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BOUNCING BALLS IN NON-LINEAR POTENTIALS

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Dedicated to Yakov Pesin on occasion of his 60th birthday

Abstract. We consider a ball bouncing off infinitely heavy periodically moving plate in the presence of a potential force. Assuming that the potential equals to a power of the ball's height we present conditions guaranteeing recurrence in the sense that the total energy of almost every trajectory does not go to infinity.

1. **Introduction.** Consider a point mass falling vertically on an infinitely heavy horizontal plate which oscillates periodically with period 2π in the vertical direction and interacts with the particle by the law of elastic reflection. The acceleration of the particle between collisions is driven by a force with potential $U(x)$. We would like to know for which potentials the total energy of the particle tends to infinity and for which potentials there are arbitrary large times when it falls below a certain threshold.

Let us introduce the notation.

Let $f(t)$ denote the plate position at time t. Suppose that the $n+1$st collision of the particle with the plate occurs at time t_n and that the particle has velocity v_n immediately after the collision. Since the force is periodic we are only interested in the value of t_n modulo the period so we regard t_n as a cyclic coordinate and so the phase space of our system is a cylinder. Let

$$\mathcal{E} = \{(v_0, t_0) : v_n \to \infty\}.$$

We would like to know how large is this set. Two cases have been studied before.

(I) *Gravity field.* $U(x) = x$. The following remarkable result has been proven by Pustylnikov for motion in the constant gravity field.

Theorem 1. *([16]) There is an open set of plate motions $f(t)$ (in the space of analytic functions admitting an analytic continuation to a given strip $|\Im t| \leq \varepsilon$) such that* $\text{mes}(\mathcal{E}) = \infty$.

(Here and below mes denotes the invariant measure of our system.)

(II) *Impact oscillator.* $U(x) = cx^2/2$. This problem was investigated by R. Ortega.

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Theorem 2. *([12, 13]) (a) If* $2\sqrt{c} \notin \mathbb{Q}$ *and* $\int_0^{2\pi} f(t)dt \neq 0$ *then* $\mathcal{E} = \emptyset$.

(b) If $2\sqrt{c} = p/q$, *let* $\Phi(\tau) = \sum_{j=0}^{2q-1} f(\tau + \pi j)$. *Then if* Φ *changes sign and zeroes of* Φ *are non-degenerate then all solutions with sufficiently large energy belong to* $\mathcal{E}$. *If* Φ *does not change sign, then* $\mathcal{E} = \emptyset$.

We would like to understand what happens for other power potentials. In this paper we consider the case where the motion of the plate and the potential take especially simple form, namely the vertical position of the plate at time t is $B + A\sin t$ with $0 < A < B$ and $U(x) = x^\alpha$.

Conjecture 1. *If* $\alpha \neq 1, 2$ *then* $\mathrm{mes}(\mathcal{E}) = 0$.

In this note we present some evidence for this conjecture.

Theorem 3. *If* $\alpha > 1$, $\alpha \neq 2$ *then* $\mathcal{E} = \emptyset$.

For $\alpha < 1$ $\mathcal{E}$ is non-empty (in fact, it has Hausdorff dimension 2 [3]). However Conjecture 1 holds at least for small α.

Theorem 4. *If* $\alpha < 1/3$ *then* $\mathrm{mes}(\mathcal{E}) = 0$.

Theorems 2 and 3 show that for $\alpha > 1$ typically ergodic components are small. A much more ambitious form of Conjecture 1 would be the following.

Conjecture 2. *For* $\alpha < 1$ *there is an ergodic component of infinite measure.*

This conjecture would imply in particular that for $\alpha < 1$ there is a large (infinite measure) set of oscillating trajectories where

$$\liminf_{n\to\infty} v_n < \infty, \quad \limsup_{n\to\infty} v_n = \infty.$$

However Conjecture 2 seems much more difficult than Conjecture 1. Indeed according to [3] there is a large set of parameters where elliptic islands appear at arbitrary large heights.

The proof of Theorem 3 follows a standard route of applying Kolmogorov-Arnold-Moser Theory (cf. e.g. [16, 17, 8, 11]). The proof of Theorem 4 relies on ideas from (partially) hyperbolic dynamics (see [2, 4, 5, 6, 15]). It seems interesting to axiomatize infinite measure systems where the method developed in this paper works similarly to the axiomatic approach to non-uniformly hyperbolic systems (cf. [14, 21]).

2. Background.

2.1. KAM theory.

The proof of Theorem 3 relies on KAM theory for twist maps. We are going to study the "successor map" $(v_n, t_n) \to (v_{n+1}, t_{n+1})$. Therefore we recall in this section results about the existence of invariant curves for twist maps of the cylinder. Details can be found in [9, 10, 7, 19, 20, 11]. Depending on the value of α we need two results about the existence of invariant curves.

Proposition 2.1 (Moser Small Twist Theorem)**.** *Let* $Q : \mathbb{R}_+ \to \mathbb{R}_+$ *be a* C^5-*function. Then for any numbers* a, b *such that* $Q'(r) \neq 0$ *for* $r \in [a, b]$ *for any* K *there is* ε_0 *such that if* F_ε *are exact mappings of the annulus* $\mathbb{R}_+ \times \mathbb{S}^1$ *of the form*

$$F_\varepsilon(r, \phi) = (r + \varepsilon^{1+\delta} P(r, \phi), \phi + \varepsilon Q(r) + \varepsilon^{1+\delta} R(r, \phi))$$

where

$$||P||_{C^5([a,b]\times\mathbb{S}^1)} \leq K, \quad ||R||_{C^5([a,b]\times\mathbb{S}^1)} \leq K$$

then for $\varepsilon \leq \varepsilon_0$ F_ε *has (many) invariant curve(s) separating* $[a,b] \times \mathbb{S}^1$ *into two parts.*

Proposition 2.2 (Moser Invariant Curve Theorem). *Let* $Q : \mathbb{R}_+ \to \mathbb{R}_+$ *be a* C^5*-function. Then for any numbers* a, b *such that* $Q'(r) \neq 0$ *for* $r \in [a,b]$ *there is* ε_0 *such that if* F *is an exact mapping of the annulus* $\mathbb{R}_+ \times \mathbb{S}^1$ *of the form*

$$F(r,\phi) = (r + P(r,\phi), \phi + Q(r) + R(r,\phi))$$

where

$$||P||_{C^5([a,b]\times\mathbb{S}^1)} \leq \varepsilon_0, \quad ||R||_{C^5([a,b]\times\mathbb{S}^1)} \leq \varepsilon_0$$

then F *has (many) invariant curve(s) separating* $[a,b] \times \mathbb{S}^1$ *into two parts.*

In our case the exactness required in the above propositions follows from the fact that the transformations we consider are Poincare maps for Hamiltonian systems (see e.g. [11, 17]).

The existence of the invariant curves separating the phase space into two parts will prevent the trajectories from escaping to infinity.

2.2. **Biased random walks.** For the proof of Theorem 4 we need several elementary facts about simple biased random walks which can be found e.g. in [1].

