GLOBAL OBSERVABLES FOR RANDOM WALKS: LAW OF LARGE NUMBERS.

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1. Introduction

1.1. **Motivation.** Ergodic theory was created in the beginning of the last century motivated by the needs of homogenization (more specifically the quest to justify the kinetic equations of statistical mechanics). By now ergodic theory is a flourishing subject. Namely, ergodic theorems are established under very general conditions and ergodic properties of a large number of smooth systems are known (see e.g. [17]). Moreover, ergodicity turns out to be useful in the questions of averaging and homogenization (see e.g. [16, 26, 28, 32]). However, many dynamical systems appearing in applications preserve infinite invariant measure and ergodic theory of infinite measure preserving systems is much less developed. In fact, most of the work in infinite ergodic theory (see e.g. [1]) deals with local (L^1) observables while from physical point of view it is more natural to consider extensive observables ([19, 31]) which admit an infinite volume limit. One explanation for this is that while for local observables ergodic theorems can be obtained with minimal regularity assumptions on the observable, this is not the case for global observables as the present paper shows. The study of ergodic properties of infinite measure transformations with respect to extensive functions started relatively recently [21]. In particular, mixing properties of several systems with respect to global observables were obtained in [12, 21, 22]. A natural question is thus to investigate the law of large numbers for global observables. A first step in this direction was recently taken in [25]. In this paper we carry out a detailed analysis in the simplest possible setting: random walks on $\mathbb{Z}^d$. Our goal is to ascertain the correct spaces for the law of large numbers in various cases.

1.2. **Results.** Let $X_1, X_2, ...$ be an iid sequence of $\mathbb{Z}^d$ valued random variables. Let $S_0 = 0$ and $S_N = \sum_{n=1}^N X_n$ be the corresponding random walk. We assume that

(1) (non-degeneracy) the smallest group supporting the range of X_1 is $\mathbb{Z}^d$
(2) (aperiodicity) $g.c.d.\{n > 0 : \mathbb{P}(S_n = 0) > 0\} = 1$

We will also assume that S_n is in the normal domain of attraction of a stable law with some index α. That is, there is a non-degenerate d dimensional random variable Y such that

$$\frac{S_n}{n^{1/\alpha}} \Rightarrow Y \text{ if } \alpha \in (0,1) \text{ and } \frac{S_n - n\mathbb{E}(X_1)}{n^{1/\alpha}} \Rightarrow Y \text{ if } \alpha \in (1,2]$$

To avoid uninteresting minor technical difficulties, we will mostly assume that $\alpha \neq 1$.

We define several function spaces which proved to be useful in the previous studies of global observables [12]. Without further notice, we always assume that all functions are bounded.

Given a non-empty subset $V \subset \mathbb{Z}^d$ and $F \in L^\infty(\mathbb{Z}^d, \mathbb{R})$, we write

$$\bar{F}_V = \frac{1}{|V|} \sum_{v \in V} F(v).$$

Given $(a_1, b_1, \dots a_d, b_d)$ with $a_i \le b_i$, for $j = 1, \dots d$, let

$$V(a_1, b_1, \dots a_d, b_d) = \{x \in \mathbb{Z}^d : \ x_j \in [a_j, b_j] \text{ for } j = 1, \dots d\}.$$

Let $\mathbf{G}_+$ be the space of bounded functions on $\mathbb{Z}$ such that the limit $\bar{F}_+ = \lim_{v \to \infty} \bar{F}_{[0,v]}$ exists and $\mathbf{G}_-$ be the space of bounded functions on $\mathbb{Z}$ such that the limit $\bar{F}_- = \lim_{v \to \infty} \bar{F}_{[-v,0]}$ exists. Set $\mathbf{G}_\pm = \mathbf{G}_+ \cap \mathbf{G}_-$. Define

$$\mathbf{G}_0 = \{F \in L^\infty(\mathbb{Z}^d, \mathbb{R}) : \exists \bar{F} \ \ \forall a_1, b_1, \dots a_d, b_d \quad \lim_{L \to \infty} \bar{F}_{V(a_1 L, b_1 L, \dots, a_d L, b_d L)} = \bar{F}\}.$$

Note that in dimension 1, $\mathbf{G}_0 = \{F \in \mathbf{G}_\pm : \bar{F}_+ = \bar{F}_-\}$. Let $\mathbf{G}_U$ be the space of functions such that for each ε there is L such that for all cubes V with side larger than L we have

$$|\bar{F}_V - \bar{F}| \le \varepsilon. \tag{1.1}$$

Let $\mathbf{G}_\gamma$ be the set of functions where (1.1) only holds if the center of V is within distance L^γ of the origin. Thus $\mathbf{G}_U \subset \mathbf{G}_\gamma$. Also, $\mathbf{G}_\gamma \subset \mathbf{G}_0$ if $\gamma > 1$. Finally, let

$$\mathbf{G}_\gamma^\beta = \{F \in L^\infty(\mathbb{Z}^d, \mathbb{R}) : \exists \bar{F} \ \ \forall a_1, b_1, ..., a_d, b_d \ \ \exists C : \forall L, \forall z \in \mathbb{Z}^d, |z| < L^\gamma,$$
$$|\bar{F}_{z + V(a_1 L, b_1 L, \dots, a_d L, b_d L)} - \bar{F}| < C L^{d(\beta - 1)}\}.$$

Clearly, $\mathbf{G}_\gamma^\beta \subset \mathbf{G}_\gamma$ for any $\beta < 1$. Also, let $\mathbf{G}_\infty^\beta = \bigcap_{\gamma > 0} \mathbf{G}_\gamma^\beta$.

Our goal is to study Birkhoff sums

$$T_N = \sum_{n=1}^N F(S_n).$$

In particular we would like to know if $\frac{T_N}{N}$ converges to $\bar{F}$ for F in each of the spaces $\mathbf{G}_*$ introduced above.

Our results could be summarized as follows.

Theorem 1.1. *Suppose that $\mathbb{E}(X) = 0$ and that S_N is in the normal domain of attraction of a stable law of some index $\alpha > 1$. Then for all $F \in \mathbf{G}_0$, $\dfrac{T_N}{N} \Rightarrow \bar{F}$ in law as $N \to \infty$.*

Theorem 1.2. *Suppose that $d = 1$, $\mathbb{E}(X) = 0$ and $V(X) < \infty$. Then for all $F \in \mathbf{G}_\pm$, $\dfrac{T_N}{N}$ converges in law as $N \to \infty$. In particular, if $\bar{F}_- = 0$ and $\bar{F}_+ = 1$ then the limiting law has arcsine distribution: for $z \in [0, 1]$*

$$\lim_{N \to \infty} \mathbb{P}\left(\frac{T_N}{N} \le z\right) = \frac{2}{\pi} \arcsin \sqrt{z}.$$

Note that Theorem 1.2 is a simple homogenization result: it says that the limit distribution of $\frac{T_N}{N}$ remains the same if the oscillatory function F is replaced by a more regular function $\bar{F}_- 1_{x<0} + \bar{F}_+ 1_{x\geq 0}$ (see [24]). This confirms the usefulness of global observables in applications.

Theorem 1.3. *Suppose that S_N is in the normal domain of attraction of the stable law of some index α. Suppose that either*
(i) $1 < \alpha \leq 2$, $\mathbb{E}(X_1) \neq 0$ and $\gamma > 1$ or
(ii) $1 < \alpha \leq 2$, $\mathbb{E}(X_1) = 0$ and $\gamma > 1/\alpha$ or
(iii) $\alpha \leq 1$ and $\gamma > 1/\alpha$.
Then, for all $F \in \mathbf{G}_\gamma$, $\frac{T_N}{N} \to \bar{F}$ almost surely.

Theorem 1.4. *Suppose $E(X_1) = 0$ and $E(|X_1|^k) < \infty$ for all $k \in \mathbb{N}$. For $d \in \mathbb{N}$, let*

$$\rho_d(\beta) := \begin{cases} \frac{1}{2} & \text{if } \beta \leq \frac{d-1}{d} \\ \frac{d}{2}(\beta - 1) + 1 & \text{if } \beta > \frac{d-1}{d}, \end{cases} \qquad \gamma(d, \beta, \varepsilon) := \begin{cases} \frac{2}{\beta} & \text{if } d = 1 \\ \frac{1}{\varepsilon} & \text{if } d \geq 2. \end{cases}$$

Then for every $d \in \mathbb{N}$, for every $\beta \in [0, 1)$ every $\varepsilon > 0$ and any $F \in \mathbf{G}^\beta_{\gamma(d,\beta,\varepsilon)}$ with $\bar{F} = 0$, we have

$$\frac{T_N}{N^{\rho_d(\beta)+\varepsilon}} \to 0 \text{ almost surely as } N \to \infty.$$

Corollary 1.5. *If $d = 1$ and $F \in \mathbf{G}^\beta_{2/\beta}$ or $d \geq 2$ and $F \in \mathbf{G}^\beta_\infty$ then with probability 1, for all ε*

$$\lim_{N\to\infty} \frac{T_N}{N^{\rho_d(\beta)+\varepsilon}} = 0.$$

Remark 1.6. Let us discuss two special cases.

(A) (Random walk in random scenery) If $F(x)$, $x \in \mathbb{Z}^d$ are bounded and iid with expectation 0, then by moderate deviation estimates, for every $\gamma < \infty$ and for every $\varepsilon > 0$, $F \in \mathbf{G}_\gamma^{\frac{1}{2}+\varepsilon}$ holds with $\bar{F} = 0$ almost surely. Now assuming that the random walker has zero expectation and finite moments of every order, Theorem 1.4 implies $\frac{T_N}{N^{\frac{3}{4}+\varepsilon}} \to 0$ almost surely in dimension $d = 1$. Note that in this case, $\frac{T_N}{N^{\frac{3}{4}}}$ has a non-trivial weak limit by [18]. If $d \geq 2$, Theorem 1.4 gives $\frac{T_N}{N^{\frac{1}{2}+\varepsilon}} \to 0$ almost surely while $\frac{T_N}{N^{\frac{1}{2}}}$ ($\frac{T_N}{\sqrt{N \ln N}}$ if $d = 2$) has a non-trivial weak limit ([18]). We note that Theorem 1.4 is not new for F as above (see [20, 15]) however, we would like to emphasize that our space $\mathbf{G}_\gamma^{\frac{1}{2}+\varepsilon}$ includes many more functions than just realizations of iid process, so both the result and the proof of Theorem 1.4 are new even for $\mathbf{G}_\gamma^{\frac{1}{2}+\varepsilon}$.

