E}(\check{S}_{N,j})) + i\check{\Phi}'_{N,j}(t_j)\big)x + \check{\Phi}_{N,j}(t_j)\gamma_{N,j}x^3/6.$$

Proof. Let $t_j = \frac{2\pi l}{m}$ be a resonant point with $M_N(m) \leq R(2,K)\ln V_N$. Recall that $\mathbf{C}_j(k)$ are given by (4.22). First, in order to compute the term corresponding to $\sigma_{N_0,N}^{-2}$ we need only to consider the case $s \leq 1$ in (4.23). Using (3.2) we end up with the following contribution of the interval containing t_j,

$$\sqrt{(2\pi)^{-1}}e^{-it_jk}\tilde{\Phi}_{N,j}(t_j)\sigma_{N_0,N}^{-1}\left(\int_{-\infty}^{\infty} e^{-ih(k-\mathbb{E}[\tilde{S}_{N,j}])/\sigma_{N,j}}e^{-h^2/2}dh\right.$$
$$+\sigma_{N,j}^{-1}\int_{-\infty}^{\infty} e^{-ih(k-\mathbb{E}[\tilde{S}_{N,j}])/\sigma_{N,j}}\left(\frac{ih^3}{6}\mathbb{E}\left[\big(\tilde{S}_{N,j} - \mathbb{E}(\tilde{S}_{N,j})\big)^3\right]\sigma_{N,j}^{-3}\right)dh$$
$$\left.+\sigma_{N,j}^{-1}(C_{1,N}\check{\Phi}_{N,j}(t_j) + \check{\Phi}'_{N,j}(t_j))\int_{-\infty}^{\infty} e^{-ih(k-\mathbb{E}(\tilde{S}_{N,j}))/\sigma_{N,j}}he^{-h^2/2}dh\right)$$
$$= e^{-it_jk}\tilde{\Phi}_{N,j}(t_j)\sqrt{(2\pi)^{-1}}e^{-k_{N,j}^2/2}\sigma_{N,j}^{-1}\Big(\check{\Phi}_{N,j}(t_j) + i\big(C_{1,N,t_j}\check{\Phi}_{N,j}(t_j) + \check{\Phi}'_{N,j}(t_j)\big)$$
$$\times k_{N,j}\sigma_{N,j}^{-1} + \check{\Phi}_{N,j}(t_j)(k_{N,j}^3 - 3k_{N,j})\gamma_{N,j}\sigma_{N,j}^{-1}/6\Big)$$

where $\sigma_{N,j} = \sqrt{V(\tilde{S}_{N,j})}$, $k_{N,j} = (k - \mathbb{E}(\tilde{S}_{N,j}))/\sigma_{N,j}$ and $\gamma_{N,j} = \frac{\mathbb{E}[(\tilde{S}_{N,j} - \mathbb{E}(\tilde{S}_{N,j}))^3]}{\sigma_{N,j}^2}$ (which is uniformly bounded).

As before we shall only consider the case where $|k_N| \le V_N^\varepsilon$ with $\varepsilon = 0.01$ since otherwise both the LHS and the RHS (9.1) are $O(\sigma_N^{-r})$ for all r. Then, the last display can be rewritten as $I + I\!I$ where

$$I = \frac{e^{-it_jk}}{\sqrt{2\pi}\sigma_{N,j}}\, e^{-k_{N,j}^2/2}\, \Phi_N(t_j); \tag{9.2}$$

$$I\!I = \frac{e^{-it_jk}}{\sqrt{2\pi}\sigma_{N,j}^2}\, e^{-k_{N,j}^2/2}\times$$

$$\left[\Phi_N(t_j)\left(iC_{1,N,t_j}k_{N,j} + \frac{\gamma_{N,j}}{6}\left(k_{N,j}^3 - 3k_{N,j}\right)\right) + i\check{\Phi}'_{N,j}(t_j)\tilde{\Phi}_{N,j}(t_j)k_{N,j}\right].$$

In the region $|k_N| \le V_N^\varepsilon$ we have

$$I = \frac{e^{-it_jk}}{\sqrt{2\pi}\sigma_N}\, e^{-k_N^2/2}\ \left[1 - q_{N,j}k_N\right]\Phi_N(t_j) + o\left(\sigma_N^{-2}\right)$$

where

$$q_{N,j} = \mathbb{E}(\check{S}_{N,j})/\sigma_{N,j} = \mathbb{E}(\check{S}_{N,j})/\sigma_N + O(\ln\sigma_N/\sigma_N^3) = O(\ln\sigma_N/\sigma_N)$$

while

$$I\!I = \frac{e^{-it_jk}}{\sqrt{2\pi}\sigma_N^2}\, e^{-k_N^2/2}\times$$

$$\left[\Phi_N(t_j)\left(iC_{1,N,t_j}k_N + \frac{\gamma_{N,j}\left(k_N^3 - 3k_N\right)}{6}\right) + i\check{\Phi}'_{N,j}(t_j)\tilde{\Phi}_{N,j}(t_j)k_N\right]$$

$$+o\left(\sigma_N^{-2}\right).$$

This yields (9.1) with $\mathcal{R}_N$ in place of $\mathcal{R}$, where $\mathcal{R}_N$ is the set on nonzero resonant points $t_j = 2\pi l/m$ such that $M_N(m) \le R(2,K)\ln V_N$. Next, (3.8) shows that if $M_N(m) \ge R(2,k)\ln V_N$ then

$$\sup_{t\in I_j}|\Phi_N(t)| \le e^{-c_0M_N(m)} = o(\sigma_N^{-2})$$

and so the contribution of I_j to the right hand side of (9.1) is $o(\sigma_N^{-2})$. Finally, the contribution coming from $t_j = 0$ is

$$e^{-k_N^2/2}\left(\sigma_N^{-1} + \sigma_N^{-2}\gamma_N^3k_N^3/6\right)$$

and the proof of the proposition is complete. □

Remark 9.2. Suppose that $M_N(m) \ge R(2,K)\ln V_N$ and let N_0 is the number of n's between 1 to N so that $q_n(m) \ge \frac{1}{8K}$. Then using (3.8) we also have