Proposition 2.3. *Let* $\tilde{\xi}_1, \tilde{\xi}_2 \dots \tilde{\xi}_n \dots$ *be iid random variables such that* $\tilde{\xi}_n \in \{-1, 1\}$ *and* $\mathbb{P}(\tilde{\xi}_n = -1) = p > 1/2$. *Let* $\tilde{\mathcal{X}}_n = \tilde{\xi}_1 + \tilde{\xi}_2 + \cdots + \tilde{\xi}_n$. *Then*

(a) $\mathbb{P}(\tilde{\mathcal{X}}_n \leq 0$ *for all* $n) > 0$.

(b) For each $c > 1 - 2p$ *there are constants* $C > 0$ *and* $\theta < 1$ *such that*

$$\mathbb{P}(\tilde{\mathcal{X}}_n > cn) \leq C\theta^n.$$

Taking $c < 0$ we see that a simple random walk with a bias to the left tends to $-\infty$ with probability 1. We would need the same conclusion about more complicated processes (whose increments are not independent). To this end we shall use the following comparison criterion.

Proposition 2.4. *Suppose* $\xi_1, \xi_2 \dots \xi_n \dots$ *is a random process such that* $\xi_n = \pm 1$ *and for all* n

$$\mathbb{P}(\xi_n = -1 | \xi_1 \dots \xi_{n-1}) \geq p.$$

Let $\tilde{\xi}_1, \tilde{\xi}_2 \dots \tilde{\xi}_n \dots$ *be iid random variables such that* $\tilde{\xi}_n = \pm 1$, *and* $\mathbb{P}(\tilde{\xi}_n = -1) = p$. *Let*

$$\mathcal{X}_n = \sum_{j=1}^{n} \xi_n \quad \tilde{\mathcal{X}}_n = \sum_{j=1}^{n} \tilde{\xi}_n.$$

Then for any n, m_1, m_2

$$\mathbb{P}(\max_{k\leq n} \tilde{\mathcal{X}}_k \leq m_1, \quad \min_{k\leq n} \tilde{\mathcal{X}}_k \leq m_2) \leq \mathbb{P}(\max_{k\leq n} \mathcal{X}_k \leq m_1, \quad \min_{k\leq n} \mathcal{X}_k \leq m_2).$$

Proof of Proposition 2.4. Let $U_1, U_2 \dots U_n \dots$ be random variables which are independent and identically distributed on $[0,1]$. Define $\xi_n^* = -1$ if $U_n < \mathbb{P}(\xi_n = -1 | \xi_1 = \xi_1^*, \dots, \xi_{n-1} = \xi_{n-1}^*)$ and $\xi_n = 1$ otherwise. Also let $\tilde{\xi}_n^* = -1$ if $U_n < p$ and $\tilde{\xi}_n^* = 1$ otherwise. Let

$$\mathcal{X}_n^* = \sum_{j=1}^{n} \xi_j^*, \quad \tilde{\mathcal{X}}_n^* = \sum_{j=1}^{n} \tilde{\xi}_j^*.$$

Then $\{\mathcal{X}_n^*\}$ has the same distribution as $\{\mathcal{X}_n\}$, $\{\tilde{\mathcal{X}}_n^*\}$ has the same distribution as $\{\tilde{\mathcal{X}}_n\}$ and $\mathcal{X}_n^* \leq \tilde{\mathcal{X}}_n^*$. □

3. **The mapping.** In this section we describe the mapping relating (v_{n+1}, t_{n+1}) to (v_n, t_n). In the computations below we assume that v_n is sufficiently large so that the outgoing velocity of the ball after the collision is larger than the maximal possible velocity of the plate and hence there are no recollisions before the ball reaches its maximal height and its velocity changes sign. The reader will see that we shall use the formulas derived below only in situations where v_n is large.

We note that the computations of this section work for all $\alpha > 0$. The restriction on α comes from later sections.

Let v_{n+1}^- denote the velocity immediately before the $n+1$st collision. Since the energy is preserved between the collisions we have

$$\frac{(v_{n+1}^-)^2}{2} + (B + A\sin t_{n+1})^\alpha = \frac{(v_n)^2}{2} + (B + A\sin t_n)^\alpha.$$

Thus

$$v_{n+1}^- = \sqrt{v_n^2 + 2(B + A\sin t_n)^\alpha - 2(B + A\sin t_{n+1})^\alpha}.$$

The law of elastic collision reads

$$v_{n+1} - v_{n+1}^- = 2A\cos t_{n+1}.$$

Combining the last two formulas we get

$$v_{n+1} = G(v_n, t_n, t_{n+1}),$$

where

$$G(v, t, \bar{t}) = 2A\cos\bar{t} + \sqrt{v_n^2 + 2(B + A\sin t)^\alpha - 2(B + A\sin\bar{t})^\alpha}. \tag{1}$$

Observe that the expression for v_n^- can be simplified as follows.

$$v_{n+1}^- = \sqrt{v_n^2 + \Delta_n} = v_n + \frac{\Delta_n}{2v_n} + v_n q(\Delta_n/v_n^2)$$

where

$$\Delta_n = (B + A\sin t_n)^\alpha - (B + A\sin t_{n+1})^\alpha$$

and q is a smooth function satisfying

$$q(\xi) = \mathcal{O}(\xi^2), \text{ as } \xi \to 0. \tag{2}$$

Summarizing we get

$$v_{n+1} - v_n = 2A\cos t_{n+1} + \frac{\Delta_n}{2v_n} + v_n q(\Delta_n/v_n^2). \tag{3}$$

It remains to find t_{n+1}. Let t_{n+1}^* be the first time the particle would return to the height $B + A\sin t_n$ if we remove the plate. Then $t_{n+1}^* = t_n + b(v_n, t_n)$. To compute b let E denote the energy of the ball between the collisions. Then the maximal height the particle reaches is $E^{1/\alpha}$ and since

$$\frac{dx}{dt} = \sqrt{2(E - x^\alpha)}$$

we get

$$b(v,t) = \sqrt{2}\int_{B+A\sin t}^{E^{1/\alpha}} \frac{dx}{\sqrt{E - x^\alpha}} = \sqrt{2}\int_0^{E^{1/\alpha}} \frac{dx}{\sqrt{E - x^\alpha}} - \sqrt{2}\int_0^{B+A\sin t} \frac{dx}{\sqrt{E - x^\alpha}}.$$

Making the change of variables $x = E^{1/\alpha} y$ we obtain that the first term here equals $E^{1/\alpha - 1/2}\int_0^1 \frac{dy}{\sqrt{1-y^\alpha}}$. Next

$$t_{n+1} = t_{n+1}^* + \bar{c}(t_{n+1}^*, 1/v). \tag{4}$$

Incorporating the last part of b into c and remembering that

$$E = \frac{v^2}{2} + (B + A\sin t)^\alpha = \frac{v^2}{2}(1 + \bar{g}(t, 1/v))$$

we obtain

$$t_{n+1} = t_n + v_n^\gamma\,[k + g(t_n, 1/v_n)] + c(t_n, k + g(t_n, 1/v_n), 1/v) \tag{5}$$

where $k > 0$ is a constant

$$\gamma = \frac{2}{\alpha} - 1 \tag{6}$$

and $g(t, 0) = 0$. Let F be the mapping given by (3), (5).