(B) If $F(x)$ is periodic, $\bar{F} = 0$, then $F \in \mathbf{G}_\infty^{\frac{d-1}{d}}$. Thus, assuming that the random walker has zero expectation and finite moments of every order, Corollary 1.5 implies

$$\frac{T_N}{N^{\frac{1}{2}+\varepsilon}} \to 0 \tag{1.2}$$

almost surely for all d. Note that by the central limit theorem for finite Markov chains, $\frac{T_N}{\sqrt{N}}$ has a Gaussian weak limit.

In fact, our results also give (1.2) for quasi-periodic observables. That is, given $d \in \mathbb{N}$ and a C^∞ function $\mathfrak{F} : \mathbb{T}^d \to \mathbb{R}$, let $\hat{\mathfrak{F}} : \mathbb{R}^d \to \mathbb{R}$ be the $[0,1]^d$-periodic extension of $\mathfrak{F}$. Furthermore, given d vectors $\alpha_{(1)}, \dots \alpha_{(d)} \in \mathbb{R}^d$ and an initial phase $\omega \in [0,1]^d$, let

$$F(x) = \hat{\mathfrak{F}} \left(\omega + \sum_{j=1}^{d} x_j \alpha_{(j)} \right)$$

where $(x_1, \dots x_d)$ are coordinates of vector $x \in \mathbb{Z}^d$. We say that a vector $\alpha \in \mathbb{Z}^d$ is Diophantine, if there are constants K and σ such that for each $m \in \mathbb{Z}^d$,

$$\left| e^{2\pi \langle m, \alpha \rangle} - 1 \right| \geq \frac{K}{|m|^\sigma}.$$

If $\alpha_{(j)}$ is Diophantine for all $j = 1, ..., d$, then $F \in \mathbf{G}^0_\infty$ (see e.g. [17, §2.9]) so (1.2) holds.

Thus in both cases (A) and (B) our results give an optimal exponent for the growth rate of T_N.

Remark 1.7. Periodic (and quasi-periodic) observables are special case of stationary ergodic observables. More precisely, let $\mathfrak{T}_1, \dots \mathfrak{T}_d$ be commuting measurable maps of a space Ω preserving a probability measure ν. Given a bounded measurable function $\mathfrak{F}$ on Ω and an initial condition $\omega \in \Omega$, define

$$F_\omega(k) = \mathfrak{F}(\mathfrak{T}^k \omega), \tag{1.3}$$

where for $k = (k_1, \dots k_d) \in \mathbb{Z}^d$ we let $\mathfrak{T}^k = \mathfrak{T}_d^{k_d} \dots \mathfrak{T}_1^{k_1}$. If the family $\mathfrak{T}^k$ is ergodic, then the ergodic theorem tells us that for almost all ω, $F_\omega \in \mathbf{G}_0$ and $\bar{F} = \nu(\mathfrak{F})$. For the observables given by (1.3) the strong law of large numbers for almost every ω follows from ergodicity of the environment viewed by the particle process ([5]). Theorem 1.1 only gives a weak law of large numbers, (except in dimension 1 in the ballistic case, see Theorem 1.8 below). On the other hand our result gives valuable additional information even for stationary ergodic environments. Namely, the set of full measure where the weak law of large numbers holds contains all environments where ergodic averages of $\mathfrak{F}$ exist. We also note that Theorem 1.4 provides new and non-trivial information even in the stationary ergodic case.

Theorem 1.8. *Suppose that $d = 1$, $v = \mathbb{E}(X_1) > 0$ and for all $t \geq 1$, $\mathbb{E}(|X_1| > t) \leq C/t^\beta$ for some $C > 0$ and $\beta > 1$. If $F \in \mathbf{G}_+$, then $\frac{T_N}{N} \to \bar{F}_+$ almost surely.*

The next theorem shows that in general the strong law of large numbers fails in $\mathbf{G}_0$.

Theorem 1.9. *Suppose that S_N is in the normal domain of attraction of the stable law of some index α. Moreover assume that one of the following assumptions is satisfied*

(a) $\alpha > 1$ and $\mathbb{E}(X_1) = 0$; or

(b) $\alpha < 1$.

Then there exists $F \in \mathbf{G}_0$ such that, with probability 1, $\frac{T_N}{N}$ does not converge as $N \to \infty$.

Remark 1.10. The same conclusion holds in case (a) even if $\mathbb{E}(X_1) \neq 0$. However, in this case $\mathbf{G}_0$ is not an appropriate space to look at since we even do not have a weak law of large numbers in $\mathbf{G}_0$.

2. Weak convergence.

Here we prove Theorems 1.1 and 1.2.

2.1. **Preliminaries.** First, we recall two useful results.

Theorem 2.1. *(*[14, Section 50]*) Under the assumptions of Theorem 1.1, S_n satisfies the local limit theorem, i.e. there is a continuous probability density g such that*

$$\lim_{n\to\infty} \sup_{l\in\mathbb{Z}^d} |n^{d/\alpha}\mathbb{P}(S_n = l) - g(l/n^{1/\alpha})| = 0.$$

Theorem 2.2 (Local global mixing, [12])**.** *Under the assumptions of Theorem 1.1, S_n is local global mixing, i.e.*

$$\lim_{n\to\infty} \mathbb{E}(F(S_n)) = \bar{F}.$$

2.2. **Proof of Theorem 1.1.** Replacing F by $F - \bar{F}$, we can assume that $\bar{F} = 0$. By Theorem 2.2, we have $\lim_{N\to\infty} \dfrac{\mathbb{E}(T_N)}{N} = 0$. Thus in order to prove Theorem 1.1, it suffices to verify that $\lim_{N\to\infty} \dfrac{\mathbb{E}(T_N^2)}{N^2} = 0$. Let us fix some $\varepsilon > 0$ and prove that $\mathbb{E}(T_N^2) < \varepsilon N^2$ for all sufficiently large N. We have

$$\mathbb{E}(T_N^2) = 2 \sum_{0\le n_1<n_2\le N} \mathbb{E}(F(S_{n_1})F(S_{n_2})) + \sum_{n=1}^{N} \mathbb{E}(F^2(S_n))$$

Now writing $\varepsilon_1 = \frac{\varepsilon}{50\|F\|_\infty^2}$, we have

$$\mathbb{E}(T_N^2) \le \frac{\varepsilon}{10}N^2 + \left| 2 \sum_{\varepsilon_1 N<n_1<n_1+\varepsilon_1 N<n_2\le N} \mathbb{E}(F(S_{n_1})F(S_{n_2})) \right|$$

Choose a constant K such that $\mathbb{P}(|S_N| > KN^{1/\alpha}/2) < \frac{\varepsilon}{10\|F\|_\infty^2}$ for all sufficiently large N. Thus we have

$$\mathbb{E}(T_N^2) \le \frac{2\varepsilon}{10}N^2 + \left| 2 \sum_{\varepsilon_1 N<n_1<n_1+\varepsilon_1 N<n_2\le N} \mathbb{E}(1_{\{|S_{n_1}|,|S_{n_2}|<KN^{1/\alpha}\}}F(S_{n_1})F(S_{n_2})) \right|.$$

Observe that by the Markov property of the random walk, we have

$$\mathbb{E}(F(S_{n_1})F(S_{n_2})) = \mathbb{E}(F(S_{n_1})\mathbb{E}_{S_{n_1}}(F(\tilde{S}_{n_2-n_1}))).$$

where $\tilde{S}_n = \tilde{S}_0 + \sum_{k=1}^n \tilde{X}_k$, $\{\tilde{X}_k\}$ are iid and have the same distribution as X_1 and $\mathbb{P}_x$ is the measure defined by $\mathbb{P}_x(\tilde{S}_0 = x) = 1$. Let us write

$$e_1 = \mathbb{E}(1_{\{|S_{n_1}|,|S_{n_2}|<KN^{1/\alpha}\}}F(S_{n_1})F(S_{n_2})),$$

$$e_2 = \mathbb{E}(1_{\{|S_{n_1}|<KN^{1/\alpha}\}}F(S_{n_1})\mathbb{E}_{S_{n_1}}(1_{\{|\tilde{S}_{n_2-n_1}|<2KN^{1/\alpha}\}}F(\tilde{S}_{n_2-n_1}))),$$

$$\mathcal{A} = \{x, y : |x|, |y| < KN^{1/\alpha}\}, \quad \mathcal{B} = \{x, y : |x| < KN^{1/\alpha}, |y - x| < 2KN^{1/\alpha}\}.$$

Note that $\mathcal{A} \subset \mathcal{B}$ and set $\mathcal{C} = \mathcal{B} \setminus \mathcal{A}$. Then

$$e_2 - e_1 = \sum_{(x,y)\in\mathcal{C}} \mathbb{P}(S_{n_1} = x, S_{n_2} = y)F(x)F(y)$$

and we find

$$|e_2 - e_1| \le \|F\|_\infty^2 P(S_{n_1} > KN^{1/\alpha}) \le \frac{\varepsilon}{10}.$$

Consequently,

$$|e_1| \le |e_2| + \frac{\varepsilon}{10}.$$

Recall that g is the density function of the limiting distribution of $S_n/n^{1/\alpha}$ as in Theorem 2.1. Now we choose δ so that the oscillation of g on any cube of side length $\delta/\varepsilon_1^{1/\alpha}$ within distance $2K$ from the origin is less than $\eta := \frac{\varepsilon\varepsilon_1^{d/\alpha}}{20\cdot(2K)^d\|F\|_\infty^2}$. Also note that if $F \in \mathbf{G}_0$ and $\bar{F} = 0$, then

$$\lim_{N\to\infty} \sup_{-2K<a<2K} |\bar{F}_{[Na,N(a+\delta)]^d}| = 0. \tag{2.1}$$

Now given n_1, n_2 with $\varepsilon_1 N < n_1 < n_1 + \varepsilon_1 N < n_2 < N$ $x < KN^{1/\alpha}$, we write

$$\mathbb{E}_x(1_{\{|\tilde{S}_{n_2-n_1}|<2KN^{1/\alpha}\}} F(\tilde{S}_{n_2-n_1}))) = \sum_k \mathbb{E}_x(1_{\{\tilde{S}_{n_2-n_1}\in B_k\}} F(\tilde{S}_{n_2-n_1})), \tag{2.2}$$

where B_k's are cubes of side length $\delta N^{1/\alpha}$ partitioning $[-KN^{1/\alpha}, KN^{1/\alpha}]^d$. For brevity, we write $m = n_2 - n_1$. By Theorem 2.1 and the choice of δ, we have

$$\mathbb{E}_x(1_{\{\tilde{S}_m\in B_k\}} F(\tilde{S}_m)) = \sum_{y\in B_k} (p_k + e_{x,k,m,y}) m^{-d/\alpha} F(y), \tag{2.3}$$

where $p_k = g\left(\frac{z_k}{m^{1/\alpha}}\right)$, z_k is the center of B_k and $e_{x,k,m,y} < 2\eta$ for m sufficiently large (uniformly in x, k, y as above). Consequently,

$$\sum_k \sum_y e_{x,k,m,y} m^{-d/\alpha} \le m^{-d/\alpha} (2KN^{1/\alpha})^d 2\eta \le \frac{\varepsilon}{10\|F\|_\infty^2}$$

for sufficiently large m. Thus dropping $e_{x,k,m,y}$ from the right hand side of (2.3) gives a negligible error. The remaining term is $p_k m^{-d/\alpha} \sum_{y\in B_k} F(y)$, which when summed over k, is small by (2.1). Thus the absolute value of (2.2) is smaller than $\varepsilon/5$ for N sufficiently large, completing the proof of Theorem 1.1.