$$|\check{\Phi}'_{N,j}(t_j)\tilde{\Phi}_{N,j}(t_j)| \le \sum_{n\in\mathcal{B}_{\bar{\varepsilon}}(m)}|\mathbb{E}[X_ne^{it_jX_n}]|\cdot|\Phi_{N;n}(t_j)|$$

$$\le CN_0(N,t_j,\bar{\varepsilon})e^{-c_0M_N(m)} \le C'M_N(m)e^{-c_0M_N(m)},$$

where

$$\mathcal{B}_{N,\bar\varepsilon}(m)=\{1\le n\le N: q_n(m)>\bar\varepsilon\}.$$

Since $M_N(m)\ge R(2,K)\ln V_N$, for any $0<c_1<c_0$, when N is large enough we have

$$M_N(m)e^{-c_0M_N(m)}\le C_1e^{-c_1M_N(m)}=o(\sigma_N^{-2}).$$

Similarly, $|\mathbb{E}(\check S_{N,j})\Phi_N(t_j)|=o(\sigma_N^{-2})$ and

$$C_{1,N,t_j}\Phi_N(t_j)=O(M_N(m))\Phi_N(t_j)=o(\sigma_N^{-2}).$$

Therefore we get (9.1) when $\tilde S_{N,j}$ and $\check S_{N,j}$ are defined in the same way as in the case $M_N(m)\le R(2,K)\ln V_N$.

Under additional assumptions the order 2 expansion can be simplified.

Corollary 9.3. *If S_N satisfies SLLT then*

$$\sqrt{2\pi}\mathbb{P}(S_N=k)=\frac{e^{-k_N^2/2}}{\sigma_N}\left(1+\sum_{t_j\in\mathcal{R}}e^{-it_jk}\Phi_N(t_j)-\frac{\gamma_Nk_N^3}{6\sigma_N}\right)+o(\sigma_N^{-2}).$$

Proof. The estimates of §8.1 together with (3.8) show that if S_N satisfies the SLLT then for all j

$$(1+M_N(m))\tilde\Phi_{N,j}(t_j)\check\Phi_{N,j}(t_j)=o(1)\text{ and }\tilde\Phi_{N,j}(t_j)\check\Phi'_{N,j}(t_j)=o(1).$$

Thus all terms in the second line of (9.1) except the first one make a negligible contribution, and so they could be omitted. □

Next, assume that S_N satisfies the LLT but not SLLT. According to Proposition 1.2, in this case there exists m such that $M_N(m)$ is bounded and for $k=1,\dots,m-1$ there exists $n=n(k)$ such that $\phi_n(k/m)=0$. Let $\mathcal{R}_s$ denote the set of nonzero resonant points $t_j=\frac{2\pi k}{m}$ so that $M_N(m)$ is bounded and $\phi_{\ell_j}(t_j)=0$ for unique ℓ_j.

Corollary 9.4. *Uniformly in k, we have*

$$\sqrt{2\pi}\mathbb{P}(S_N=k)=\Big(1+\sum_{t_j\in\mathcal{R}}e^{-it_jk}\Phi_N(t_j)\Big)\sigma_N^{-1}e^{-k_N^2/2}$$

$$-\sigma_N^{-2}e^{-k_N^2/2}\left(\gamma_Nk_N^3/6+\sum_{t_j\in\mathcal{R}_s}ie^{-it_jk}\Phi_{N;\ell_j}(t_j)\phi'_{\ell_j}(t_j)k_N\right)+o(\sigma_N^{-2}).$$

Proof. As in the proof of Corollary 9.3 we see that the contribution of the terms with k/m with $M_N(m)\to\infty$ is negligible. Next, for terms in $\mathcal{R}_s$ the only non-zero term in the second line in (9.1) corresponds to $\Phi_{N;\ell_j}(t_j)\phi'_{\ell_j}(t_j)$ while for the resonant points such that $\phi_\ell(t_j)=0$ for two different ℓs all terms vanish. □

10. Examples.

Example 10.1. Suppose X_n are iid integer valued with step $h > 1$. That is there is $s \in \mathbb{Z}$ such that $\mathbb{P}(X_n \in s + h\mathbb{Z}) = 1$ and h is the smallest number with this property. In this case [18, Theorem 4.5.4] (see also [12, Theorem 5]) shows that there are polynomials P_b such that

$$\mathbb{P}(S_N = k) = \sum_{b=1}^{r} \frac{P_b((k - \mathbb{E}[S_N])/\sigma_N)}{\sigma_N^b} \mathfrak{g}\left(\frac{k - \mathbb{E}(S_N)}{\sigma_N}\right) + o(\sigma_N^{-r}) \tag{10.1}$$

for all $k \in sN + h\mathbb{Z}$. Then

$$\sum_{a=0}^{h-1} \sum_{b=1}^{r} e^{2\pi i a(k-sN)/h} \frac{P_b((k - \mathbb{E}[S_N])/\sigma_N)}{\sigma_N^b} \mathfrak{g}((k - \mathbb{E}(S_N))/\sigma_N)$$

provides $o(\sigma_N^{-r})$ approximation to $\mathbb{P}(S_N = k)$ which is valid for *all* $k \in \mathbb{Z}$.

Next let $\bar{S}_N = X_0 + S_N$ where X_0 is bounded and arithmetic with step 1. Then using the identity

$$\mathbb{P}(\bar{S}_N = k) = \sum_{u \equiv k - sN \bmod h} \mathbb{P}(X_0 = u)\mathbb{P}(S_N = k - u), \tag{10.2}$$

invoking (10.1) and expanding $\mathfrak{g}\left(\frac{k - u - \mathbb{E}(S_N)}{\sigma_N}\right)$ in the Taylor series about $\frac{k - \mathbb{E}(S_N)}{\sigma_N}$ we conclude that there are polynomials $P_{b,j}$ such that we have for $k \in j + h\mathbb{Z}$,

$$\mathbb{P}(\bar{S}_N = k) = \sum_{b=1}^{r} \frac{P_{b,j}((k - \mathbb{E}[S_N])/\sigma_N)}{\sigma_N^b} \mathfrak{g}\left(\frac{k - \mathbb{E}(S_N)}{\sigma_N}\right) + o(\sigma_N^{-r}).$$

Again

$$\sum_{a=0}^{h-1} \sum_{j=0}^{h-1} e^{2\pi i a(k-j)/h} \sum_{b=1}^{r} \frac{P_{b,j}((k - \mathbb{E}[S_N])/\sigma_N)}{\sigma_N^b} \mathfrak{g}\left(\frac{k - \mathbb{E}(S_N)}{\sigma_N}\right)$$

provides the oscillatory expansion valid for all integers.