For our analysis it is important to note that the functions q, g and c describing subleading terms in (3) and (5) have the property that they tend to 0 together with their partial derivatives up to the fifth order as $v \to \infty$. For the functions themselves this follows by direct inspection and to get the same conclusion about the derivatives we use the fact that the function are sums of Laurent series convergent at infinity, so taking the derivative can only improve their decay.

Finally we shall use the fact that F preserves measure with density

$$\boldsymbol{\rho}(v, t) = v - A\cos t \tag{7}$$

(To see this make a change of variables

$$y = x - (B + A\sin t), \quad p = \frac{dy}{dt} = v - A\cos t$$

so that a collision happens if $y = 0$ and the elasticity of collisions reads $p_{n+1} = -p_{n+1}^-$. Then on the collision set $\{y = 0\}$ we have $\boldsymbol{\rho} dp dt = d\boldsymbol{\lambda}$ where $\boldsymbol{\lambda}$ stands for Poincare–Cartan form $p dy + \tilde{H} dt$ and

$$\tilde{H}(y, p, t) = \frac{p^2}{2} + (y + B + A\sin t)^\alpha - Ay\sin t$$

is the Hamiltonian in (y, p)-coordinates.)

4. **Strong potentials.** In this section we prove Theorem 3. Accordingly we work in the regime $\gamma < 1$. We want to know what happens for $v \gg 1$. Make a change of variables

$$v = au \tag{8}$$

where $a \gg 1$ is a normalizing constant and $u_0 \sim 1$. F takes the following form

$$u_{n+1} - u_n = \frac{1}{a}\left(2A\cos t_{n+1} + \frac{\Delta_n}{2au_n}\right) + \hat{q}(\Delta_n, u_n, 1/a). \tag{9}$$

$$t_{n+1} - t_n = \tag{10}$$
$$a^\gamma u^\gamma k + a^\gamma u^\gamma \hat{g}(t_n, u_n, 1/a) + \hat{c}(t_n, t_n + a^\gamma u_n(k + \hat{g}), u_n, 1/a).$$

We consider two cases.

(I) $\alpha > 2$. In this case we have $-1 < \gamma < 0$. In this case both increments of t and u are small but since $\gamma > -1$ the change of u is much smaller. Therefore the result follows from Moser Small Twist Theorem. (Here the twist condition follows from the fact that k is a positive constant and the norm estimates follow from the expansions of Section 3.)

(II) $1 < \alpha < 2$. In this case F has strong twist. Accordingly we introduce a new variable $z_n = a^\gamma(u_n - u_0)$. We get

$$z_{n+1} - z_n = \frac{1}{a^{1-\gamma}}\left[A\cos t_{n+1} + \frac{\Delta_n}{2au_n}\right] + a^\gamma \hat{q}\left(t_n, u_0 + \frac{z_n}{a^\gamma}, \frac{1}{a}\right),$$

$$t_{n+1} - t_n = ka^\gamma u_0^\gamma + k\gamma u_0^{\gamma-1} z_n + ka^\gamma u_0^\gamma \beta\left(z_n, \frac{1}{a^\gamma}\right) + a^\gamma\left(u_0 + \frac{z_n}{a^\gamma}\right)^\gamma \tilde{c}\left(t_n, u_0 + \frac{z_n}{a^\gamma}, \frac{1}{a}\right).$$

Here $\tilde{c}$ incorporates both $\hat{c}$ and $\hat{g}$ terms and β comes from higher order terms in the expansion of $(u_0 + \frac{z_n}{a^\gamma})^\gamma$. In particular, β is quadratic in $\frac{1}{a^\gamma}$. It follows that as $a \to \infty$ the partial derivatives of the RHS are uniformly bounded. Take $u_0 = 1$ and let a_m be such that $ka_m^\gamma = 2\pi m + 2\pi\rho$. Then for large m the above map is a small perturbation of the limit transformation

$$z_{n+1} = z_n, \quad t_{n+1} = t_n + 2\pi\rho + k\gamma z_n.$$

Now the result follows from Moser Invariant Curve Theorem.

Our proof shows why the values $\alpha = 1$ and 2 are special. Note that for the motion in the potential x^α with reflection at 0 all trajectories are periodic with the period $T(v)$ which is asymptotically proportional to v^γ where γ is given by (6) (v denotes the velocity at the moment of reflection). For large v our system can be considered as a small perturbation of this motion. Accordingly for $\alpha > 2$ we have a positive twist and for $\alpha < 2$ negative twist. Therefore to make our asymptotic computations work for α close to 2 we need to take larger and larger initial velocities. $\alpha = 2$ corresponds to the harmonic oscillator for which the frequency is independent of the energy. To see if we have twist for $\alpha = 2$ we need to know if the average hight on which the ball collides with the plate is positive or negative which explains the formulas in Theorem 2. Finally for $\alpha \leq 1$ $\frac{dT}{dv} \not\to 0$ as $v \to \infty$ meaning that the frequency changes significantly from one iteration to the next. Therefore the system is not close to integrable for large v. Again for $\alpha > 1$ this means that the first invariant curve appears higher and higher as α approaches 1 finally disappearing at infinity for $\alpha = 1$.

5. **Weak potentials. Critical set.** The proof of Theorem 4 relies on hyperbolic theory. Our first goal is to define a critical set $\mathbf{C}$ so that the orbits avoiding this set are hyperbolic. To show the hyperbolicity we foliate the complement of the critical set by curves which are expanded by the iterations of F. The precise definitions of the the critical set and the foliation involve quite cumbersome formulas and in fact there are several possible definitions. For our arguments it is important that these objects satisfy Lemmas 5.1–5.3 below. In the main body of the paper we show how these lemmas imply Theorem 4 postponing straightforward but lengthy derivation of the lemmas to appendices.

We expect that the conditions of Lemmas 5.1–5.3 can be verified for many other infinite measure systems thus implying that the set of the trajectories going to infinity has zero measure for those systems. In particular we hope those lemmas still hold for all $\alpha < 1$ (with more elaborate definitions of $\mathbf{C}$ and the foliation) thus establishing our main conjecture in full generality.

Let us describe our construction. The idea is very straightforward. We would like to define an invariant family of cones on the most of the phase space (see (16) below). Since the expression for derivatives of F involves large quantities we hope that the hyperbolicity is strong on most of the phase space and so a big part of the tangent space is mapped into the cone. So a good guess for invariant cones is obtained by choosing an arbitrary field of directions (we choose vertical lines) and defining the cone to be a small neighborhood of the image of this field (see equations (11) and (16)) below).