2.3. **Proof of Theorem 1.2.** We prove the second statement. The first one is a trivial corollary. Indeed, given $F \in \mathbf{G}_\pm$ with $\bar{F}_- = \bar{F}_+$, the convergence follows from Theorem 1.1. On the other hand if $\bar{F}_- \ne \bar{F}_+$, then we can consider $\tilde{F}(x) = \dfrac{F(x) - \bar{F}_-}{\bar{F}_+ - \bar{F}_-}$ and note that $\bar{\tilde{F}}_- = 0$, $\bar{\tilde{F}}_+ = 1$ and $\tilde{T}_N = \displaystyle\sum_{n=1}^N \tilde{F}(S_n) = \frac{T_N - N\bar{F}_-}{\bar{F}_+ - \bar{F}_-}$, whereby one derives the limit distribution of T_N.

Thus we assume $\bar{F}_- = 0$, $\bar{F}_+ = 1$. The proof is similar to that of Theorem 1.1 (with $\alpha = 2$, $d = 1$). The difference is that, as the limit distribution is now non-degenerate, we need to verify the convergence of all moments, not just the first two. Since the arcsine distribution is compactly supported, this implies the weak convergence.

We start with the following lemma.

Lemma 2.3. *Fix $F \in \mathbf{G}_\pm$ with $\bar{F}_- = 0$, $\bar{F}_+ = 1$. Then for any continuous and compactly supported function ϕ, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} F(x)\phi\left(\frac{x}{n}\right) = \int_0^\infty \phi(x)dx.$$

Furthermore, for any compact interval I, the convergence is uniform for ϕ lying in a compact subset of $\mathcal{C}(I,\mathbb{R})$.

Proof. The proof is a simple generalization of the computation at the end of the proof of Theorem 1.1. Let $L = \max\{|x| : \phi(x) \neq 0\}$. Given $\varepsilon > 0$, we choose δ so that the oscillation of ϕ on intervals of length δ is less than $\varepsilon\|F\|_\infty(L+1)$. Let K be the smallest positive integer so that $K > L/\delta$. Then

$$\sum_{x>0} F(x)\phi\left(\frac{x}{n}\right) = \sum_{k=0}^{K-1} \sum_{x=\lfloor kn/\delta\rfloor+1}^{\lfloor (k+1)n/\delta\rfloor} F(x)\phi\left(\frac{x}{n}\right)$$

Next, by the choice of δ,

$$\left|\sum_{k=0}^{K-1} \sum_{x=\lfloor kn/\delta\rfloor+1}^{\lfloor (k+1)n/\delta\rfloor} F(x)\left[\phi\left(\frac{x}{n}\right) - \phi\left(\frac{k}{\delta}\right)\right]\right| \leq \varepsilon n.$$

Since $\bar{F}_+ = 1$, we have

$$\sup_{0=1,\ldots,K-1} \left|\frac{1}{n}\phi\left(\frac{k}{\delta}\right) \sum_{x=\lfloor kn/\delta\rfloor+1}^{\lfloor (k+1)n/\delta\rfloor} F(x) - \frac{1}{\delta}\phi\left(\frac{k}{\delta}\right)\right| < \varepsilon$$

for n sufficiently large. Thus replacing a Riemann sum by an integral (further reducing δ if necessary), we find

$$\left|\frac{1}{n}\sum_{x>0} F(x)\phi\left(\frac{x}{n}\right) - \int_0^\infty \phi(x)dx\right| < 3\varepsilon.$$

Note that all the above estimates are uniform over compact subsets of $\mathcal{C}(I,\mathbb{R})$. The computation for $x < 0$ is similar but easier as $\bar{F}_- = 0$. □

Now we prove the convergence of the moments by induction. The inductive hypothesis is the following:

($\mathbf{H_k}$) for every $\varepsilon > 0$ and $K < \infty$ there exists some N_0 so that for all $N > N_0$ and for all $x \in [-K\sqrt{N}, K\sqrt{N}]$,

$$|N^{-k}\mathbb{E}_x(T_N^k) - E(\tau^k_{x/\sqrt{N}})| < \varepsilon.$$

Here, τ_a is the total time spent on the positive half-line by a Brownian motion $(B_t)_{t\in[0,1]}$ with $B_0 \equiv a$ and such that $V(B_1) = V(X_1)$. Furthermore, as in Theorem 1.1, $E_x(.)$ means that the random walk starts from $x \in \mathbb{Z}$ (more precisely, in the definition of T_N, one replaces S with $\tilde{S}$, where $\tilde{S}_0 \equiv x$ and $\tilde{S}_n - \tilde{S}_0$ has the same distribution as S_n).

Clearly, $(\mathbf{H_0})$ holds. Now assume that $(\mathbf{H_k})$ holds for some k. To prove $(\mathbf{H_{k+1}})$, first observe that

$$\begin{aligned}\frac{\mathbb{E}_x(T_N^{k+1})}{N^{k+1}} &= \frac{1}{N}\mathbb{E}_x\left(\sum_{n=1}^N F(\tilde S_n)\frac{1}{N^k}\sum_{n_1,\dots n_k=1}^N F(\tilde S_{n_1})\dots F(\tilde S_{n_k})\right)\\ &\approx \frac{k+1}{N}\mathbb{E}_x\left(\sum_{n=1}^N F(\tilde S_n)\frac{1}{N^k}\sum_{n_1,\dots n_k=n}^N F(\tilde S_{n_1})\dots F(\tilde S_{n_k})\right)\\ &\approx \frac{k+1}{N}\sum_{n=\delta N}^{(1-\delta)N}\sum_{y=-\lfloor K_1\sqrt{n}\rfloor}^{\lfloor K_1\sqrt{n}\rfloor}\mathbb{P}_x(\tilde S_n=y)F(y)N^{-k}\mathbb{E}_y(T^k_{N-n})\\ &= \frac{k+1}{N}\sum_{n=\delta N}^{(1-\delta)N}\left(1-\frac{n}{N}\right)^k\sum_{y=-\lfloor K_1\sqrt{n}\rfloor}^{\lfloor K_1\sqrt{n}\rfloor}\mathbb{P}_x(\tilde S_n=y)F(y)(N-n)^{-k}\mathbb{E}_y(T^k_{N-n}).\end{aligned}$$

Here, $a_{x,\delta,K_1}(N)\approx b_{x,\delta,K_1}(N)$ means that for every $\varepsilon>0$, $K<\infty$ there exists some δ, K_1, N_0 so that for all $N>N_0$, all $x\in[-K\sqrt{N},K\sqrt{N}]$ we have $|a_{x,\delta,K_1}(N)-b_{x,\delta,K_1}(N)|<\varepsilon$ and the factor $k+1$ appears in the second line since we imposed an additional restriction that n is the smallest among $n, n_1,\dots n_k$.

Now using the inductive hypothesis and the local limit theorem, we find that

$$\frac{\mathbb{E}_x(T_N^{k+1})}{N^{k+1}}\approx (k+1)\frac{1}{N}\sum_{n=\delta N}^{(1-\delta)N}\left(1-\frac{n}{N}\right)^k\sum_{y\in\mathbb{Z}}\frac{1}{\sqrt{n}}F(y)\phi_{x,N,n}\left(\frac{y}{\sqrt{n}}\right),$$

where

$$\phi_{x,N,n}(\mathfrak{y})=\mathfrak{g}\left(\mathfrak{y}-\frac{x}{\sqrt{n}}\right)E\left(\tau^k_{\mathfrak{y}\sqrt{\frac{n}{N-n}}}\right)1_{|\mathfrak{y}|<K_1}$$

and $\mathfrak{g}$ is the centered Gaussian density with variance $V(X)$. Now Lemma 2.3 implies that

$$\frac{\mathbb{E}_x(T_N^{k+1})}{N^{k+1}}\approx (k+1)\frac{1}{N}\sum_{n=\delta N}^{(1-\delta)N}\left(1-\frac{n}{N}\right)^k\int_{y=0}^{K_1}\phi_{x,N,n}(y)dy. \tag{2.4}$$

It remains to check that

$$E(\tau^k_{x/\sqrt{N}})\approx (k+1)\frac{1}{N}\sum_{n=\delta N}^{(1-\delta)N}\left(1-\frac{n}{N}\right)^k\int_{y=0}^{K_1}\phi_{x,N,n}(y)dy. \tag{2.5}$$

To prove (2.5), we observe that in particular (2.4) holds for the heaviside function $F(x)=H(x)=1_{x>0}$. By classical theory [13, §XII.8], T_N/N converges weakly to the arcsine law for $F(x)=H(x)$. Since T_N/N is bounded, the moments also converge and thus the left hand sides of (2.4) and (2.5) are asymptotically equivalent for $F(x)=H(x)$. Since the right hand side does not depend on the specific choice of F, (2.5) follows. This completes the proof of Theorem 1.2.

3. SLLN IN $\mathbf{G}_\gamma$.

Here we prove Theorem 1.3.