Example 10.2. Our next example is a small variation of the previous one. Fix a positive integer m. Let X' be a random variable such that $X' \bmod m$ is uniformly distributed. Then its characteristic function satisfies $\phi_{X'}(\frac{2\pi a}{m}) = 0$ for $a = 1, \dots, m-1$. We also assume that $\phi'_{X'}(\frac{2\pi a}{m}) \neq 0$ for a as above (for example one can suppose that X' takes the values Lm, $1, 2, \dots, m-1$ with equal probabilities where L is a large integer). Let X'' take values in $m\mathbb{Z}$ and have zero mean. We

also assume that X'' does not take values at $m_0\mathbb{Z}$ for a larger m_0. Then $q(X'', m_0) > 0$ for any $m_0 \neq m$. Fix $r \in \mathbb{N}$ and let

$$X_n = \begin{cases} X' & n \leq r, \\ X'' & n > r. \end{cases}$$

Then $M_N(m_0)$ grows linearly fast in N if $m_0 \neq m$ and $M_N(m)$ is bounded in N. We claim that S_N admits the Edgeworth expansion of order r but does not admit Edgeworth expansion of order $r+1$. The first statement holds due to Theorem 1.5, since $\Phi_N^{(\ell)}(\frac{2\pi a}{m}) = 0$ for each $a \in \mathbb{Z}$ and each $\ell < r$. On the other hand, since $\Phi_N^{(\ell)}(\frac{2\pi a}{m}) = 0$ for any $\ell < r$, using Lemma 7.6 we see that the conditions of Lemma 7.2 are satisfied with $r+1$ in place of r. Moreover, with $t_j = 2\pi a/m$, $a \neq 0$ we have $\mathcal{H}_{N,r+1,s}(x, t_j) \equiv 0$ for any $q \leq r+1$ and $s = 2, 3, 4$ while $\mathcal{H}_{N,q,w}(x, t_j) \equiv 0$ for any $q \leq r$ and $w = 1, 2, 3, 4$. Furthermore, when $N \geq r$ we have

$$\begin{aligned} \mathcal{H}_{N,r+1,1}(x; t_j) &= \frac{i^r H_r(x)\big(\phi_{X''}(2\pi a/m)\big)^{N-r}\Phi_r^{(r)}(2\pi a/m)}{r!} \\ &= (i)^r H_r(x)\big(\Phi'_{X'}(2\pi a/m)\big)^r. \end{aligned}$$

We conclude that

$$\begin{aligned} &\mathbb{P}(S_N = k) \\ &= \frac{e^{-k_N^2/2}}{\sqrt{2\pi}}\left[\mathcal{E}_{r+1}(k_N) + \frac{i^r}{\sigma_N^{r+1}}\sum_{a=1}^{m-1} e^{-2\pi i a k/m}\left(\phi'_{X'}\left(\frac{2\pi a}{m}\right)\right)^r H_r(k_N)\right] \\ &\qquad + o(\sigma_N^{-r-1}) \end{aligned}$$

where $\mathcal{E}_{r+1}$ the Edgeworth polynomial (i.e. the contribution of 0) and $H_r(x)$ is the Hermite polynomial.

Observe that since the uniform distribution on $\mathbb{Z}/m\mathbb{Z}$ is shift invariant, S_N are uniformly distributed mod m for all $N \in \mathbb{N}$. This shows that for $r \geq 1$, one can not characterize Edgeworth expansions just in term of the distributions of S_N mod m, so the additional assumptions in Theorems 1.6 and 1.8 are necessary.

Next, consider a more general case where for each n, X_n equals in law to either X' or X'', however, now we assume that X' appears infinitely often. In this case S_N obeys Edgeworth expansions of all orders since for large N, $\Phi_N(t)$ has zeroes of order greater N at all points of the form $\frac{2\pi a}{m}$, $a = 1, \dots, m-1$. In fact, the Edgeworth expansions hold in the superstable way since removing a finite number of terms does not make the order of zero to fall below r.

Example 10.3. Let $\mathfrak{p}_n = \min(1, \frac{\theta}{n})$ and let X_n take value 0 with probability $\mathfrak{p}_n$ and values ± 1 with probability $\frac{1-\mathfrak{p}_n}{2}$. In this example the only non-zero resonant point is $\pi = 2\pi \times \frac{1}{2}$. Then for small θ the contributions of $P_{1,b,N}$ (the only non-zero a is 1) are significant and as a result S_N does not admit the ordinary Edgeworth expansion. Increasing θ we can make S_N to admit Edgeworth expansions of higher and higher orders. Namely we get that for large n, $\phi_n(\pi) = \dfrac{2\theta}{n} - 1$. Accordingly

$$\ln(-\phi_n(\pi)) = -\frac{2\theta}{n} + O\left(\frac{1}{n^2}\right).$$

Now the asymptotic relation

$$\sum_{n=1}^{N} \frac{1}{n} = \ln N + \mathfrak{c} + O\left(\frac{1}{N}\right),$$

where $\mathfrak{c}$ is the Euler-Mascheroni constant, implies that that there is a constant $\Gamma(\theta)$ such that

$$\Phi_N(\pi) = \frac{(-1)^N e^{\Gamma(\theta)}}{N^{2\theta}} \left(1 + O(1/N)\right).$$