Denote by M the phase space of our system. Set $\beta = \frac{\gamma-1}{2}$. Recall that we assume that $\alpha < 1/3$, so by (6) $\beta > 2$. Let G be the function given by (1). We shall denote by $\partial_j G$ the derivatives of G with respect to the jth variable. Let

$$E(v,t) = \frac{v^2}{2} + (B + A\sin t)^\alpha$$

denote the energy of the ball at the time of collision. Set

$$h(v,t) = \partial_3 G(F^{-1}(t,v),t). \tag{11}$$

Let K be a large constant to be specified later. Define the *critical set*

$$\mathbf{C} = \left\{ \left| \frac{\partial E}{\partial t} + \frac{\partial E}{\partial v} h \right| < K v^{1-\beta} \right\}.$$

Thus a more explicit description of $\mathbf{C}$ is

$$\mathbf{C} = \left\{ \left| \frac{\alpha A \cos t}{(B + A\sin t)^{1-\alpha}} + h(v,t)v \right| < K v^{1-\beta} \right\}.$$

The following two properties of $\mathbf{C}$ play a crucial role in our analysis. First, given any V_0 we can take K so large that

$$\mathbf{C} \supset \{v \le V_0\}. \tag{12}$$

The verification of this property is straightforward and it will allow us to use asymptotic expansions in $1/v$ outside $\mathbf{C}$. The second property is

$$\mathrm{mes}(\mathbf{C}) < \infty. \tag{13}$$

To check (13) observe that

$$h(v,t) = -2A\sin t - \frac{A\alpha\cos t}{v(B + A\sin t)^{1-\alpha}} + \mathcal{O}\left(\frac{1}{v^3}\right) \tag{14}$$

and so

$$\mathbf{C} \subset \left\{ \left| 2A\sin t + \frac{A\alpha\cos t}{v(B + A\sin t)^{1-\alpha}} \right| \le \mathrm{Const}\left(\frac{1}{v^3} + \frac{1}{v^\beta}\right) \right\}.$$

Now (13) follows from (7). In fact this argument gives more. Namely consider a neighborhood of $\mathbf{C}$

$$\hat{\mathbf{C}} = \left\{ (v,t) : \mathrm{dist}((v,t),\mathbf{C}) \le \frac{10}{v^\beta} \right\}.$$

Then

$$\mathrm{mes}(\hat{\mathbf{C}}) < \infty. \tag{15}$$

$\hat{\mathbf{C}}$ will be handy in our analysis because while dynamics outside $\mathbf{C}$ is hyperbolic it is non-Markov since $F(\partial\mathbf{C}) \not\subset \partial\mathbf{C}$. (15) allows us to make critical set fuzzy, sometimes taking it as large as $\hat{\mathbf{C}}$ and sometimes as small as $\mathbf{C}$. This remedies the lack of Markov property.

Next, consider a family of cones

$$\mathcal{C} = \left\{ (\delta v, \delta t) : \left| \frac{\delta v}{\delta t} - h \right| < \frac{1}{v^\beta} \right\}. \tag{16}$$

A curve $\Gamma \in M - \mathbf{C}$ will be called a *standard curve* if it satisfies requirements (17)–(20) below. First it is neither too short nor too long. Namely

$$\delta \leq \text{length}(\Gamma) \leq 2\delta \tag{17}$$

for a sufficiently small δ (e.g. $\delta = 0.01$ will do). Secondly we demand a Markov type property

$$F(\partial\Gamma) \subset \partial\mathbf{C} \tag{18}$$

Third we want

$$\Gamma' \in \mathcal{C}. \tag{19}$$

(19) implies that Γ is a graph of a function $v = \psi(t)$. In fact (14) shows that ψ is close to $2A\cos t + C$ for some constant $C = C(\Gamma)$. Our fourth requirement extends this closeness condition to the second derivative. Namely,

$$|\psi'' + 2A\cos t| \leq 0.1. \tag{20}$$

A probability density $\rho(t)$ on Γ is called a *standard density* if

$$\left|\frac{d}{dt}\ln\rho\right| \leq C_1 \tag{21}$$

for a constant C_1 to be specified below.

We call a pair $\ell = (\Gamma, \rho)$ where Γ is a standard curve and ρ is a density on it a *standard pair*. Given ℓ we denote by $\mathbb{P}_\ell$ the δ-measure on Γ

$$\mathbb{P}_\ell(\Omega) = \int_\Gamma 1_\Omega \rho dt.$$

We shall write

$$V(\ell) = V(\Gamma) = \inf_\Gamma v.$$

Invariance properties of standard curves play a key role in our argument.

Lemma 5.1 (**Invariance**). *(a) If Γ is a standard curve then*

$$||dF|\Gamma'|| \geq C_2 V^\beta(\ell).$$

(b) $F\Gamma = Z \bigcup \left(\bigcup_j \Gamma_j\right)$ where Γ_j are standard curves and

$$\left(F\Gamma \bigcap \mathbf{C}\right) \subset Z \subset \hat{\mathbf{C}}.$$

(c) If ρ is a standard density on Γ then its pushforward, properly renormalized, is a standard density on Γ_j.

(d) $\mathbb{P}_\ell(Fx \in Z) \leq C_3 V^{-\beta}(\ell)$.

Remark. The part (d) of this lemma is not really needed in the proof. See remark after the proof of Lemma 7.1. However part (d) allows to get a conclusion stronger than Theorem 4. Namely, it shows that in fact for most orbits visit regions where the energy is of order 1. We include part (d) because the above conclusion is of independent interest and also because our method might be useful to prove that for other systems the measure of points escaping to infinity is infinite and in that case part (d) would be essential.

Our next goal is to show that there are many standard curves. To this end it is convenient to relax (20). We say that Γ is a *prestandard pair* if it satisfies (17), (19) and

$$|\phi''(t)| \leq C_4 V^{2\beta}(\Gamma) \tag{22}$$

Likewise we call a density ρ on Γ *prestandard density* if

$$\left|\frac{d}{dt}\ln\rho\right| \leq C_5 V^{2\beta}(\Gamma).$$

We call a pair $\ell = (\Gamma, \rho)$ *prestandard pair* if Γ is prestandard curve and ρ is prestandard density on it. The definition $\mathbb{P}_\ell$ extends without changes to prestandard pairs.

Lemma 5.2 (**Abundance**). *(a) Let $\ell = (\Gamma, \rho)$ be a prestandard pair. Then*

$$F^3\Gamma = Z \bigcup \left(\bigcup_k \bar{\Gamma}_k\right)$$

where $\bar{\Gamma}_k$ are standard curves, the pushforward of ρ, properly renormalized, is a standard density on $\bar{\Gamma}_k$ and

$$Z \subset \left(\hat{\mathbf{C}} \bigcup F^{-1}\hat{\mathbf{C}} \bigcup F^{-2}\hat{\mathbf{C}} \bigcup \left\{(v,t) : \text{dist}((v,t), \partial\Gamma) < \frac{10}{V^\beta(\ell)}\right\}\right).$$

(b) If ν is the restriction of the Lebesgue measure to $M - \mathbf{C}$ then ν is a convex combination of prestandard pair measures

$$\nu = \int \mathbb{P}_{\ell_\alpha} d\lambda(\alpha)$$

for some factor measure λ on the set of prestandard pairs.