Fix $F \in \mathbf{G}_\gamma$. As before, we can assume w.l.o.g. that $\bar{F} = 0$. Since $F \in \mathbf{G}_\gamma$, the proof of Theorem 2.3 of [12] shows that, for any $\eta \in (0,1)$,

$$\lim_{k\to\infty} \sup_{|x|\le k^{\eta\gamma}} \left| \frac{\mathbb{E}_x(T_k)}{k} \right| = 0, \tag{3.1}$$

where $\mathbb{E}_x$ denotes the expectation in the case where $S_0 = x$.

Lemma 3.1. *Suppose that $1 < \gamma_1$ in case (i) or $1/\alpha < \gamma_1$ in cases (ii) and (iii). Then with probability one we have that, for large N,*

$$\max_{0\le k\le N} |S_k| \le N^{\gamma_1}.$$

Proof. In case (i) the statement follows from the Law of Large Numbers, so we only need to consider cases (ii) and (iii). We have for any $\varepsilon > 0$ that $|S_N| > N^{1/\alpha+\varepsilon}$ holds only finitely many times almost surely by [27] in case (ii) and by [23] in case (iii). The lemma follows. □

Choose $\gamma_1 < \gamma$ as in Lemma 3.1 and $\eta < 1$ such that $\gamma_1 < \gamma\eta$. For $j = 0, 1, \dots, \lfloor N^{1-\eta} \rfloor$, set $\tilde{T}_j := \tilde{T}_{N,j} := N^{-\eta}\, T_{\lfloor jN^\eta \rfloor}$ (with the convention $T_0 \equiv 0$) and denote by $\tilde{\mathcal{F}}_j := \tilde{\mathcal{F}}_{N,j}$ the σ-algebra generated by $\{S_k\}_{k=0}^{\lfloor jN^\eta \rfloor}$. Denote $\mathcal{A}_{j,N} = \{|S_{\lfloor jN^\eta \rfloor}| \le N^{\gamma_1}\}$. Fix $\varepsilon > 0$. We claim that there exists $N_0 = N_0(\varepsilon)$ such that, for all $N \ge N_0$ and $j < \lfloor N^{1-\eta} \rfloor$,

$$\left| \mathbb{E}\Big(1_{\mathcal{A}_{j,N}} \Big(\tilde{T}_{j+1} - \tilde{T}_j \Big) \Big| \tilde{\mathcal{F}}_j \Big) \right| \le \varepsilon. \tag{3.2}$$

Indeed if $\mathcal{A}_{j,N}$ occurs then (3.2) holds due to (3.1), otherwise it holds since the LHS is zero.

Setting

$$Y_j := 1_{\mathcal{A}_{j,N}} \Big(\tilde{T}_{j+1} - \tilde{T}_j \Big) - D_j \tag{3.3}$$

where

$$D_j := \mathbb{E}\Big(1_{\mathcal{A}_{j,N}} \Big(\tilde{T}_{j+1} - \tilde{T}_j \Big) \Big| \tilde{\mathcal{F}}_j \Big) \tag{3.4}$$

defines a martingale difference, w.r.t. $\{\tilde{\mathcal{F}}_j\}$, with $|Y_j| \le \|F\|_\infty + \varepsilon$. Applying Azuma's inequality we get that, for all $\delta > 0$,

$$\mathbb{P}\left(\left| \sum_{j=0}^{N^{1-\eta}-1} Y_j \right| \ge \delta N^{1-\eta} \right) \le 2\exp\left(-\frac{\delta^2 N^{1-\eta}}{2(\|F\|_\infty + \varepsilon)^2} \right).$$

Therefore, by Borel-Cantelli,

$$\limsup_{N\to\infty} \frac{1}{N^{1-\eta}} \left| \sum_{j=0}^{N^{1-\eta}-1} Y_j \right| \le \delta \quad \text{a.s.}$$

Since δ is arbitrary, with probability one we have

$$\lim_{N\to\infty} \frac{1}{N^{1-\eta}} \sum_{j=0}^{N^{1-\eta}-1} Y_j = 0. \tag{3.5}$$

On the other hand, definitions (3.3)-(3.4) and Lemma 3.1 show that, with probability one, for all large N depending on the realization of the walk,

$$T_N = N^{\eta} \left(\sum_{j=0}^{N^{1-\eta}-1} Y_j + \sum_{j=0}^{N^{1-\eta}-1} D_j \right). \tag{3.6}$$

In view of (3.2), (3.5), and (3.6) we have:

$$\begin{aligned} \limsup_{N\to\infty} \left| \frac{T_N}{N} \right| &= \limsup_{N\to\infty} \frac{1}{N^{1-\eta}} \left| \sum_{j=0}^{N^{1-\eta}-1} (Y_j + D_j) \right| \\ &= \limsup_{N\to\infty} \frac{1}{N^{1-\eta}} \left| \sum_{j=0}^{N^{1-\eta}-1} D_j \right| \le \varepsilon \quad \text{a.s.} \end{aligned}$$

Since ε is arbitrary, $\lim_{N\to\infty} \frac{T_N}{N} = 0$ almost surely.

4. Speed of convergence in $\mathbf{G}_\gamma^\beta$.

Here we prove Theorem 1.4.

Note that $\mathbf{G}_\gamma^{\beta_1} \subset \mathbf{G}_\gamma^{\beta_2}$ whenever $\beta_1 < \beta_2$. Since $\rho_d(\beta)$ is constant for $\beta \in [0, (d-1)/d]$ and is continuous at $(d-1)/d$, it is sufficient to prove the theorem for

$$\beta > \frac{d-1}{d}. \tag{4.1}$$

Let $\mathbb{P}_x(.) = \mathbb{P}(.|S_0 = x)$, $\mathbb{E}_x(.) = \mathbb{E}(.|S_0 = x)$.

We start with the following

Proposition 4.1. *Under the conditions of Theorem 1.4, for every $d \in \mathbb{N}$, every $\beta \in ((d-1)/d, 1)$ and every $\varepsilon > 0$ there exists some $\delta > 0$ so that*

$$\sup_{x_0 : |x_0| \le N^{1/2+\delta}} \mathbb{E}_{x_0}(T_N^2) < C N^{2\rho_d(\beta) + 2\varepsilon}. \tag{4.2}$$

Note that Proposition 4.1 combined with Chebyshev's inequality implies that

$$\frac{T_N}{N^{\rho_d(\beta)+\varepsilon}} \Rightarrow 0 \text{ in law as } N \to \infty.$$

Section 4 is divided into three parts. In §4.1 we derive Theorem 1.4 from Proposition 4.1. In §4.2 we prove Proposition 4.1 for $d = 1$. In §4.3, we extend the proof of Proposition 4.1 to arbitrary dimension d.

4.1. **Proof of Theorem 1.4.** Here, we derive the theorem from Proposition 4.1. For simplicity we write $\rho = \rho_d(\beta)$.

We will show that

$$\mathbb{P}(\exists n \leq N : |T_n| > 2N^{\rho+\varepsilon/2}) \leq CN^{-\varepsilon/2} \tag{4.3}$$

If (4.3) holds, then writing $N_k = 2^k$, we find

$$\mathbb{P}(\exists n = N_{k-1}, ...N_k : |T_n| > 2N_k^{\rho+\varepsilon/2}) \leq CN_k^{-\varepsilon/2}.$$

and the theorem follows from Borel Cantelli lemma. To prove (4.3), let us write

$$\tau_N = \min\{\min\{n : |T_n| > 2N^{\rho+\varepsilon/2}\}, N\}.$$

Then

$$\mathbb{P}(\exists n \leq N : |T_n| > 2N^{\rho+\varepsilon/2})$$
$$\leq \mathbb{P}(|T_N| > N^{\rho+\varepsilon/2}) + \mathbb{P}(|T_{\tau_N}| > 2N^{\rho+\varepsilon/2}, |T_N| \leq N^{\rho+\varepsilon/2}) =: p_1 + p_2$$

By Proposition 4.1 for $x_0 = 0$ and by Chebyshev's inequality, we have $p_1 \leq CN^{-\varepsilon/2}$. To bound p_2, we distinguish two cases: $S_{\tau_N} > N^{1/2+\delta}$ and $S_{\tau_N} \leq N^{1/2+\delta}$. The first case has negligible probablity by moderate deviation bound for random walks (see formula (4.5) below). In the second case we compute

$$\mathbb{P}(|T_{\tau_N}| > 2N^{\rho+\varepsilon/2}, |T_N| \leq N^{\rho+\varepsilon/2}, |S_{\tau_N}| \leq N^{1/2+\delta})$$
$$\leq \sup_{x_0 : |x_0| \leq N^{1/2+\delta}} \max_{n=1,\ldots,N} \mathbb{P}_{x_0}(|T_n| \geq N^{\rho+\varepsilon/2})$$

which is again bounded by $CN^{-\varepsilon/2}$ by Proposition 4.1 and Chebyshev's inequality. We have verified (4.3) and finished the proof of the theorem.

4.2. **Proof of Proposition 4.1 for** $d = 1$**.** We have $\beta \in (0, 1)$ and $2\rho_d(\beta) = \beta + 1$. We start by recalling some results on expansions in the LLT in case all moments are finite (a.k.a. Edgeworth expansion).

Theorem 4.2. *(*[14, Section 51]*) Under the assumptions of Theorem 1.4, there are polynomials $Q_1, Q_2, ...$ so that for any $M \in \mathbb{N}$*

$$\mathbb{P}(S_n = l) = \frac{1}{\sqrt{n}}\mathfrak{g}(l/\sqrt{n})(1 + \sum_{m=1}^{M} Q_m(l/\sqrt{n})n^{-m/2}) + e_{n,l,M} \tag{4.4}$$

where $\mathfrak{g}$ is a Gaussian density and

$$\limsup_{n\to\infty} \sup_{l\in\mathbb{Z}} e_{n,l,M} n^{M/2+1} < \infty.$$

Note that (4.4) implies the following estimate: for any $\eta > 0$ there exists $C < \infty$ such that

$$\mathbb{P}(|S_n| > n^{1/2+\eta}) < Cn^{-1/\eta}. \tag{4.5}$$

Given a function $h = h(n, x) : \mathbb{Z}_+ \times \mathbb{Z} \to \mathbb{R}$, we write ∇h for the discrete derivative in the second coordinate, i.e.

$$\nabla h(n, x) = h(n, x) - h(n, x - 1).$$

Note that

$$\nabla(gh)(n,x) = (\nabla g)(n,x)h(n,x) + g(n,x-1)(\nabla h)(n,x). \tag{4.6}$$

We will also write $\nabla^k h$ for the kth discrete derivative.