Therefore S_N admits the Edgeworth expansions of order r iff $\theta > \dfrac{r-1}{4}$. Moreover, if $\theta \in \left(\dfrac{r-2}{4}, \dfrac{r-1}{4}\right]$, then Corollary 7.4 shows that

$$\mathbb{P}(S_N = k) = \frac{e^{-k_N^2/2}}{\sqrt{2\pi}} \left[\mathcal{E}_r(k_N) + \frac{(-1)^{N+k} e^{\Gamma(\theta)}}{N^{2\theta+(1/2)}} + O\left(\frac{1}{N^{2\theta+1}}\right)\right]$$

where $\mathcal{E}_r$ is the Edgeworth polynomial of order r. In particular if $\theta \in (0, 1/4)$ then using that

$$V_N = N + O(\ln N) = N\left(1 + O\left(\frac{\ln N}{N}\right)\right) \tag{10.3}$$

and hence

$$\sigma_N = \sqrt{N}\left(1 + O\left(\frac{\ln N}{N}\right)\right) \tag{10.4}$$

we conclude that

$$\mathbb{P}(S_N = k) = \frac{e^{-k^2/(2N)}}{\sqrt{2\pi}} \left[\frac{1}{\sqrt{N}} + \frac{(-1)^{N+k} e^{\Gamma(\theta)}}{N^{2\theta+(1/2)}} + O\left(\frac{1}{N^{2\theta+1}}\right)\right].$$

Next, take $\mathfrak{p}_n = \min\left(1, \frac{\theta}{n^2}\right)$. Then the SLLT does not hold, since the Prokhorov condition fails. Instead we have (6.1) with $\mathcal{R} = \{\pi\}$.

Namely, uniformly in k we have

$$\sqrt{2}\pi\mathbb{P}(S_N=k)=\left(1+(-1)^k\prod_{u=1}^{N}(2\mathfrak{p}_u-1)\right)\sigma_N^{-1}e^{-k^2/2V_N}+o(\sigma_N^{-1}).$$

Next, $\mathfrak{p}_u$ is summable and moreover

$$\prod_{u=1}^{N}(2\mathfrak{p}_u-1)=(-1)^N U(1+O(1/N))$$

where $U=\prod_{n=1}^{\infty}(1-2\mathfrak{p}_u)$. We conclude that

$$\sqrt{2}\pi\mathbb{P}(S_N=k)=\left(1+(-1)^{k+N}U\right)\sigma_N^{-1}e^{-k^2/2V_N}+O\left(\sigma_N^{-2}\right) \tag{10.5}$$

uniformly in k. In this case the usual LLT holds true if and only if $U=0$ in agreement with Proposition 1.2.

In fact, in this case we have a faster rate of convergence. To see this we consider expansions of order 2 for $\mathfrak{p}_n$ as above. We observe that $q_m(2)=\mathfrak{p}_n$ for large n. Thus

$$|\mathbb{E}(e^{\pi i X_n})|=1-2\mathfrak{p}_n$$

and so $|\mathbb{E}(e^{\pi X_n})|\geq\frac{1}{2}$ when $n\geq N_\theta$ for some minimal N_θ. Therefore, we can take $N_0=N_\theta$. Note also that we have $Y_n=X_n \bmod 2-1$. We conclude that for $n>N_0$ we have

$$a_n=a_{n,j}=\frac{\mathbb{E}[((-1)^{Y_n}-1)X_n]}{\mathbb{E}[(-1)^{Y_n}]}=0$$

and so the term $C_{1,N}$ vanishes. Next, we observe that

$$\gamma_{N,j}=\frac{\sum_{n=N_0+1}^{N}\mathbb{E}(X_n^3)}{\sum_{n=N_0+1}^{N}(1-\mathfrak{p}_n)}=0.$$

Finally, we note that $\mathbb{E}[(-1)^{X_n}X_n]=0$, and hence $\Phi'_{N_0}(\pi)=0$. Therefore, the second term in (9.1) vanishes and we have

$$\sqrt{2}\pi\mathbb{P}(S_N=k)=\left(1+(-1)^{k+N}U\right)\sigma_N^{-1}e^{-k^2/(2V_N)}+O\left(\sigma_N^{-3}\right).$$

Taking into account (10.3) and (10.4) we obtain

$$\sqrt{2}\pi\mathbb{P}(S_N=k)=\frac{1+(-1)^{k+N}U}{\sqrt{N}}\,e^{-k^2/(2N)}+O\left(N^{-3/2}\right).$$

In particular, (10.5) holds with the stronger rate $O\left(\sigma_N^{-3}\right)$.

Example 10.4. The last example exhibited significant simplifications. Namely, there was only one resonant point, and, in addition, the second term vanished due to the symmetry. We now show how a similar analysis could be performed when the above simplifications are not present. Let us assume that X_n takes the values $-1, 0$ and 3 with probabilities a_n, b_n and c_n so that $a_n + b_n + c_n = 1$. Let us also assume that $b_n < \frac{1}{8}$ and that $a_n, c_n \geq \rho > 0$ for some constant ρ. Then

$$V(X_n) = 9(c_n - c_n^2) + 6a_n c_n + (a_n - a_n^2) \geq 6\rho^2$$

and so V_N grows linearly fast in N.

Next, since we can take $K = 3$, the denominators m of the nonzero resonant points can only be $2, 3, 4, 5$ or 6. An easy check shows that for $m = 3, 5, 6$ we have $q_n(m) \geq \rho$, and that for $m = 2, 4$ we have $q_n(m) = b_n$. Therefore, for $m = 3, 5, 6$ we have $M_N(m) \geq \rho N$, and so we can disregard all the nonzero resonant points except for $\pi/2, \pi$ and $3\pi/2$. For the latter points we have

$$\phi_n\left(\frac{\pi}{2}\right) = b_n - i(1 - b_n), \tag{10.6}$$

$$\phi_n(\pi) = 2b_n - 1, \quad \phi_n\left(\frac{3\pi}{2}\right) = b_n + i(1 - b_n). \tag{10.7}$$

Hence, denoting $\eta_n = b_n(1 - b_n)$, we have

$$\left|\phi_n\left(\frac{\pi}{2}\right)\right|^2 = \left|\phi_n\left(\frac{3\pi}{2}\right)\right|^2 = 1 - 2\eta_n, \quad \left|\phi_n(\pi)\right|^2 = 1 - 4\eta_n.$$