(c)

$$F^2_*\nu = \int \mathbb{P}_{\ell_{\bar{\alpha}}} d\bar{\lambda}(\bar{\alpha}) + \hat{\nu}$$

where $\ell_{\bar{\alpha}}$ are standard pairs, $\bar{\lambda}$ is a factor measure on a set of standard pairs and and $\hat{\nu}$ is concentrated on $\hat{\mathbf{C}} \bigcup F^{-1}\hat{\mathbf{C}} \bigcup F^{-2}\hat{\mathbf{C}}$.

To formulate our next result recall that a change of variables (8) brings our system to a slow-fast form (9)–(10) where the slow variable is the rescaled velocity and the fast variable is the plate phase. The key to that follows is the fact that the fast variable becomes uniformly distributed before the value of the slow variable changes significantly. Namely the following result holds.

Lemma 5.3 (**Equidistribution**). *(a) Let $\ell = (\Gamma, \rho)$ be a standard pair where Γ is a graph of a function $v = \psi(t)$. Let $\mathcal{A} = \mathcal{A}(t)$ be a C^1 function which does not depend on v. Then*

$$\int_\Gamma \rho(t)\mathcal{A}(F(\psi(t),t))dt - \frac{1}{2\pi}\int_0^{2\pi} \mathcal{A}(t)dt = \mathcal{O}(V^{-\beta/2}(\Gamma)).$$

(b) Moreover if $\mathcal{B} : \Gamma \to \mathbb{R}$ is a C^1 function then

$$\int_\Gamma \rho(t)\mathcal{B}(t)\mathcal{A}(F(\psi(t),t))dt - \left(\int_\Gamma \rho(t)\mathcal{B}(t)dt\right)\left(\frac{1}{2\pi}\int_0^{2\pi} \mathcal{A}(t)dt\right)$$

$$= \mathcal{O}(||\mathcal{B}||_{C^1} V^{-\beta/2}(\Gamma)).$$

6. **Proof of Theorem 4.** In this section we reduce Theorem 4 to our main technical estimate (Lemma 6.2).

Lemma 6.1. *If K is large enough then*

$$\text{mes}((v_0, t_0) : (v_n, t_n) \notin \hat{\mathbf{C}} \textit{ for all } n \in \mathbb{N}) = 0.$$

Proof of Theorem 4. By Lemma 6.1 it suffices to show that $\text{mes}(\mathcal{E} \bigcap \hat{\mathbf{C}}) = 0$. Let $T : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ denote the first return map (which is well defined due to Lemma 6.1). Denote $(v^{(j)}, t^{(j)}) = T^j(v, t)$. Due to (15) we can apply Poincare Recurrence Theorem which tells us that for almost every point there are infinitely many j such that $v^{(j)} < v + 1$. In particular $v^{(j)} \not\to \infty$. □

To prove Lemma 6.1 we study the dynamics outside of the critical set.

Let Γ be a standard pair. We define $\tau : \Gamma \to \mathbb{N}$ as follows. By Lemma 5.1(b)

$$F\Gamma = Z \bigcup \left(\bigcup_j \Gamma_j \right).$$

We define $\tau = 1$ on Z. To define τ on $\bigcup_j \Gamma_j$ we apply Lemma 5.1(b) to each Γ_j getting

$$F\Gamma_j = Z_j \bigcup \left(\bigcup_k \Gamma_{jk} \right).$$

We let $\tau = 2$ on $\bigcup_j Z_j$ and continue this procedure inductively. Then $F^{\tau(v,t)}(v, t) \in \hat{\mathbf{C}}$ so it suffices to prove that

$$\mathbb{P}_\ell(\tau < \infty) = 1. \tag{23}$$

for every smooth standard pair. The following auxiliary estimate is our main technical result.

Lemma 6.2. *There exists $\kappa > 0$ such that for any smooth standard pair ℓ we have*

$$\mathbb{P}_\ell(\tau < V^3(\ell)) \geq \kappa.$$

Lemma 6.2 is proven in the next section. Here we show how it implies Lemma 6.1 and, hence, Theorem 4.

Proof of Lemma 6.1. Combining Lemmas 5.1 and 6.2 we conclude that for any $k \in \mathbb{N}$ for any standard pair ℓ there are functions $n_k(v_0, t_0)$ such that

$$\mathbb{P}_\ell(\tau > n_k) < (1 - \kappa)^k.$$

Indeed we let $n_1 = V^3(\ell)$. Next if $\tau(t_0, v_0) \leq n_1$ we let $n_2 = n_1$. Otherwise $F^{n_1}(t_0, v_0)$ belongs to some standard pair Γ_{j1}. We let $n_2 = n_1 + V^3(\Gamma_{j1})$. Continuing this procedure recursively we construct n_k. Since k is arbitrary (23) follows.

Now to complete the proof of Lemma 6.1 we apply (23) to each standard pair $\ell_{\bar{\alpha}}$ in the decomposition of Lemma 5.2(c). □

7. **Proof of Lemma 6.2.** Let us outline the idea of the proof. We want to show that a typical orbit eventually enters the region where velocity is small and so it has a significant chance to fall into a critical set. To see why we expect velocity to decrease eventually recall that Lemma 5.3 states that the dynamics of the fast variable is chaotic. Therefore we expect (due to (3)) that the change of velocity obeys the Central Limit Theorem, that is, it can be approximated by the Brownian Motion. Since the Brownian Motion is recurrent we expect the same for our system. In this paper, however, we do not prove the Central Limit Theorem, because, first, it would make the paper much longer, and, second, the Central Limit Theorem does not immediately imply recurrence because the Brownian approximation works on large scale whereas recurrence depends on the behavior of the system at small scale. Instead we directly mimic one of the proofs of the recurrence of the Brownian Motion. This proof is based on the fact that the Brownian Motion is a martingale so we show that v_n is an approximate martingale (with respect to the filtration generated by the partition given by Lemma 5.1) and then conclude as in the Brownian case.

Given ℓ let $R_k = 2^k V(\ell)$. Consider the decomposition

$$F^n\Gamma = \bigcup_j \Gamma_{jn} \bigcup \{\tau < n\}.$$

Let τ_1 be the first time when either $\tau(t,v) = n$ or $(t_n, v_n) \in \Gamma_{jn}$ where $V(\Gamma_{jn}) \le R_{-1}$ or $V(\Gamma_{jn}) \ge R_1$. We next define τ_k by induction. Namely if $\tau \le \tau_1$ we let $\tau_2 = \tau_1$. Otherwise v_{τ_1} is near R_{σ_1} where $\sigma_1 = \pm 1$ (more precisely due to (3) and (17) there is a constant C such that $|v_{\tau_1} - R_{\sigma_1}| \le C$). We let τ_2 be the first time after τ_1 when either $(t_n, v_n) \in \mathbf{C}$ or $(t_n, v_n) \in \Gamma_{jn}$ where $V(\Gamma_{jn}) \le R_{\sigma_1 - 1}$ or $V(\Gamma_{jn}) \ge R_{\sigma_1+1}$. Then we define τ_k recursively by the same procedure.