Denote

$$H(n,x) = \mathbb{P}(S_n = x). \tag{4.7}$$

With this notation, (4.5) can be rewritten as

$$\sum_{x:|x|\geq n^{1/2+\eta}} H(n,x) < C_\eta n^{-1/\eta} \text{ for any } \eta > 0. \tag{4.8}$$

Also (4.4) implies that there is a constant c so that, for every $k = 0,1,2$,

$$\sup_{x\in\mathbb{Z}} |\nabla^k H(n,x)| \leq c n^{-\frac{k+1}{2}}. \tag{4.9}$$

Observe that

$$\mathbb{E}_{x_0}(T_N^2) = \sum_{0\leq n_1\leq n_2\leq N} c_{n_1,n_2} E_{n_1,n_2}(x_0) \tag{4.10}$$

where $c_{n_1,n_2} = 1$ if $n_1 = n_2$ and $c_{n_1,n_2} = 2$ otherwise and

$$E_{n_1,n_2}(x_0) = \mathbb{E}_{x_0}(F(S_{n_1})F(S_{n_2}))$$
$$= \sum_{x_1,x_2\in\mathbb{Z}} \mathbb{P}(S_{n_2-n_1} = x_2 - x_1)\mathbb{P}_{x_0}(S_{n_1} = x_1)F(x_1)F(x_2)$$

We will show the following: for any $0 \leq n_1 \leq n_2 \leq N$ such that

$$n_1 > N^\alpha, \quad n_2 - n_1 > N^\alpha \text{ where } \alpha = 1/\gamma = \beta/2 \tag{4.11}$$

we have

$$|E_{n_1,n_2}(x_0)| \leq \tag{4.12}$$
$$C n_1^{\frac{\beta-1}{2}+\varepsilon}(n_2-n_1)^{\frac{\beta-1}{2}+\varepsilon} + C n_1^{\frac{\beta}{2}+\varepsilon}(n_2-n_1)^{\frac{\beta-2}{2}+\varepsilon}.$$

Summing the estimate (4.12) for n_1, n_2 satisfying (4.11) we obtain $N^{\beta+1+2\varepsilon}$ as needed. To complete the proof of the proposition, it remains to

(I) prove (4.12)

(II) verify that the contribution of (n_1,n_2)'s that do not satisfy (4.11) is also negligible;

We start with (I).

We will use the following lemma:

Lemma 4.3. *There is a constant $\hat{C}$ such that for any positive integer n and any constants A, B, the following holds. If $g(x) : \mathbb{Z} \to \mathbb{R}$ satisfies*

(H1) $\sup_{x:|x|\leq n^{1/2+\delta}} |g(x)| \leq A$

(H2) $\sup_{x:|x|\leq n^{1/2+\delta}} |\nabla g(x)| \leq B$

for some sufficiently small δ, then, for $i=0,1$,

$$\sup_{y:|y|\le n^{\frac{1/2+\delta}{\alpha}}}\left|\sum_{z\in\mathbb{Z}}\nabla^i H(n,z-y)g(z)F(z)\right|\le \hat{C}(\|g\|_\infty n^{-10}+An^{\frac{\beta-i-1}{2}+\varepsilon}+Bn^{\frac{\beta-i}{2}+\varepsilon}). \tag{4.13}$$

Proof. For the rest of the section C will denote a constant (independent of A and B) whose value may change from line to line. By (4.8) and since F is bounded, the sum for z's with $|z-y|>n^{1/2+\delta}$ is bounded by $C\|g\|_\infty n^{-10}$. Denote $I(x)=\sum_{w=y-2n^{1/2+\delta}}^{x}F(w)$. Using summation by parts and (H1), we find

$$\begin{aligned}\sum_{z:\,|z-y|\le n^{1/2+\delta}}\nabla^i H(n,z-y)g(z)F(z)&=O\left(An^{-10}\right)+\\ -\sum_{z:\,|z-y|\le n^{1/2+\delta}}&I(z-1)\nabla_z\left(\nabla^i H(n,z-y)g(z)\right).\end{aligned} \tag{4.14}$$

Using (4.6), (4.9), (H1) and (H2) we find that

$$\nabla_z\left(\nabla^i H(n,z-y)g(z)\right)\le C(An^{-\frac{i+2}{2}}+Bn^{-\frac{i+1}{2}}). \tag{4.15}$$

Next, using $F\in\mathbf{G}^\beta_\gamma$, with $\bar{F}=0$, we find

$$|I(z)|\le Cn^{\frac{\beta}{2}+\varepsilon} \tag{4.16}$$

(assuming that $\delta=\delta(\varepsilon)$ is small enough). The last two estimates imply that the sum in (4.14) is bounded by $C(An^{\frac{\beta-1-i}{2}+\varepsilon}+Bn^{\frac{\beta-i}{2}+\varepsilon})$. □

Now we are ready to estimate $E_{n_1,n_2}(x_0)$. First, let

$$g_1(x_1):=g_{1,n_2-n_1}(x_1):=\sum_{x_2\in\mathbb{Z}}H(n_2-n_1,x_2-x_1)F(x_2). \tag{4.17}$$

By definition, $\|g_1\|_\infty\le\|F\|_\infty$. Applying Lemma 4.3 with $i=0$, $g=1$, $n=n_2-n_1$, $A=1$, $B=0$ and using $n_2-n_1>N^\alpha$, we find

$$\sup_{x_1:|x_1|\le N^{1/2+\delta}}|g_1(x_1)|\le C(n_2-n_1)^{\frac{\beta-1}{2}+\varepsilon}. \tag{4.18}$$

Using Lemma 4.3 the same way but now with $i=1$, we find

$$\sup_{x_1:|x_1|\le N^{1/2+\delta}}|\nabla g_1(x_1)|\le C(n_2-n_1)^{\frac{\beta-2}{2}+\varepsilon} \tag{4.19}$$

Next, set

$$g_2(x_0):=g_{2,n_1}(x_0):=\sum_{x_1\in\mathbb{Z}}H(n_1-n_0,x_1-x_0)g_1(x_1)F(x_1) \tag{4.20}$$

Now we use Lemma 4.3 with $i=0$, $n=n_1$, $g=g_1$, $A=(n_2-n_1)^{\frac{\beta-1}{2}+\varepsilon}$, $B=(n_2-n_1)^{\frac{\beta-2}{2}+\varepsilon}$. Since $n\le N$, (4.18) and (4.19) give (H1) and (H2). Also using that

$\|g_1\|_\infty \le \|F\|_\infty^2$ and $n_1 - n_2 > N^\alpha$, we get

$$\sup_{x_0 : |x_0| \le N^{1/2+\delta}} |g_2(x_0)| \le \tag{4.21}$$

$$Cn_1^{\frac{\beta-1}{2}+\varepsilon}(n_2-n_1)^{\frac{\beta-1}{2}+\varepsilon} + Cn_1^{\frac{\beta}{2}+\varepsilon}(n_2-n_1)^{\frac{\beta-2}{2}+\varepsilon}$$

which gives (4.12).

It remains to verify (II), that is that the contribution of pairs (n_1, n_2)'s that do not satisfy (4.11) is negligible.

First, assume that $n_1 > N^\alpha$ and $n_2 - n_1 \le N^\alpha$. Then we derive as in (4.21) but using the trivial bounds $A = 1 + \|F\|_\infty$, $B = 2(1 + \|F\|_\infty)$ that

$$\sup_{x_0 : |x_0| \le N^{1/2+\delta}} |g_2(x_0)| \le Cn_1^{\frac{\beta-1}{2}+\varepsilon} + Cn_1^{\frac{\beta}{2}+\varepsilon}.$$

Summing this estimate for $n_1 = N^\alpha, ..., N$ and multiplying by N^α for the number of choices of n_2, we obtain

$$O(N^\alpha N^{\frac{\beta+1}{2}}) = O(N^{\beta+\frac{1}{2}}) = o(N^{\beta+1}). \tag{4.22}$$

Next, assume that $n_1 < N^\alpha$, $n_2 - n_1 < N^\alpha$. Using the bound $|E_{n_1,n_2}(x_0)| \le \|F\|_\infty^2$ we obtain

$$\sum_{n_1=0}^{N^\alpha} \sum_{n_2=n_1}^{n_1+N^\alpha} |E_{n_1,n_2}(x_0)| \le CN^{2\alpha} = CN^\beta = o(N^{\beta+1}).$$

Finally, assume that $n_1 \le N^\alpha$ and $n_2 - n_1 > N^\alpha$. By (4.8), we can assume that $S_{n_1} - S_{n_0} \le N^{1/2+\delta}$. Then (4.18) still holds and we conclude that

$$|E_{n_1,n_2}(x_0)| \le C(n_2-n_1)^{\frac{\beta-1}{2}+\varepsilon}. \tag{4.23}$$

Summing for $n_2 = n_1 + N^\alpha, ..., N$ and multiplying by N^α, we obtain the same error term as in (4.22).

This completes the proof.

4.3. **Proof of Proposition 4.1 for $d \ge 2$.**

4.3.1. *Preliminary estimates.* In dimension d, we have

$$\sup_{x \in \mathbb{Z}^2} |\nabla_{i_1} ... \nabla_{i_k} H(n, x)| \le cn^{-\frac{k+d}{2}}. \tag{4.24}$$

for any $i_1, ... i_k = 1, ..., d$, where ∇_i denotes the discrete derivative with respect to x_i, the i-th component of x. We apply a similar approach as in $d = 1$. That is, we perform summations by parts to estimate g_1 and g_2 defined by (4.17) and (4.20). Each time we need d summations by parts. For example, if $d = 2$, then

$$\begin{aligned} g_{1,m}(0) &= \sum_{|x|<m^{1/2+\delta}} \sum_{|y|<m^{1/2+\delta}} H(m,(x,y))F(x,y) \\ &\approx \sum_{|x|<m^{1/2+\delta}} \sum_{|y|<m^{1/2+\delta}} \nabla_2 H(m,(x,y))I_1(x,y) \\ &\approx \sum_{|x|<n^{1/2+\delta}} \sum_{|y|<m^{1/2+\delta}} \nabla_1\nabla_2 H(m,(x,y))I(x,y) \end{aligned}$$

where $I_1(x,y) = \sum_{z=0}^{y} F(x,z)$ and $I(x,y) = \sum_{w=0}^{x}\sum_{z=0}^{y} F(w,z)$ and $a_m \approx b_m$ means that the $a_m - b_m$ is superpolynomially small in m. Using that $|I(x,y)| \leq Cm^{\beta(\frac{1}{2}+\delta)}$, we find

$$|g_{1,m}(0)| \leq Cm^{\beta-1+\varepsilon}$$

(with $m = n_2 - n_1$).