Since we suppose that $\eta_n \leq b_n < \frac{1}{8}$ it follows that $1 - 4\eta_n \geq \frac{1}{2}$. Then for the above three resonant points we can take $N_0 = 0$. Now Proposition 7.1 and a simple calculation show that for any r we get the Edgeworth expansions of order r if and only if

$$\prod_{n=1}^{N}(1 - 2\eta_n) = o\left(N^{1-r}\right).$$

Let us focus for the moment on the case when $b_n = \gamma/n$ for n large enough where $\gamma > 0$ is a constant. Rewriting (10.6), (10.7) as

$$\frac{\phi_n\left(\frac{\pi}{2}\right)}{-i} = (1 - b_n) + ib_n, \quad \frac{\phi_n\left(\frac{3\pi}{2}\right)}{i} = (1 - b_n) - ib_n, \tag{10.8}$$

$$-\phi_n(\pi) = 1 - 2b_n$$

and, using that the condition $b_n < \frac{1}{8}$ implies that that $\phi_n(t) \neq 0$ for all $n \in \mathbb{N}$ and all $t \in \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$, we conclude similarly to Example 10.3

that there are non-zero complex numbers κ_1, κ_3 and a non-zero real number κ_2 such that

$$\Phi_N\left(\frac{\pi}{2}\right) = \frac{(-i)^N \kappa_1}{N^\gamma} e^{i\gamma \ln N}\left(1 + O\left(\frac{1}{N}\right)\right),$$

$$\Phi_N\left(\frac{3\pi}{2}\right) = \frac{i^N \kappa_3}{N^\gamma} e^{-i\gamma \ln N}\left(1 + O\left(\frac{1}{N}\right)\right),$$

$$\Phi_N(\pi) = (-1)^N \frac{\kappa_2 e^{2\gamma w_N}}{N^{2\gamma}}\left(1 + O\left(\frac{1}{N}\right)\right).$$

It follows that S_N admits Edgeworth expansion of order r iff $\gamma > \frac{r-1}{2}$. In fact if $\frac{r-1}{2} < \gamma \leq \frac{r}{2}$ then Corollary 7.4 shows that

$$\mathbb{P}(S_N = k) = \frac{e^{-k_N^2/2}}{\sqrt{2\pi}}\left[\mathcal{E}_r(k_N) + \frac{\kappa_1 e^{i\gamma \ln N}}{N^\gamma \sigma_N} + \frac{\kappa_3 e^{-i\gamma \ln N}}{N^\gamma \sigma_N} + O\left(N^{-\eta}\right)\right]$$

where $\mathcal{E}_r$ is the Edgeworth polynomial of order r and η=min $\left(2\gamma, \frac{r}{2}\right) + \frac{1}{2}$.

To give a specific example, let us suppose that $\frac{1}{2} \leq \gamma < 1$ and that $E(X_n) = 0$ which means that

$$a_n = \frac{3(1-b_n)}{4}, \quad c_n = \frac{1-b_n}{4}. \tag{10.9}$$

Then

$$V_N = 3N - 3\gamma \ln N + O(1), \quad E(S_N^3) = 6N - 6\gamma \ln N + O(1), \tag{10.10}$$

so Proposition 9.1 gives

$$\sqrt{2\pi}\mathbb{P}(S_N = k) =$$

$$e^{-k^2/6N}\left[\frac{1}{\sqrt{3N}}\left(1 + \frac{\kappa_1 i^{k-N} e^{i\gamma \ln N} + \kappa_3 i^{N-k} e^{-i\gamma \ln N}}{N^\gamma}\right) - \frac{k^3}{81\sqrt{3N^5}}\right]$$

$$+ O\left(N^{-3/2}\right)$$

Next, let us provide the second order trigonometric expansions under the sole assumption that $1 - 4\eta_n \geq \frac{1}{2}$ and $a_n, c_n \geq \rho$. As we have mentioned, we only need to consider the nonzero resonant points $\pi/2, \pi, 3\pi/2$ and for these points we have $N_0 = 0$. Therefore, the term involving the derivative in the right hand side of (9.1) vanishes. Now, a direct calculation shows that

$$C_{1,N,\pi} = \sum_{n=1}^{N} \frac{\mathbb{E}(e^{i\pi X_n}\bar{X}_n)}{\mathbb{E}(e^{i\pi X_n})} = 2\sum_{n=1}^{N} \frac{(a_n - 3c_n)b_n}{2b_n - 1}$$

and

$$C_{1,N,\pi/2} = \sum_{n=1}^{N} \frac{(a_n - 3c_n)(1+i)b_n}{b_n - i(1-b_n)}, \quad C_{1,N,3\pi/2} = \sum_{n=1}^{N} \frac{(a_n - 3c_n)(1-i)b_n}{b_n + i(1-b_n)}.$$

Note that $3c_n - a_n = \mathbb{E}(X_n)$. Set

$$\Gamma_{1,N} = \prod_{n=1}^{N}(b_n - i(1-b_n)), \quad \Gamma_{2,N} = \prod_{n=1}^{N}(2b_n - 1), \quad \Gamma_{3,N} = \prod_{n=1}^{N}(b_n + i(1-b_n)).$$

Then $\Gamma_{s,N} = \mathbb{E}(e^{\frac{s\pi i}{2}S_N})$. We also set

$$\Theta_{s,N} = C_{1,N,s\pi/2}\Gamma_{s,N},\ s = 1,2,3$$

and

$$\Gamma_N(k) = \sum_{j=1}^{3} e^{-j\pi ik/2}\Gamma_{j,N}, \quad \Theta_N(k) = \sum_{j=1}^{3} e^{-j\pi ik/2}\Theta_{j,N}.$$

Then by Proposition 9.1 and Remark 9.2, uniformly in k we have

$$\sqrt{2\pi}P(S_N = k) = \sigma_N^{-1}\left(1 + \Gamma_N(k)\right)e^{-k_N^2/2} \tag{10.11}$$
$$-\sigma_N^{-2}\left(k_N^3 T_N\big(1 + \Gamma_N(k)\big) + ik_N\Theta_N(k)\right)e^{-k_N^2/2} + o(\sigma_N^{-2})$$

where $T_N = \frac{\sum_{n=1}^{N}\mathbb{E}(\bar{X}_n^3)}{6V_N}$, $\bar{X}_n = X_n - \mathbb{E}(X_n)$.