Observe that a repeated application of Lemma 5.1 shows that if $\ell = (\Gamma, \rho)$ is a standard pair then

$$F^{\tau_k}\Gamma = \bigcup_m \Gamma_m \bigcup \{\tau \le \tau_k\} \tag{24}$$

where Γ_m are standard curves and pushforwards of ρ to Γ_m, properly renormalized, are standard densities.

Lemma 7.1. *If K is large enough then*

(a) $\mathbb{P}_\ell(V(\Gamma_{jn}) < R_{-1}) \ge 0.6$

(b) There exists $\theta < 1$ such that $\mathbb{P}_\ell(\tau_1 \ge s) \le C\theta^{s/V^2(\ell)}$.

We now define ξ_n as follows. If $\tau > \tau_k$ then v_{τ_k} is close to some $R_{\mathcal{X}_k}$ and $v_{\tau_{k-1}}$ is close to some $R_{\mathcal{X}_{k-1}}$. In this case we let $\xi_k = \mathcal{X}_k - \mathcal{X}_{k-1}$. If $\tau \le \tau_k$ we let $\xi_k = -1$.

Comparing $\mathcal{X}_k$ with a simple random walk moving down with probability 0.6 and up with probability 0.4 (see section 2.2) we conclude that there exist constant κ such that

$$\mathbb{P}(\Omega) \ge 2\kappa$$

where

$$\Omega = \{ \max_{k \le 100 \log_2 V(\ell)} \mathcal{X}_k \le 0 \qquad \min_{k \le 100 \log_2 V(\ell)} \mathcal{X}_k \le -\log_2 V(\ell)\}.$$

Next Lemma 7.1(b) applied to those curves in (24) which belong to Ω implies that for each k

$$\mathbb{P}(\tau_{k+1} - \tau_k > V^{5/2}(\ell) | v_{\tau_k} < V(\ell)) \le C\theta^{\sqrt{V(\ell)}}.$$

Hence

$$\mathbb{P}(\max_{k\leq 100\log_2 V(\ell)}(\tau_{k+1}-\tau_k)>V^{5/2}(\ell)|\Omega)\leq \bar{C}\,[\log_2 V(\ell)]\,\theta^{\sqrt{V(\ell)}}.$$

On the other hand if K is sufficiently large then $\mathcal{X}_k < -\log_2 V(\ell)$ implies that $(t_{\tau_k}, v_{\tau_k})\in \mathbf{C}$. This completes the proof of Lemma 6.1. It remains to prove Lemma 7.1.

Proof of Lemma 7.1. Given a standard pair we shall denote by

$$\mathbb{E}_\ell(\mathcal{A})=\int_\Gamma \mathcal{A}(\psi(t),t)\rho(t)dt$$

the expectation with respect to $\mathbb{P}_\ell$. We shall write $V=V(\ell)$. Observe that parts (b) and (c) of Lemma 5.1 imply that for any n

$$\mathbb{E}_\ell((\mathcal{A}\circ F^n)1_{\tau_1>n})=\sum_j c_j\mathbb{E}_{\ell_j}(A) \tag{25}$$

where $\ell_j=(\Gamma_j,\rho_j)$ are standard pairs and c_j are positive numbers such that

$$\sum_j c_j=\mathbb{P}_\ell(\tau_1>n).$$

Indeed let

$$F^n\Gamma=\bigcup_j \Gamma_j\bigcup\{\tau_1\leq n\}$$

be the decomposition obtained by repeated application of Lemma 5.1(b). Defining

$$c_j=\mathbb{P}_\ell(F^n x\in\Gamma_j),$$

$$\rho_j=\left(\rho\circ F^{-n}\right)\left(\frac{dt_0}{dt_n}\right)/c_j,$$

we obtain (25)

Let $\zeta_k=2A\cos t_{k+1}1_{\tau_1>k}$. We claim that

$$\mathbb{E}_\ell(\zeta_k)=\mathcal{O}\left(\mathbb{P}_\ell(\tau_1>k)V^{-\beta/2}\right). \tag{26}$$

Indeed (25) gives

$$\mathbb{E}_\ell(\zeta_k)=\sum_j c_{jk}\mathbb{E}_{\ell_{jk}}(2A\cos t_1).$$

By Lemma 5.3(a)

$$\sum_j c_{jk}\mathbb{E}_{\ell_{jk}}(2A\cos t_1)=\sum_j c_{jk}\mathcal{O}\left(V^{-\beta/2}\right)=\mathbb{P}_\ell(\tau_1>k)\mathcal{O}\left(V^{-\beta/2}\right)$$

as claimed. (Here and below we use the fact that for $n\leq\tau_1$ we have $v_n\geq\frac{V(\ell)}{2}-C$.) The same argument shows that

$$\mathbb{E}_\ell(\zeta_k^2)=2A^2\mathbb{P}_\ell(\tau_1>k)\left(1+O\left(V^{-\beta/2}\right)\right). \tag{27}$$

Next we claim that for $m<V^3$

$$\mathbb{E}_\ell\left(\left(\sum_{k=0}^{m-1}\zeta_k\right)\zeta_m\right)=\mathcal{O}\left(\mathbb{P}_\ell(\tau_1>m)V^{1-\beta/2}\right). \tag{28}$$

Indeed by (25)

$$\mathbb{E}_\ell \left(\left(\sum_{k=0}^{m-1} \zeta_k \right) \zeta_m \right) = \sum_j c_{jm} \mathbb{E}_{\ell_{jm}} \left(\left[\sum_{k=0}^{m-1} \zeta_{-k} \right] 2A \cos t_1 \right) \tag{29}$$

for some $\ell_{jm} = (\Gamma_{jm}, \rho_{jm})$. Next, (2) and (3) imply

$$\sum_{k=0}^{m-1} 2A \cos t_{k+1} = v_m - v_0 + \mathcal{O}(1) + \mathcal{O}\left(\frac{m}{V^3}\right) \tag{30}$$

Therefore

$$\left\| \sum_{k=0}^{m-1} \zeta_{-k} \right\|_{C^1(\Gamma_{jm})} \leq \text{Const} V \tag{31}$$

(the main contribution comes from C^0-norm, the norm of the derivative is much smaller since F^{-k} strongly contracts Γ_{jm}).

Now we derive (28) from (29) the same way as (26) was derived from (25) except we use (31) and Lemma 5.3(b) instead of Lemma 5.3(a). Let $n = LV^2(\ell)$ where L is a large constant. From (27) and (28) we obtain

$$\mathbb{E}_\ell \left(\left(\sum_{k=1}^{n} \zeta_k \right)^2 \right) \geq \text{Const} n \mathbb{P}_\ell \left(\tau_1 > n\right) + \mathcal{O}\left(nV^{1-\beta/2}\right) \tag{32}$$

$$= \text{Const} n \mathbb{P}_\ell(\tau_1 > n) + o\left(V^2\right)$$

since $\beta > 2$. On the other hand by (30)

$$\mathbb{E}_\ell \left(\left(\sum_{k=1}^{n} \zeta_k \right)^2 \right) \leq V^2 + \mathcal{O}\left(V\right). \tag{33}$$

Combining (32) and (33) we obtain

$$V^2(1 + o_{V\to\infty}(1)) \geq n\mathbb{P}_\ell(\tau_1 > n)$$

If L is large enough this inequality implies

$$\mathbb{P}(\tau_1 \geq n) \leq \bar{\theta} \text{ for some } \bar{\theta} < 1.$$

Now using Lemma 5.1(b) we obtain by induction that for any k

$$\mathbb{P}(\tau_1 \geq kn) \leq \bar{\theta}^k.$$

This proves part (b) of Lemma 7.1.