To simplify formulas, we will use the notation

$$a_N \lesssim b_N \text{ if } a_N \leq Cb_N N^{\varepsilon}.$$

Lemma 4.4. *For any $a \in (0,1]$, if $F \in \mathbf{G}^{\beta}_{1/a}$, then for all n_1, n_2 satisfying $n_1 \geq N^a$, $n_2 - n_1 \geq N^a$ we have*

$$\sup_{x_0\in\mathbb{Z}^d:|x_0|\leq N^{1/2+\delta}} |E_{n_1,n_2}(x_0)| \lesssim \sum_{j=0}^{d} n_1^{\frac{d\beta-j}{2}}(n_2-n_1)^{\frac{d\beta-2d+j}{2}}. \tag{4.25}$$

Proof. Since the proof of the lemma is similar to that of (4.12), we only mention the main difference. That is, now j can take values $0,1,...,d$ and in dimension $d = 1$ it could only take values $0,1$. This follows from the fact that when applying d summations by parts to the function $H(n_1 - n_0, x_1 - x_0)g_1(x_1)$, we obtain

$$\nabla_1...\nabla_d(Hg) = \sum_{\{i_1,...,i_I\}\subset\{1,...,d\}} (\nabla_{i_1}...\nabla_{i_I}H)(\nabla_{j_1}...\nabla_{j_J}g)$$

where $\{j_1,...,j_J\} = \{1,...,d\} \setminus \{i_1,...,i_I\}$.

In the proof of (4.12), we only used the definition of $\mathbf{G}^{\beta}_{\gamma}$ for boxes with side length $L \geq N^{1/2+\delta}$ (specifically in deriving (4.16)). In order to extend that proof to the present setting, we only need to replace $F(.)$ by $F(.-x_0)$. Thus assuming $\gamma > 1/a$ and using the definition of $\mathbf{G}^{\beta}_{\gamma}$, we can repeat the previous proof. □

Next we show that the extreme terms in the right hand side of (4.25) provide the main contribution.

Set $m_1 = n_1$, $m_2 = n_2 - n_1$. By Lemma 4.4, we have for $m_1 \geq N^a$, $m_2 \geq N^a$, $|x_0| \leq N^{1/2+\delta}$ that

$$|E_{n_1,n_2}(x_0)| \lesssim \begin{cases} m_1^{\frac{(\beta-1)d}{2}} m_2^{\frac{(\beta-1)d}{2}} & \text{if } m_2 \geq m_1 \\ m_1^{\frac{\beta d}{2}} m_2^{\frac{d\beta-2d}{2}} & \text{if } m_2 < m_1. \end{cases} \tag{4.26}$$

Note that the second bound is quite bad if $m_2 \ll m_1$. However we can improve it by bootstrap. Namely we have

Lemma 4.5. *If* $N^a < m_2 < m_1$ *then*

$$|E_{n_1,n_2}(x_0)| \lesssim m_2^{(\beta-1)d}$$

Proof. If $m_1 \le 2m_2$ then the result follows from (4.26). If $m_1 > 2m_2$, let $k = m_1 - m_2$ and note that $k > m_2$ and

$$E_{n_1,n_2}(x_0) = \sum_{y\in\mathbb{Z}^d} H(k, y - x_0) E_{m_2,2m_2}(y).$$

The sum of the terms where $|y| > 2N^{1/2+\delta}$ decays faster than N^{-r} for any r. The terms where $|y| \le 2N^{1/2+\delta}$ can be estimated by (4.26) with $m_1 = m_2$ giving the result. □

We now combine the foregoing results in different regimes in case where $a = \varepsilon$. Then if $\gamma = 1/\varepsilon$, $F \in \mathbf{G}_\gamma^\beta$ we gather that

$$|E_{n_1,n_2}(x_0)| \lesssim \begin{cases} m_1^{\frac{(\beta-1)d}{2}} m_2^{\frac{(\beta-1)d}{2}} & \text{if } m_2 \ge m_1 \ge N^\varepsilon, \\ m_2^{(\beta-1)d} & \text{if } m_1 > m_2 \ge N^\varepsilon, \\ 1 & \text{if } \min(m_1, m_2) < N^\varepsilon. \end{cases} \tag{4.27}$$

Summing the bounds of (4.27) for $m_1, m_2 \in \{1 \dots N\}$ we obtain Proposition 4.1.

5. SLLN in dimension 1.

5.1. **Reduction to occupation times sum.** Here we prove Theorem 1.8.

Let $\ell_n(x)$ be time spent by the walker at site x before time n. Set $\ell_\infty(x) := \lim_{n\to\infty} \ell_n(x)$. Thus $\ell_\infty(x)$ is the total time spent by the walker at site x.

Lemma 5.1. *There exist* $C, c > 0$, $\mathfrak{p} \in (0,1)$, ε_1 *such that for all* $x \in \mathbb{N}$ *and* $m \in \mathbb{N}$

$$\mathbb{P}(\ell_\infty(x) > m) < Ce^{-cm}. \tag{5.1}$$

Furthermore

$$\left|\mathbb{E}(\ell_\infty(x)) - \frac{1}{v}\right| \le \frac{C}{x^{\varepsilon_1}}, \quad |\mathbb{P}(\ell_\infty(x) = 0) - \mathfrak{p}| \le \frac{C}{x^{\varepsilon_1}}. \tag{5.2}$$

Proof. (5.2) follows from quantitative renewal theorem [30]. (5.1) holds since for $k \ge 1$, $\mathbb{P}(\ell_\infty(x) = k) = \mathbb{P}(\ell_\infty(x) \ne 0)\, \mathfrak{p}_0^{k-1}(1 - \mathfrak{p}_0)$ where $\mathfrak{p}_0$ is the probability that S_n returns to the origin at some positive moment of time. □

Let $\tilde{T}_N = \sum_{x=1}^N \ell_\infty(x) F(x)$. We will show that with probability 1

$$\frac{\tilde{T}_N}{N} \to \frac{\bar{F}}{v}. \tag{5.3}$$

We first deduce Theorem 1.8 from (5.3) and then prove (5.3). Denote $\mathbb{L}_N = \sum_{x=1}^N \ell_\infty(x)$.

By the strong law of large numbers for S_N

$$\frac{\mathbb{L}_N}{N} \to \frac{1}{v}. \tag{5.4}$$

On the other hand for each ε and for almost every ω, there is some $N_0 = N_0(\varepsilon, \omega)$ so that for all $N > N_0$,

$$\left| T_N - \tilde{T}_{Nv(1-\varepsilon)} \right| \leq ||F||_\infty \left(\mathbb{L}^- + [\mathbb{L}_{Nv(1+\varepsilon)} - \mathbb{L}_{Nv(1-\varepsilon)}] \right),$$

where $\mathbb{L}^- = \sum_{x=-\infty}^{0} \ell_\infty(x)$ is the total time spent on the negative halfline. In view of (5.4), $\frac{T_N - \tilde{T}_{Nv(1-\varepsilon)}}{N}$ can be made as small as we wish by taking ε small. Hence Theorem 1.8 follows from (5.3).

In order to prove (5.3) we observe that by Lemma 5.1

$$\mathbb{E}\left(\frac{\tilde{T}_N}{N}\right) = \frac{1}{N}\sum_{x=1}^{N} F(x)\mathbb{E}(\ell_\infty(x)) = \frac{1}{v}\left[\frac{1}{N}\sum_{x=1}^{N} F(x)\right] + O\left(N^{-\varepsilon_1}\right) = \frac{\bar{F}}{v} + o(1),$$

as $N \to \infty$.

We need the following bound, which will be proved in §5.2.

Lemma 5.2. *There are constants C and ε_2 such that for each $n_1 < n_2$*

$$|\mathrm{Cov}(\ell_\infty(n_1), \ell_\infty(n_2))| \leq C\left(\frac{1}{n_1^{\varepsilon_2}} + \frac{1}{(n_2 - n_1)^{\varepsilon_2}}\right).$$

Lemma 5.2 implies that

$$\mathrm{Var}\left(\frac{\tilde{T}_N}{N}\right) \leq \frac{C}{N^{\varepsilon_2}}$$

and so

$$\mathbb{P}\left(\frac{|\tilde{T}_N - \mathbb{E}(\tilde{T}_N)|}{N} \geq \delta\right) \leq \frac{C}{\delta^2 N^{\varepsilon_2}}.$$

Set $r = 2/\varepsilon_2$. By Borel-Cantelli Lemma

$$\frac{\tilde{T}_{n^r}}{n^r} \to \frac{\bar{F}}{v} \text{ as } n \to \infty$$

almost surely. On the other hand, (5.1) and the Borel-Cantelli Lemma imply that, with probability 1, for all sufficiently large x, $\ell_\infty(x) \leq \ln^2 x$. Given N, take n such that $n^r \leq N < (n+1)^r$. Then

$$\left| \tilde{T}_N - \tilde{T}_{n^r} \right| \leq ||F||_\infty \left(\mathbb{L}_N - \mathbb{L}_{n^r} \right) \leq C N^{(r-1)/r} \ln^2 N.$$

It follows that $\frac{\tilde{T}_N}{N} = \frac{\tilde{T}_{n^r}}{n^r} + o(1)$, for $N \to \infty$, proving (5.3).

5.2. **Covariance of occupation times.** The proof of Lemma 5.2 relies on the following estimates.

Lemma 5.3. *There are constants C and ε_3 such that for all $m \geq 1$,*

$$\mathbb{P}(\min_n(S_n) \leq -m) \leq \frac{C}{m^{\varepsilon_3}}.$$

Lemma 5.3 (with $\varepsilon_3 = \beta - 1$) follows from Theorem 2(B) of [33].