Let us now consider a more specific situation. Namely we suppose that $b_n = \frac{\gamma}{n^{3/2}}$ for large n and that $E(X_n) = 0$. Then (10.9) shows that $C_{1,N,s\pi/2} = 0$. Next (10.8) gives

$$\frac{\Phi_N\left(\frac{\pi}{2}\right)}{(-i)^N} = \prod_{n=1}^{N}\left[(1-b_n) + ib_n\right] = \frac{\bar{\kappa}_1}{\prod_{n=N+1}^{\infty}\left[(1-b_n) + ib_n\right]}$$

$$= \bar{\kappa}_1\left(1 + \frac{2\gamma(1-i)}{\sqrt{N}} + O\left(\frac{1}{N}\right)\right)$$

where $\bar{\kappa}_1 = \prod_{n=1}^{\infty}\left[(1-b_n) + ib_n\right]$. Likewise

$$\frac{\Phi_N\left(\frac{3\pi}{2}\right)}{i^N} = \bar{\kappa}_3\left(1 + \frac{2\gamma(1+i)}{\sqrt{N}} + O\left(\frac{1}{N}\right)\right)$$

and

$$\frac{\Phi_N(\pi)}{(-1)^N} = \bar{\kappa}_2\left(1 + \frac{4\gamma}{\sqrt{N}} + O\left(\frac{1}{N}\right)\right).$$

Taking into account (10.10) we can reduce (10.11) to the following expansion

$$\sqrt{2\pi}\mathbb{P}(S_N=k)=e^{-k^2/6N}\left[\frac{1}{\sqrt{3N}}\left(1+\sum_{s=1}^{3}\bar{\kappa}_s i^{s(k-N)}\right)\right.$$
$$\left.+\frac{1}{N}\left(-\frac{\tilde{k}_N^3}{3}+\sum_{s=1}^{3}\bar{\kappa}_s i^{s(k-N)}\left(\frac{2\gamma(1-i^{-s})}{\sqrt{3}}-\frac{\tilde{k}_N^3}{3}\right)\right)\right]+O\left(\frac{1}{N^{3/2}}\right)$$

where $\tilde{k}_N=k/\sqrt{3N}$.

Example 10.5. Let X' take value ±1 with probability $\frac{1}{2}$, X'' take values 0 and 1 with probability $\frac{1}{2}$, and X^δ, $\delta\in[0,1]$ be the mixture of X' and X'' with weights δ and $1-\delta$. Thus X^δ take value -1 with probability $\frac{\delta}{2}$, the value 0 with probability $\frac{1-\delta}{2}$ and value 1 with probability $\frac{1}{2}$. Therefore, $\mathbb{E}(e^{\pi i X^\delta})=-\delta$. We suppose that X_{2m} and X_{2m-1} have the same law which we call Y_m. The distribution of Y_m is defined as follows. Set $k_j=3^{3^j}$, and let Y_{k_j} have the same distribution as X^{δ_j} where $\delta_j=\frac{1}{\sqrt{k_{j+1}}}$. When $m\notin\{k_j\}$ we let Y_m have the distribution of X'. It is clear that V_N grows linearly fast in N. Note also that $\mathbb{E}(e^{\pi i Y_m})=-\delta_j$ when $m=k_j$ for some j, and otherwise $\mathbb{E}(e^{\pi i Y_m})=-1$. Now, take $N\in\mathbb{N}$ such that $N>2k_2$, and let J_N be so that $2k_{J_N}\le N<2k_{J_N+1}$. Then

$$|\Phi_N(\pi)|\le\prod_{j=1}^{J_N}(k_{j+1})^{-1}.$$

Since $k_{J_N}\le\frac{N}{2}<k_{J_N+1}$ and $k_j=(k_{j+1})^{1/3}$ we have $k_{J_N+1}^{-1}\le 2N^{-1}$ and $k_{J_N+1-m}\le 2^{3^{-m}}N^{-3^{-m}}$ for any $0<m\le J_N$. Denote $\alpha_N=\sum_{j=1}^{J_N-1}3^{-j}$. Since $\alpha_N>1/3$ we get that

$$|\Phi_N(\pi)|\le 2^{3/2}N^{-\alpha_N}=o(N^{-1-1/3}).$$

Similarly, for each $j_1,j_2\le N$,

$$|\Phi_{N:j_1}(\pi)|\le 2^{3/2}N^{-1/2-\alpha_N}=o(N^{-1/2-1/3}) \tag{10.12}$$

and

$$|\Phi_{N:j_1,j_2}(\pi)|\le 2^{3/2}N^{-\alpha_N}=o(N^{-1/3}). \tag{10.13}$$

Indeed, the largest possible values are obtained for $j_1=2k_{J_N}$ (or $j_1=2k_{J_{N+1}}-1$ if it is smaller than $N+1$) and $j_2=2k_{J_N}-1$ (or $j_2=2k_{J_N}$). Using the same estimates as in the proof of of Theorem 1.8 we conclude from (10.12) that $\Phi'_N(\pi)=o\left(1/\sqrt{N}\right)$ and we conclude from (10.13)

that $\Phi''_N(\pi) = o(1)$. It follows from Lemma 7.6 and Proposition 7.1 that S_N satisfies an Edgeworth expansion of order 3. The same conclusion holds if we remove a finite number of terms from the beginning of the sequence $\{X_n\}$ because the smallness of $\Phi_N(\pi)$ comes from the terms X_{2k_j-1} and X_{2k_j} for arbitrary large j's.

On the other hand

$$\left|\Phi_{2k_j;2k_j,2k_j-1,2k_{j-1},2k_{j-1}-1}(\pi)\right| = \prod_{s=2}^{j-1}\left(3^{-3^s}\right)$$

$$= 3^{-(3^j-9)/2} = \frac{3^{9/2}}{\sqrt{k_j}} \gg \frac{1}{k_j} = 3^{-3^j}.$$

It follows that $S_{2k_j;2k_j,2k_j-1,2k_{j-1},2k_{j-1}-1}$ does not obey the Edgeworth expansion of order 3. Accordingly, stable Edgeworth expansions need not be superstable if $r = 3$. A similar argument allows to construct examples showing that those notions are different for all $r > 2$.

11. Extension for uniformly bounded integer-valued triangular arrays

In this section we will describe our results for arrays of independent random variables. We refer to [13], [26, 27] and [11], [35], [29] and [10] for results for triangular arrays of inhomogeneous Markov chains. Example where Markov arrays appear naturally include the theory of large deviations for inhomogeneous systems (see [34, 31, 14] and references wherein), random walks in random scenery [4, 16], and statistical mechanics [19].