To prove (a) observe that by part (b) and (26)

$$\mathbb{E}_\ell \left(\sum_{k=1}^{\infty} \zeta_k \right) = \mathcal{O}\left(V^{-\beta/2}\right) \mathbb{E}_\ell(\tau_1) = \mathcal{O}\left(V^{2-\beta/2}\right) = o(V). \tag{34}$$

Since the probability to enter Z during each iteration is $\mathcal{O}\left(V^{-\beta}\right)$ due to Lemma 5.1(d) we have

$$\mathbb{P}_\ell(\tau_1 \leq \tau) \leq \text{Const} \frac{\mathbb{E}(\tau_1)}{V^\beta} = \mathcal{O}(V^{2-\beta}).$$

Since

$$\sum_{k=1}^{\infty} \zeta_k = v_{\tau_1} - v_0 + \mathcal{O}(1) + \mathcal{O}\left(\frac{\tau_1}{V^3}\right).$$

we obtain

$$o(V) = \tag{35}$$
$$V\,\mathbb{P}_\ell(v_{\tau_1} \text{ is close to } 2V) - \frac{V}{2}\mathbb{P}_\ell\left(v_{\tau_1} \text{ is close to } \frac{V}{2}\right) + \mathcal{O}(1).$$

Therefore by choosing K large the probability of the second alternative can be made as close to $2/3$ as we wish. □

Remark. We used Lemma 5.1(d) to obtain (35) but (35) is not really necessary to conclude the proof. Without Lemma 5.1(d) we would obtain a weaker bound

$$\mathbb{P}_\ell\left(v_{\tau_1} \text{ is close to } \frac{V(\ell)}{2}\right) + \mathbb{P}_\ell\left((v_{\tau_1}, t_{\tau_1}) \in \hat{\mathbf{C}}\right) \approx \frac{2}{3}$$

which however suffices for our purposes.

Appendix A. **Invariance properties of standard pairs.** In this section we prove Lemmas 5.1 and 5.2.

Proof of Lemma 5.1. Let (v_n, t_n) be a point on our standard curve and

$$(v_{n+1}, t_{n+1}) = F(v_n, t_n).$$

Let h_n be the slope of the curve. We shall write $V = V(\ell)$. Denote $L_n = \frac{dt_{n+1}}{dt_n}$. To establish (a) it suffices to show

$$L_n \geq C v_n^\beta. \tag{36}$$

For other parts of Lemma 5.1 it is important to check that choosing K sufficiently large C can be made as large as we wish. By (4) it suffices to check that $\frac{dt^*_{n+1}}{dt_n} \geq \bar{C} v^\beta$ where $\bar{C}$ can be made as large as we wish. We have

$$\frac{dt^*_{n+1}}{dt_n} = \text{Const} E_n^{1/\alpha - 3/2}\frac{dE_n}{dt_n} + \mathcal{O}(1).$$

Since $E_n^{1/\alpha-3/2} \sim v_n^{2\beta-1}$ it is enough to bound

$$\frac{dE_n}{dt_n} = h_n v_n + \frac{\partial E_n}{\partial t_n} = h(v_n, t_n) v_n + \frac{\partial E_n}{\partial t_n} + v_n \varepsilon_n$$

where $|\varepsilon_n| < V^{-\beta}$. Since $(v_n, t_n) \notin \mathbf{C}$ the sum of the first two terms is larger that $K v^{1-\beta}$ so part (a) follows.

Our next goal is to estimate h_{n+1} and its derivative. Since $v_{n+1} = G(v_n, t_n, t_{n+1})$ we have

$$h_{n+1} = \partial_3 G + \frac{1}{L_n}\left((\partial_1 G)\, h_n + \partial_2 G\right).$$

Since $|h_n| < 2A$ the second term is less than $v_n^{-\beta}$ if K is large enough. Hence $(F\Gamma)' \in \mathcal{C}$. Next

$$\frac{dh_{n+1}}{dt_{n+1}} = \partial^2_{33} G + 2\frac{\left(\partial^2_{13} G\right) h_n + \partial^2_{23} G}{L_n} \tag{37}$$
$$+ \frac{\left(\partial^2_{11} G\right) h_n^2 + 2\left(\partial^2_{12} G\right) h_n + \partial^2_{22} G}{L_n^2} + \frac{(\partial_1 G)\frac{dh_n}{dt_n}}{L_n^2} + \left(\frac{(\partial_1 G)\, h_n + \partial_2 G}{L_n^3}\right)\left(\frac{d^2 t_{n+1}}{dt_n^2}\right).$$

The first term here can be made as close to $-2A\cos t_{n+1}$ as we wish by choosing K large (and thus making V_0 in (12) as large as we wish). The second and the third terms are $\mathcal{O}(V^{-\beta})$ and $\mathcal{O}(V^{-2\beta})$ respectively. The fourth term is $\mathcal{O}\left(V^{-2\beta}\right)$ by (20). Next, (4) shows that for large v

$$\frac{d^2 t_{n+1}}{dt_n^2} \leq 2\left|\frac{d^2 t^*_{n+1}}{dt_n^2}\right| + \mathcal{O}\left(\frac{dh_n}{dt_n}\right).$$

Now a direct computation shows that

$$\frac{d^2 t^*_{n+1}}{dt_n^2} = \mathcal{O}\left(v_n^{2\beta}\frac{dh_n}{dt_n}\right). \tag{38}$$

Combining the above inequalities we obtain

$$\frac{d^2 t_{n+1}}{dt_n^2} = \mathcal{O}(V^{2\beta}) \tag{39}$$

so that the last term is $\mathcal{O}(V^{-\beta})$.

To prove (c) let $r_n = \frac{d}{dt_n}\ln\rho$. Then

$$r_{n+1} = \frac{r_n}{L_n} + \left(\frac{1}{L_n^2}\right)\frac{d^2 t_{n+1}}{dt_n^2}. \tag{40}$$

Now part (c) follows from part (a) and (39).