Lemma 5.4. *For each $\delta > 0$ there is a constant $C(\delta)$ such that the following holds. Consider a Markov chain with states $\{1, 2, 3\}$ and transition matrix*

$$\begin{pmatrix} p_1 & q_1 & \eta_1 \\ q_2 & p_2 & \eta_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and initial distribution (π_1, π_2, π_3). Assume that

$$q_1 > \delta, \quad \text{and} \quad \eta_2 > \delta. \tag{5.5}$$

Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ denote the occupation times of sites 1 and 2. Then

$$|\mathrm{Cov}(\mathfrak{l}_1, \mathfrak{l}_2)| \leq C(\delta) \left(\frac{q_1}{q_1 + \eta_1}(1 - \pi_1) - \pi_2 + q_2 \right).$$

In the special case where $\eta_1 = 0$, $\pi_1 = 1$, Lemma 5.4 follows from [7, Lemma 3.9(a)]. In this case the statement simplifies significantly since the first term in the RHS vanishes.

The proof of Lemma 5.4 will be given in §5.3.

We apply Lemma 5.4 to the states (n_1, n_2, ∞) with $n_1 < n_2$. This means that we define a 3-state Markov chain as a function of the random walk (S_k), such that the chain starts in the state 1, if the random walk visits n_1 for the first time before it visits n_2; or in the state 2, if the random walk visits n_2 for the first time before it visits n_1; or in in the state 3 if the random walk never visits n_1 or n_2. After that, the chain transitions to state 1, 2 or 3, respectively, if the next return of the random walk to the set $\{n_1, n_2\}$ occurs at n_1, n_2, or never does. Clearly 3 is an absorbing state for this chain. So

$$\pi_1 = \mathbb{P}(n_1 \text{ is visited before } n_2), \quad \pi_2 = \mathbb{P}(n_2 \text{ is visited before } n_1),$$

$$\pi_3 = \mathbb{P}(n_1 \text{ and } n_2 \text{ are not visited}).$$

Let V_n be the event that n is visited by our random walk. Note that Lemma 5.3 implies $q_2 = O\left((n_2 - n_1)^{-\varepsilon_3}\right)$. Hence the probability that both n_1 and n_2 are visited with n_2 being the first is also $O\left((n_2 - n_1)^{-\varepsilon_3}\right)$. Therefore

$$\pi_1 \asymp \mathfrak{q}, \quad \frac{q_1}{q_1 + \eta_1} = \mathbb{P}_{n_1}(V_{n_2}) \asymp \mathfrak{q}, \quad \pi_2 \asymp \mathbb{P}(V_{n_2}) - \mathbb{P}(V_{n_1} \cap V_{n_2}) \asymp \mathfrak{q} - \mathfrak{q}^2,$$

where $\mathfrak{q} = 1 - \mathfrak{p}$ (see (5.2)) and $\asymp$ means the difference between the LHS and the RHS is

$$O\left(n_1^{-\varepsilon_2}\right) + O\left((n_2 - n_1)^{-\varepsilon_2}\right) \quad \text{where} \quad \varepsilon_2 = \min(\varepsilon_1, \varepsilon_3).$$

These estimates, combined with Lemma 5.4 imply Lemma 5.2.

5.3. **Analysis of three state chains.**

Proof of Lemma 5.4. Under the assumptions of the lemma, $\mathfrak{l}_1, \mathfrak{l}_2$ and $\mathfrak{l}_1\mathfrak{l}_2$ are uniformly integrable (the uniformity is over all chains satisfying (5.5)). Let $\bar{\mathfrak{l}}_1$ be the time spent at 1 before the first visit to another state. Then, by the uniform integrability,

$$\mathbb{E}(\mathfrak{l}_1 - \bar{\mathfrak{l}}_1) = O(q_2), \quad \mathbb{E}((\mathfrak{l}_1 - \bar{\mathfrak{l}}_1)\mathfrak{l}_2) = O(q_2), \quad \mathbb{E}_2(\mathfrak{l}_1) = O(q_2), \quad \mathbb{E}_3(\mathfrak{l}_j) = 0.$$

Hence

$$\mathbb{E}(\mathfrak{l}_1\mathfrak{l}_2) = \pi_1\mathbb{E}_1(\bar{\mathfrak{l}}_1)\frac{q_1}{q_1+\eta_1}\mathbb{E}_2(\mathfrak{l}_2) + O(q_2);$$

$$\mathbb{E}(\mathfrak{l}_1) = \pi_1\mathbb{E}_1(\bar{\mathfrak{l}}_1) + O(q_2) \quad \text{and} \quad \mathbb{E}(\mathfrak{l}_2) = (\pi_1\frac{q_1}{q_1+\eta_1} + \pi_2)\mathbb{E}_2(\mathfrak{l}_2) + O(q_2).$$

Therefore

$$\begin{aligned}\mathrm{Cov}(\mathfrak{l}_1, \mathfrak{l}_2) = & \quad \pi_1\mathbb{E}_1(\bar{\mathfrak{l}}_1)\frac{q_1}{q_1+\eta_1}\mathbb{E}_2(\mathfrak{l}_2) - \pi_1\mathbb{E}_1(\bar{\mathfrak{l}}_1)(\pi_1\frac{q_1}{q_1+\eta_1} + \pi_2)\mathbb{E}_2(\mathfrak{l}_2) + O(q_2) \\ = & \quad \pi_1\mathbb{E}_1(\bar{\mathfrak{l}}_1)\mathbb{E}_2(\mathfrak{l}_2)\left[\frac{q_1}{q_1+\eta_1}(1-\pi_1) - \pi_2\right] + O(q_2)\end{aligned}$$

as claimed. □

6. Counterexamples to the strong law.

Here we prove Theorem 1.9.

Consider first the case $d = 1$. Assume that we are given a sequence $a_n \in \mathbb{Z}$, $a_n \nearrow \infty$ with $a_n - a_{n-1} \in \{0, 1\}$ (to be specified later, see Lemma 6.1). Now define $b_1 \gg 1$, $b_{n+1} = b_n + \lfloor b_n/a_n \rfloor$. By induction, we see that

$$a_n \le n < b_n < b_{n+1} < 2^{n+1}. \tag{6.1}$$

Consider the function F defined by $F(0) = 0$,

$$F(x) = \begin{cases} 1 & \text{if } b_{2k} \le x < b_{2k+1} \text{ for some } k \\ 0 & \text{if } b_{2k+1} \le x < b_{2k+2} \text{ for some } k \end{cases} \tag{6.2}$$

for $x > 0$ and $F(x) = F(-x)$ for $x < 0$. Since $b_{n+1}/b_n \to 1$, we have $F \in \mathbf{G}_0$ with $\bar{F} = 1/2$.

Let us denote $c_n = (b_n + b_{n+1})/2$, $t_n = \lfloor c_n^\alpha \rfloor$ and

$$I_n = [c_n - \lfloor b_n/4a_n \rfloor, c_n + \lfloor b_n/4a_n \rfloor]$$

We will show that almost surely, infinitely many of the events

$$A_{2n} = \{\forall k \in [t_{2n}, 3t_{2n}] : S_k \in I_{2n}\}$$

occur, and likewise, infinitely many of the events

$$A_{2n+1} = \{\forall k \in [t_{2n+1}, 3t_{2n+1}] : S_k \in I_{2n+1}\}$$

occur. This proves the theorem as A_{2n} implies $T_{3t_{2n}} \ge 2t_{2n}$ and A_{2n+1} implies $T_{3t_{2n+1}} \le t_{2n+1}$.

To complete the proof, let us fix a sequence $D_n \nearrow \infty$ such that

$$\sum_n \mathbb{P}(|S_n| \ge D_n) < \infty.$$

Then by the Borel-Cantelli Lemma,

$$\mathbb{P}(\exists N : \forall n > N : |S_n| < D_n) = 1.$$

Now we choose a subsequence $n_k \in \mathbb{Z}$ inductively so that

$$n_{k+1} \equiv n_k \pmod 2 \tag{6.3}$$

and

$$n_{k+1} > \max\left\{\exp\left(D_{\lceil 2^{\alpha(n_k+1)}\rceil}\right), \exp\left(\frac{1}{\alpha}2^{(n_k+1)\alpha}\right)\right\}$$

These bounds, combined with (6.1), give

$$b_{n_{k+1}} > \exp(D_{t_{n_k}}) \text{ and } t_{n_{k+1}} > \exp(t_{n_k}). \tag{6.4}$$

We want to show that for every $\varepsilon > 0$ and every K,

$$\mathbb{P}\left(\bigcap_{k=K}^{\infty} A^c_{n_k}\right) < \varepsilon. \tag{6.5}$$

Since $\varepsilon > 0$ is arbitrary, it follows that infinitely many of the events A_{n_k} happen.

Choosing n_k to be even for all k we see that almost surely infinitely many of the events A_{2n} and happen. Likewise, choosing n_k to be odd for all k we see infinitely many of the events A_{2n+1} happen. Thus it remains to verify (6.5).

Given ε and K, choose $K' > K$ so that $\mathbb{P}(\mathcal{B}) < \varepsilon$, where

$$\mathcal{B} = \{\exists n > n^{\alpha}_{K'} : |S_n| > D_n\}.$$

Then we write

$$\mathbb{P}\left(\bigcap_{k=K}^{\infty} A^c_{n_k}\right) \le \mathbb{P}\left(\bigcap_{k=K'}^{\infty} A^c_{n_k}\right) \le \varepsilon + \mathbb{P}\left(\bigcap_{k=K'}^{\infty} A^c_{n_k} \cap \mathcal{B}^c\right). \tag{6.6}$$

By construction, we have

$$\begin{aligned}\mathbb{P}\left(A^c_{n_{k+1}} \cap \mathcal{B}^c \Big| \bigcap_{j=K'}^{k} A^c_{n_j} \cap \mathcal{B}^c\right) &\le 1 - \mathbb{P}\left(A_{n_{k+1}} \Big| \bigcap_{j=K'}^{k} A^c_{n_j} \cap \mathcal{B}^c\right)\\ &\le 1 - \min_{x:|x|<D_{3t_{n_k}}} \mathbb{P}(A_{n_{k+1}}|S_{3t_{n_k}} = x).\end{aligned} \tag{6.7}$$

Next, we claim

Lemma 6.1. *There is a constant K_0 and a sequence $a_n \nearrow \infty$ with $a_n - a_{n-1} \in \{0, 1\}$ such that for any $k \ge K_0$,*

$$\min_{x:|x|<D_{3t_{n_k}}} \mathbb{P}(A_{n_{k+1}}|S_{3t_{n_k}} = x) \ge \frac{1}{k}$$

Clearly, Lemma 6.1 combined with (6.7) and (6.6) implies (6.5). Thus the proof of Theorem 1.9 for the case $d = 1$ will be completed once we prove Lemma 6.1.