Let $X_n^{(N)}$, $1 \le n \le L_N$ be a triangular array such that for each fixed N, the random variables $X_n^{(N)}$ are independent and integer valued. Moreover, we assume that

$$K := \sup_N \sup_n \|X_n\|_{L^\infty} < \infty.$$

For each N we set $S_N = \sum_{n=1}^{L_N} X_n^{(N)}$. Let $V_N = \mathrm{Var}(S_N)$. We assume that $V_N \to \infty$, so that, by Lindenberg–Feller Theorem, the sequence $(S_N - \mathbb{E}(S_N))/\sigma_N$ obeys the CLT, where $\sigma_N = \sqrt{V_N}$.

We say that the array $X_n^{(N)}$ obeys the SLLT if for any k the LLT holds true for any uniformly square integrable array $Y_n^{(N)}$, $1 \le n \le L_N$,

so that $Y_n^{(N)} = X_n^{(N)}$ for all but k indexes n. Set

$$M_N := \min_{2\le h\le 2K} \sum_{n=1}^{L_N} P(X_n \neq m_n^{(N)}(h) \bmod h) \ge R \ln V_N$$

where $m_n^{(N)}(h)$ is the most likely value of $X_n^{(N)}$ modulo h. Observe now that the proofs of Proposition 3.1 and Lemmas 3.3, 3.4, 4.2 and 4.4 proceed exactly the same for arrays. Therefore, all the arguments in the proof of Theorem 2.1 proceed the same for arrays instead of a fixed sequence X_n. That is, we have

Theorem 11.1. *There $\exists J = J(K) < \infty$ and polynomials $P_{a,b,N}$ with degrees depending only on a and b, whose coefficients are uniformly bounded in N such that, for any $r \ge 1$ uniformly in $k \in \mathbb{Z}$ we have*

$$\mathbb{P}(S_N = k) - \sum_{a=0}^{J-1}\sum_{b=1}^{r} \frac{P_{a,b,N}((k-a_N)/\sigma_N)}{\sigma_N^b} \mathfrak{g}((k-a_N)/\sigma_N) e^{2\pi i a k/J} = o(\sigma_N^{-r})$$

where $a_N = \mathbb{E}(S_N)$ and $\mathfrak{g}(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$.

Moreover, $P_{0,1,N} \equiv 1$ and given K, r, there exists $R = R(K,r)$ such that if $M_N \ge R \ln V_N$ then we can choose $P_{a,b,N} = 0$ for $a \neq 0$.

All the formulas for the coefficients of the polynomials $P_{a,b,N}$ remain the same in the arrays setup. In particular, we get that, uniformly in k we have

$$\mathbb{P}(S_N = k) = \left(1 + \sum_{t\in\mathcal{R}} e^{-itk} \Phi_N(t)\right) e^{-k_N^2/2} \sigma_N^{-1} + o(\sigma_N^{-1}) \tag{11.1}$$

where $\Phi_N(t) = \mathbb{E}(e^{itS_N})$.

Next, our version for Proposition 1.2 for arrays is as follows.

Proposition 11.2. *Suppose S_N obeys LLT. Then for each integer $h \ge 2$, at least one of the following conditions occur:*

either (a) $\lim_{N\to\infty} \sum_{n=1}^{L_N} \mathbb{P}(X_n \neq m_n^{(N)}(h) \ mod\ h) = \infty$.

or (b) there exists a subsequence N_k, numbers $s \in \mathbb{N}$ and $\varepsilon_0 > 0$ and indexes $1 \le j_1^k, ..., j_{s_k}^k \le L_{N_k}$, $s_k \le s$ so that the distribution of $\sum_{u=1}^{s_k} X_{j_u}^{(N_k)}$ converges to uniform mod h, and the distance between the distribution of $S_{N_k} - \sum_{q=1}^{s_k} X_{j_q}^{(N_k)}$ and the uniform distribution mod h is at least ε_0.

Proof. First, by (11.1) and Lemma 5.1 if the LLT holds then for any nonzero resonant point t we have $\lim_{N\to\infty}|\Phi_N(t)|=0$. Now, if (a) does not hold true then there is a subsequence N_k so that $\sum_{n=1}^{L_{N_k}} q(X_n^{(N_k)},h)\le C$, where C is some constant. Set $q_n^{(N_k)}(h)=q\left(X_n^{(N_k)},h\right)$. Then there are at most $8hC$ n's between 1 and L_{N_k} so that $q_n^{(N_k)}(h)>\frac{1}{8h}$. Let us denote these n's by $n_{1,k},...,n_{s_k,k}$, $s_k\le 8hC$. Next, for any n and a nonzero resonant point $t=2\pi l/h$ we have

$$|\phi_n^{(N_k)}(t)|\ge 1-2hq_n^{(N_k)}(h)\ge e^{-2\gamma h q_n^{(N_k)}} \tag{11.2}$$

where $\phi_n^{(N_k)}$ is the characteristic function of $X_n^{(N_k)}$ and γ is such that for $\theta\in[0,1/4]$ we have $1-\theta\ge e^{-\gamma\theta}$. We thus get that

$$\prod_{n\notin\{n_{u,k}\}}|\phi_n^{(N_k)}(t)|\ge\prod_{n\notin\{n_{u,k}\}}(1-2hq_n^{(N_k)}(h))\ge C_0 \tag{11.3}$$

where $C_0>0$ is some constant. Therefore,

$$|\Phi_{N_k}(t)|\ge\prod_{u=1}^{s_k}|\phi_{n_{u,k}}^{(N_k)}(t)|\cdot C_0$$

and so we must have

$$\lim_{k\to\infty}\prod_{u=1}^{s_k}|\phi_{n_{u,k}}^{(N_k)}(t)|=0. \tag{11.4}$$

Now (b) follows from (11.3), (11.4) and Lemma 6.2. □

Using (11.1) we can now prove a version of Theorem 1.1 for arrays.

Theorem 11.3. *The SLLT holds iff for each integer $h>1$,*

$$\lim_{N\to\infty}\sum_{n=1}^{L_N}\mathbb{P}(X_n^{(N)}\ne m_n\ mod\ h)=\infty \tag{11.5}$$

where $m_n=m_n^{(N)}(h)$ is the most likely residue of $X_n^{(N)}$ modulo h.