To prove (d) we bound the measure of each component of $F\Gamma \bigcap \hat{\mathbf{C}}$. Denote

$$Z(t_{n+1}) = h_{n+1} + \frac{\alpha A \cos t_{n+1}}{v_{n+1}(B + A\sin t_{n+1})^{1-\alpha}}.$$

Now if $(v_{n+1}, t_{n+1}) \in \mathbf{C}$ then $|Z(t_{n+1})| < K v_n^{-\beta}$. Since h_{n+1} is close to $-A\sin t_{n+1}$ we infer that t_{n+1} is close to either 0 or π we conclude from (20) and part (b) that

$$\left|\frac{dZ}{dt_{n+1}}\right| \geq 1.5A.$$

Hence if $[t^-_{n+1}, t^+_{n+1}]$ is a component landing at $\mathbf{C}$ then by the Intermediate Value Theorem

$$\frac{2K}{V^\beta} \geq \left|Z(t^+_{n+1}) - Z(t^-_{n+1})\right| = \left|\frac{dZ}{dt_{n+1}}(t^\dagger_{n+1})\right| \left|t^+_{n+1} - t^-_{n+1}\right|$$

for some $t^\dagger_{n+1} \in [t^-_{n+1}, t^+_{n+1}]$. Hence

$$|t^+_{n+1} - t^-_{n+1}| \leq \frac{4K}{3AV^\beta}.$$

Let now $[\hat{t}^-_{n+1}, \hat{t}^+_{n+1}] \supset [t^-_{n+1}, t^+_{n+1}]$ be a component landing at $\hat{C}$. Then by the definition of $\hat{\mathbf{C}}$

$$\hat{t}^+_{n+1} - \hat{t}^-_{n+1} < t^+_{n+1} - t^-_{n+1} + \frac{\bar{C}_1}{V^\beta}$$

and so

$$\hat{t}^+_{n+1} - \hat{t}^-_{n+1} < \frac{\bar{C}_2}{V^\beta}$$

Combining this with part (c) we obtain

$$\mathbb{P}_\ell(Fx \in \hat{\mathbf{C}}) \leq \frac{\bar{C}_3}{V^\beta}.$$

Since $Z \subset \hat{\mathbf{C}}$ this proves (d).

It remains to prove the statement of part (b) claiming that $F\Gamma$ can be cut into pieces of length between δ and 2δ in such a way that the remainder lies in $\hat{\mathbf{C}}$. The proof of part (d) above shows that the components of $F\Gamma - \mathbf{C}$ have length uniformly bounded from below so we can them into pieces of required length. The condition that the cut points should visit $\partial \mathbf{C}$ on the next iteration poses no problem since such points are $1/V^\beta$ dense. The only issue is that happens near the endpoints of the components of $F\Gamma - \mathbf{C}$. Here we might need to chop off pieces which are $1/V^\beta$ close to the boundary. Since $F(\partial\Gamma) \in \mathbf{C}$ the removed pieces belong to $\hat{\mathbf{C}}$. The lemma follows. □

Remark. Observe that the argument used to prove part (d) of Lemma 5.1 in fact shows that for any $\xi > V^{-\beta}$ we have

$$\mathbb{P}_\ell(L_n \le \xi V^{2\beta}) \le \text{Const}\xi.$$

Accordingly (40) implies that for $\zeta > V^{-2\beta}$

$$\mathbb{P}_\ell\left(r_{n+1} \ge \frac{1}{\zeta V^{2\beta}}\right) \le \text{Const}\sqrt{\zeta}. \tag{41}$$

Proof of Lemma 5.2. The proof of part (a) is similar to the proof of parts (b) and (c) of Lemma 5.1 but there are two differences. First (38) now only gives that the RHS of (39) is $\mathcal{O}(V^{4\beta})$ and so the last term in (37) is $\mathcal{O}(V^\beta)$. However replacing n by $n+1$ (and $n+1$ by $n+2$) we can improve (39) to $\mathcal{O}(V^{3\beta})$ and so the last term in (37) can be made as small as we wish by an appropriate choice of K. Thus already $F^2\Gamma$ can be decomposed into standard curves. Second, to prove the statement about the densities observe that (40) implies $r_{n+1} = \mathcal{O}(V^{2\beta})$. Now developing

$$r_{n+3} = \frac{r_{n+1}}{L_{n+1}L_{n+2}} + \frac{1}{L_{n+2}L_{n+1}^2}\frac{d^2t_{n+2}}{dt_{n+1}^2} + \left(\frac{1}{L_{n+2}^2}\right)\frac{d^2t_{n+3}}{dt_{n+2}^2}$$

and using already established estimates

$$\frac{d^2t_{n+3}}{dt_{n+2}^2} = \mathcal{O}(V^{2\beta}) \text{ and } \frac{d^2t_{n+2}}{dt_{n+1}^2} = \mathcal{O}(V^{3\beta})$$

we prove (a). Observe that since we do not require $F(\partial\Gamma) \subset \hat{\mathbf{C}}$ for prestandard pairs we may need to remove a neighborhood of $\partial\Gamma$ at the first step explaining the difference between the statements of Lemmas 5.2(a) and 5.1(b).

To prove (b) consider the foliation of M by the integral curves of $\frac{dv}{dt} = h(v,t)$. Note that we can easily arrange that the boundaries of the prestandard pieces belong to $\hat{\mathbf{C}} \bigcup F^{-1}\hat{\mathbf{C}}$. Now part (c) follows from parts (a) and (b). □

Appendix B. Equidistribution.

Proof of Lemma 5.3. Divide $F\Gamma$ into segments I_s whose end points lie on the grid $t = 2\pi N$. Then

$$\int_\Gamma \mathcal{A}(F(\psi(t),t))\rho_{jk}(t)dt = \sum_s c_s \int_0^{2\pi} \mathcal{A}(\theta)\rho_s(\theta)d\theta. \tag{42}$$

where

$$\rho_s = \frac{(\rho \circ F^{-1}\frac{dt}{d\theta})}{c_s}$$

and $c_s = \mathbb{P}_\ell(Fx \in I_s)$. Call I_s good if

$$\left|\frac{d}{d\theta}(\ln \rho_s)\right| \le V^{-\beta}.$$

and bad otherwise. By (41)

$$\mathbb{P}_\ell(Fx \text{ belongs to a bad segment}) = \mathcal{O}(V^{-\beta/2}).$$

Therefore

$$\int_\Gamma \mathcal{A}(F(\psi(t),t))\rho_{jk}(t)dt = \sum_{s:I_s \text{ is good}} c_s \int_0^{2\pi} \mathcal{A}(\theta)\rho_s(\theta)d\theta + \mathcal{O}(V^{-\beta/2}).$$

But if I_s is good then $\rho_s(t) = \frac{1}{2\pi} + \mathcal{O}\left(V^{-\beta}\right)$ and so

$$\int_0^{2\pi} \mathcal{A}(\theta)\rho_s(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{A}(t)dt + \mathcal{O}(V^{-\beta}).$$

Since

$$\sum_{s:I_s \text{ is good}} c_s = \mathbb{P}_\ell(Fx \text{ belongs to a good segment}) = 1 + \mathcal{O}\left(V^{-\beta/2}\right)$$

part (a) follows.

The proof of part (b) is similar to the proof of part (a) since $\mathcal{B}$ can be approximated by a constant on each good segment with error $\mathcal{O}\left(\|\mathcal{B}\|_{C^1} V^{-\beta}\right)$. □

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