Proof of Lemma 6.1. Recall that the invariance principle gives

$$\frac{S_{\lfloor Nt \rfloor}}{N^{1/\alpha}} \Rightarrow Y_t, \tag{6.8}$$

where Y_t is a stable Lévy process. In particular, Y_1 is a stable random variable with parameter α and "skewness" $\beta \in [-1, 1]$ (see e.g. [3], Chapter VIII). Now we distinguish two cases.

Case 1 $\alpha > 1$ or $|\beta| \neq 1$. Set

$$q = \inf_{y \in [-1/16, 1/16]} \mathbb{P}\left(\sup_{t \leq 1} |Y_t| < \frac{1}{4}, |Y_1| < \frac{1}{16} \Big| Y_0 = y \right). \tag{6.9}$$

We claim that $q > 0$. Indeed, as $\alpha > 1$ or $|\beta| \neq 1$, the stable process Y_t cannot be a subordinator. In particular, the density of Y_t is positive everywhere for every $t > 0$ and Y_t has the scaling property (see page 216 in [3]). Thus we have

$$\liminf_{\varepsilon \searrow 0} \inf_{y \in [-1/16, 1/16]} \mathbb{P}\left(|Y_\varepsilon| < \frac{1}{16} \Big| Y_0 = y \right) = p > 0.$$

By Exercise 2 of Chapter VIII in [3], there is some $\varepsilon > 0$ such that

$$\mathbb{P}\left(\sup_{t \leq \varepsilon} |Y_t| < \frac{3}{16} \Big| Y_0 = 0 \right) > 1 - p/2.$$

Combining the last two displayed equations, we derive

$$\inf_{y \in [-1/16, 1/16]} \mathbb{P}\left(\sup_{t \leq \varepsilon} |Y_t| < \frac{1}{4}, |Y_\varepsilon| < \frac{1}{16} \Big| Y_0 = y \right) \geq p/2.$$

Applying this inequality inductively, we obtain that $q \geq (p/2)^{\lceil 1/\varepsilon \rceil}$.

Next, we claim that there exist constants $\bar{c} > 0$, $\bar{p} \in (0, 1)$ such that

$$\min_{x : |x| < D_{t_{n_k}}} \mathbb{P}(A_{n_{k+1}} | S_{t_{n_k}} = x) \geq \frac{\bar{c}}{a_{n_k}} \, \bar{p}^{a_{n_k}^\alpha}. \tag{6.10}$$

(6.10) implies the lemma since we can choose any sequence $a_n \nearrow \infty$ with $a_n - a_{n-1} \in \{0, 1\}$ such that $a_{n_k} \leq (-\log k / \log \tilde{p})^{1/\alpha}$ for a fixed $\tilde{p} \in (0, \bar{p})$.

To prove (6.10), we first use the local limit theorem and (6.4) to derive

$$\min_{x : |x| < D_{3t_{n_k}}} \mathbb{P}\left(\left| S_{t_{n_{k+1}}} - c_{n_{k+1}} \right| < \frac{b_{n_{k+1}}}{16 a_{n_{k+1}}} \Big| S_{3t_{n_k}} = x \right) > \frac{\bar{c}}{a_{n_k}}. \tag{6.11}$$

Now using (6.8) with $N = (b_{n_{k+1}} / 4 a_{n_{k+1}})^\alpha$ and (6.9) we obtain (6.10) with $p = q^{2 \cdot 4^\alpha}$.

Case 2 $\alpha < 1$ and $|\beta| = 1$. Let us assume $\beta = 1$ (otherwise apply the forthcoming argument to $-X_i$).

(6.11) still holds in case 2, however a new approach is required to estimate $\mathbb{P}(A_{n_{k+1}} | S_{t_{n_k}} = x)$ since now the process Y_t (a.k.a. stable subordinator) is non-decreasing and thus $q = 0$. Note however that in case 2, $\sup_{t \leq 1} |Y_t| = Y_1$ and thus it suffices to estimate one random variable instead of a stochastic process. Recall that the density of Y_1 is strictly positive on $\mathbb{R}^+$. Thus applying (6.8) to $|X_i|$ (which is also in the standard domain of attraction of the totally skewed α-stable distribution) we obtain the following: for any $\varepsilon > 0$ there exists $N_0(\varepsilon)$ and $\delta(\varepsilon) > 0$ such that for any $N \geq N_0(\varepsilon)$,

$$\mathbb{P}\left(\sum_{n=1}^{3N} |X_n| \leq \frac{\varepsilon}{8} N^{1/\alpha}\right) > \delta(\varepsilon). \tag{6.12}$$

Without loss of generality, we assume that N_0 and δ are, respectively, non-decreasing and non-increasing functions of ε. Now we define the sequence a_n inductively. First, let $a_1 = 1$. Now assume that a_{n_k} is defined. Let $a_m = a_{n_k}$ for $m = n_k + 1, ..., n_{k+1} - 1$. Next, we define $a_{n_{k+1}} = a_{n_k} + 1$ if both of the following conditions are satisfied:

(A) $N_0\left(\frac{1}{a_{n_k}+1}\right) < n_{k+1}^{\alpha}$ and
(B) $\frac{\bar{c}}{2a_{n_k}+1}\, \delta\left(\frac{1}{a_{n_k}+1}\right) > \frac{1}{k+1}$.

Here, $\bar{c}$ is the constant from (6.11). If either (A) or (B) fails, we put $a_{n_{k+1}} = a_{n_k}$.

Observe that by our construction, for all k, we have

$$N_0\left(\frac{1}{a_{n_k}}\right) < n_k^{\alpha}; \tag{6.13}$$

$$\frac{\bar{c}}{2a_{n_k}}\, \delta\left(\frac{1}{a_{n_k}}\right) > \frac{1}{k}. \tag{6.14}$$

Indeed, if $a_{n_{k+1}} = a_{n_k} + 1$ then (6.13) and (6.14) follow from conditions (A) and (B) above. If $a_{n_{k+1}} = a_{n_k}$ then (6.13) and (6.14) follow by induction since the LHSs of both (6.13) and (6.14) do not change when we replace k by $k + 1$, while the RHS of (6.13) increases and the RHS of (6.14) decreases.

By construction, $a_n \nearrow \infty$. Let K_0 be the smallest integer k so that $a_{n_k} = 2$. We prove that the lemma holds with this choice of K_0 and a_n. Recall that by (6.1), $b_{n_k} > n_k$ and so by (6.13), $N := b_{n_k}^{\alpha} > N_0(\varepsilon)$ with $\varepsilon = 1/a_{n_k}$. Applying (6.12) with this N and ε and using (6.14), we obtain

$$\mathbb{P}\left(\left|S_m - S_{t_{n_k}}\right| < \frac{b_{n_k}}{8a_{n_k}} \quad \forall m \leq 3b_{n_k}^{\alpha}\right) > \frac{a_{n_k}}{\bar{c}} \frac{2}{k}.$$

and since $t_{n_k} < b_{n_k}^{\alpha}$ and $k - 1 > k/2$, we arrive at

$$\mathbb{P}\left(\left|S_m - S_{t_{n_k}}\right| < \frac{b_{n_k}}{8a_{n_k}} \quad \forall m \leq 3t_{n_k}\right) > \frac{a_{n_{k-1}}}{\bar{c}} \frac{1}{k-1}. \tag{6.15}$$

Combining (6.15) with k replaced by $k + 1$ and (6.11), we obtain the estimate of the lemma. □

The above proof, with a few minor adjustments, applies to arbitrary dimension d. Specifically, we need to consider the function $\mathcal{F} \in \mathbf{G}_0$ defined by

$$\mathcal{F}(x_1, ..., x_d) = \begin{cases} F(x_1) & \text{if } |x_i| \leq |x_1| \text{ for } i = 2, ..., d \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

where F is given by (6.2) and we need to replace I_n by

$$[c_n - \lfloor b_n/4a_n \rfloor, c_n + \lfloor b_n/4a_n \rfloor] \times [-\lfloor b_n/4a_n \rfloor, \lfloor b_n/4a_n \rfloor]^{d-1}.$$

Remark 6.2. It is easy to adjust the above proof to derive the following stronger version of Theorem 1.9: There is a function $F \in \mathbf{G}_0$ so that F only takes values $\{0,1\}$, $\bar{F} = 1/2$ and for almost every ω and for any $a \in [0,1]$, there is a subsequence $n_k = n_k(a,\omega)$ such that $T_{n_k}/n_k \to a$.

7. Conclusions.

The results proven in this paper show that for random walks the weak law of large numbers holds in the largest possible space of global observables, namely $\mathbf{G}_0$. On the other hand, the strong law of large numbers fails, in general, except for the walks with drift in dimension 1. In that case the path of the walk is almost deterministic and so the ergodic theory for occupation times could be used. The good news is that the weak law of large numbers seems to be a good setting for homogenization theory (cf Theorem 1.2), so the space $\mathbf{G}_0$ could be useful for that purpose. If we have some control on fluctuations over the mesoscopic scale as provided, for example, by the space $\mathbf{G}_\gamma$, then we can ensure the strong law. If we have polynomial control on the mesoscopic scale, as provided by the space $\mathbf{G}_\gamma^\beta$ then we can estimate the rate of convergence. In particular, our results give optimal rate of convergence for two important special cases: random walks in random scenery and quasi-periodic observables.

We note that the main ingredient in most proofs is local limit theorem and its extensions, such as the Edgeworth expansion used in Section 4. This makes it plausible that similar results hold for other systems where the local limit theorem hold, including the systems described in [2, 6, 8, 9, 10, 11, 12]. Another natural research direction motivated by the present work is limit theorems for global observables. It is likely that just assuming that F belongs to an appropriate $\mathbf{G}_*$ will not be enough to derive limit theorems. For example, the computation of variance for T_N done in Sections 2 and 4 involve the expression of the form $\tilde{F}_z(x) := F(x)F(x+z)$ for a fixed $z \in \mathbb{Z}^d$. Therefore additional restrictions seem to be required to obtain limit theorems. Extending our results to more general systems as well as limit theorems for global observables will be the subject of future work.

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