Proof. First, the arguments in the proof of (3.8) show that there are constants $c_0,C>0$ so that for any nonzero resonant point $t=2\pi l/h$ we have

$$|\Phi_N(t)|\le Ce^{-c_0M_N(h)},\quad\text{where}\quad M_N(h):=\sum_{n=1}^{L_N}q(X_n^{(N)},h). \tag{11.6}$$

Let us assume that (11.5) holds for all integers $h > 1$. Consider s_N–tuples $1 \le j_1^N, ..., j_{s_N}^N \le L_N$, where $s_N \le \bar{s}$ is bounded in N. Then by applying (11.6) with $\tilde{S}_N = S_N - \sum_{l=1}^{s_N} X_{j_l^N}^{(N)}$ we have

$$\lim_{N\to\infty} |\mathbb{E}(e^{it\tilde{S}_N})| = 0. \tag{11.7}$$

Now, arguing as in the proof of Theorem 1.8(1), given a uniformly square integrable array $Y_n^{(N)}$ as in the definition of the SLLT, we still have (11.1), even though the new array is not necessarily uniformly bounded. Applying (11.7) we see that for any nonzero resonant point t we have

$$\lim_{N\to\infty} \left| \mathbb{E}\left(\exp\left[it \sum_{n=1}^{L_N} Y_n^{(N)} \right] \right) \right| = 0$$

and so $S_N Y := \sum_{n=1}^{L_N} Y_n^{(N)}$ satisfies the LLT.

Now let us assume that $M_N(h) \not\to \infty$ for some $2 \le h \le 2K$ (it not difficult to see that (11.5) holds for any $h > 2K$).

In other words after taking a subsequence we have that $M_{N_k}(h) \le L$ for some $L < \infty$. The proof of Proposition 11.2 shows that there $s < \infty$ such that after possibly removing terms $n_{1,k}, n_{2,k}, \dots, n_{s_k,k}$ with $s_k \le s$ we can obtain that $q_n^{(N_k)}(h) \le \frac{1}{8h}$, $n \notin \{n_{j,k}\}$. In this case (11.2) shows that for each ℓ

$$|\Phi_{N_k;n_{1,k},\dots,n_{s_k,k}}(2\pi\ell/h)| \ge e^{-2\gamma L}.$$

By Proposition 11.2, $S_{N_k;n_{1,k},\dots,n_{s_k,k}}$ does not satisfy the LLT. □

Next, all the other arguments in our paper proceed similarly for arrays since they essentially rely only on the specific structure of the polynomials from Theorem 2.1. For the sake of completeness, let us formulate the main (remaining) results here.

Theorem 11.4. *The following conditions are equivalent:*

(a) S_N satisfies LLT;

(b) For each $\xi \in \mathbb{R} \setminus \mathbb{Z}$, $\lim_{N\to\infty} \mathbb{E}\left(e^{2\pi i \xi S_N}\right) = 0$;

(c) For each non-zero resonant point ξ, $\lim_{N\to\infty} \mathbb{E}\left(e^{2\pi i \xi S_N}\right) = 0$;

(d) For each integer h the distribution of S_N mod h converges to uniform.

Theorem 11.5. *For each r there is $R = R(r, K)$ such that the Edgeworth expansion of order r holds true if $M_N \ge R \ln V_N$. In particular,*

S_N obeys Edgeworth expansions of all orders if

$$\lim_{N\to\infty} \frac{M_N}{\ln V_N} = \infty.$$

Theorem 11.6. *For any $r \geq 1$, the Edgeworth expansion of order r holds if and only if for any nonzero resonant point t and $0 \leq \ell < r$ we have*

$$\bar{\Phi}_N^{(\ell)}(t) = o\left(\sigma_N^{\ell+1-r}\right)$$

where $\bar{\Phi}_N(x) = \mathbb{E}[e^{ix(S_N - \mathbb{E}(S_N))}]$.

Theorem 11.7. *Suppose S_N obeys the SLLT. Then the following are equivalent:*

(a) Edgeworth expansion of order 2 holds;

(b) $|\Phi_N(t)| = o(\sigma_N^{-1})$ for each nonzero resonant point t;

(c) For each $h \leq 2K$ the distribution of S_N mod h is $o(\sigma_N^{-1})$ close to uniform.

Next, we say that an array $\{X_n^{(N)}\}$ *admits an Edgeworth expansion of order r in a superstable way* (denoted by $\{X_n^{(N)}\} \in EeSs(r)$) if for each $\bar{s}$ and each sequence $j_1^N, j_2^N, \ldots, j_{s_N}^N$ with $s_N \leq \bar{s}$ and $j_i^N \leq L_N$ there are polynomials $P_{b,N}$ whose coefficients are $O(1)$ in N and their degrees do not depend on N so that uniformly in $k \in \mathbb{Z}$ we have that

$$\mathbb{P}(S_{N;j_1^N, j_2^N, \ldots, j_{s_N}^N} = k) = \sum_{b=1}^{r} \frac{P_{b,N}(k_N)}{\sigma_N^b} \mathfrak{g}(k_N) + o(\sigma_N^{-r}) \tag{11.8}$$

and the estimates in $O(1)$ and $o(\sigma_N^{-r})$ are uniform in the choice of the tuples $j_1^N, \ldots, j_{s_N}^N$.

Let $\Phi_{N;j_1,j_2,\ldots,j_s}(t)$ be the characteristic function of $S_{N;j_1,j_2,\ldots,j_s}$.

Theorem 11.8. *(1) $S_N \in EeSs(1)$ (that is, S_N satisfies the LLT in a superstable way) if and if it satisfies the SLLT.*

(2) For arbitrary $r \geq 1$ the following conditions are equivalent:

(a) $\{X_n^{(N)}\} \in EeSs(r)$;

(b) For each $j_1^N, j_2^N, \ldots, j_{s_N}^N$ and each nonzero resonant point t we have $\Phi_{N;j_1^N, j_2^N, \ldots, j_{s_N}^N}(t) = o(\sigma_N^{1-r})$;

(c) For each $j_1^N, j_2^N, \ldots, j_{s_N}^N$, and each $h \leq 2K$ the distribution of $S_{N;j_1^N, j_2^N, \ldots, j_{s_N}^N}$ mod h is $o(\sigma_N^{1-r})$ close to uniform.

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