AN EXAMPLE OF A SMOOTH HYPERBOLIC MEASURE WITH COUNTABLY MANY ERGODIC COMPONENTS.

D. Dolgopyat, H. Hu, Ya. Pesin

I. Introduction

We construct an example of a diffeomorphism with non-zero Lyapunov exponents with respect to a smooth invariant measure which has countably many ergodic components. More precisely we will prove the following result.

Theorem. *There exists a C^∞ diffeomorphism f of the three dimensional manifold $M = \mathbb{T}^2 \times \mathbb{S}^1$ such that*

(1) *f preserves the Riemannian volume μ on M;*
(2) *μ is a hyperbolic measure;*
(3) *f has countably many ergodic components which are open* (mod 0).

II. Construction of the Diffeomorphism f.

Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a linear hyperbolic automorphism. Passing if necessary to a power of A we may assume that A has at least two fixed points p and p'. Consider the map $F = A \times \mathrm{id}$ of the manifold M. We will perturb F to obtain the desired map f.

Consider a countable collection of intervals $\{I_n\}_{n=1}^\infty$ on the circle $\mathbb{S}^1$, where

$$I_{2n} = [(n+2)^{-1}, (n+1)^{-1}], \quad I_{2n-1} = [1-(n+1)^{-1}, 1-(n+2)^{-1}].$$

Clearly, $\bigcup\limits_{n=1}^\infty I_n = (0,1)$ and $\mathrm{int}\, I_n$ are pairwise disjoint.

By Main Proposition below, for each n one can construct a C^∞ volume preserving ergodic hyperbolic diffeomorphism $f_n : \mathbb{T}^2 \times I \to \mathbb{T}^2 \times I$ satisfying: 1) $\|F - f_n\|_{C_n} \le n^{-2}$ 2) for all $0 \le m < \infty$, $D^m f_n|\mathbb{T}^2 \times \{z\} = D^m F|\mathbb{T}^2 \times \{z\}$ for $z = 0$ or 1.

Let $L_n : I_n \to I$ be the affine map and $\pi_n = (\mathrm{id}, L_n) : \mathbb{T}^2 \times I_n \to \mathbb{T}^2 \times I$. We define the map f by setting $f|\mathbb{T}^2 \times I_n = \pi_n^{-1} f_n \pi_n$ for all n and $f|\mathbb{T}^2 \times \{0\} = F|\mathbb{T}^2 \times \{0\}$. Note that

$$\big\|F|\mathbb{T}^2 \times I_n - \pi_n^{-1} f_n \pi_n\big\| \le \big\|\pi_n^{-1}\big(F - f_n\big)\pi_n\big\| \le n^{-2} \cdot n = n^{-1}.$$

It follows that f is C^∞ on M and has the required properties.

Key words and phrases. Hyperbolic measure, Lyapunov exponents, stable ergodicity, accessibility.

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III. Main Proposition

The goal of this section is to proof the following statement.

Main Proposition. *For any $k \geq 2$ and $\delta > 0$, there exists a map g such that:*

(a) *g is a C^∞ volume preserving diffeomorphism of M;*
(b) $\|F - g\|_{C_k} \leq \delta$;
(c) *for all $0 \leq m < \infty$ $g|\mathbb{T}^2 \times \{z\} = F|\mathbb{T}^2 \times \{z\}$ for $z = 0$ and 1;*
(d) *g is ergodic with respect to the Riemannian volume and has non-zero Lyapunov exponents almost everywhere.*

Before giving the formal proof let us outline the main idea. The result will be achived in two steps. First by a method of [SW] we obtain a diffeomorphism with non-zero *average* central exponent $\int \chi_c(x)d\mu(x) \neq 0$, where $\chi_c(x)$ denotes the Lyapunov eponent of x on E_c. We then further perturb this diffeomorphism using a method of [NT] to ensure that our diffeomorphism has accessibility property and is therefore ergodic.

Conjecture. *Consider a one parameter family g_ε with $g_0 = F$. Then for small ε g_ε satisfies the conditions of the Main Proposition except for a positive codimension submanifold in the space of one parameter families.*

Proof. Consider the linear hyperbolic map A. We may assume that its eigenvalues are η and η^{-1}, where $\eta > 1$. Let p and p' be fixed points of A. Choose a number $\varepsilon_0 > 0$ such that $d(p, p') \geq 3\varepsilon_0$. Consider the local stable and unstable one-dimensional manifolds for A at points p and p' of "size" ε_0 and denote them respectively by $V^s(p)$, $V^u(p)$, $V^s(p')$, and $V^u(p')$.

Let us choose the smallest positive number n_1 such that the intersection $A^{-n_1}(V^s(p')) \cap V^u(p) \cap B(p, \varepsilon_0)$ consists of a single point which we denote by q_1 (here $B(p, \varepsilon_0)$ is the ball in $\mathbb{T}^2$ of radius ε_0 centered at p). Similarly, we choose the smallest positive number n_2 such that the intersection $A^{n_2}(V^u(p')) \cap V^s(p) \cap B(p, \varepsilon_0)$ consists of a single point which we denote by q_2.

Given a sufficiently small number $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon \leq \frac{1}{2}\min\{d(p, q_1), d(p, q_2)\}$, there is $\ell \geq 2$ such that

$$A^\ell(q_1) \notin B(p, \varepsilon), \quad A^{\ell+1}(q_1) \in B(p, \varepsilon). \tag{3.1}$$

We now choose $\varepsilon' \in (0, \varepsilon)$ such that $A^{\ell+1}(q_1) \in B(p, \varepsilon')$.

Finally, let $q \in \mathbb{T}^2$ be such that

$$B(p, \varepsilon) \cap \left(A^{-n_1}(V^s(p')) \cup A^{n_2}(V^u(p'))\right) = \emptyset, \quad A^i(B(q, \varepsilon)) \cap B(q, \varepsilon) = \emptyset, \quad i = 1, \ldots, N,$$

where $N > 0$ will be determined later.

Set $\Omega_1 = B(p, \varepsilon_0) \times I$ and $\Omega_2 = B^{uc}(\bar{q}, \varepsilon_0) \times B^s(\bar{q}, \varepsilon_0)$, where $\bar{q} = (q, 1/2)$ and $B^{uc}(\bar{q}, \varepsilon_0) \subset V^u(q) \times I$ and $B^s(\bar{q}, \varepsilon_0) \subset V^s(q)$ are balls of radius ε_0 about $\bar{q}$.

After this preliminary considerations we describe the construction of the map g.

Consider the coordinate system in Ω_1 originated at $(p, 0)$ with x, y, and z-axes to be unstable, stable, and neutral directions respectively. If a point $w = (x, y, z) \in \Omega_1$ and $F(w) \in \Omega_1$ then $F(w) = (\eta x, \eta^{-1}y, z)$.

Choose a C^∞ function $\xi : I \to \mathbb{R}^+$ satisfying:

(1) $\xi(z) > 0$ on $(0,1)$;
(2) $\xi^{(i)}(0) = \xi^{(i)}(1) = 0$ for $i = 0, 1, 2, \ldots$;
(3) $\|\xi\|_{C^k} \le \delta$.

We also choose two C^∞ functions $\phi = \phi(x)$ and $\psi = \psi(y)$ which are defined on the interval $(-\varepsilon_0, \varepsilon_0)$ and satisfy

(4) $\phi(x) = \phi_0$ if $x \in (-\varepsilon', \varepsilon')$ and $\psi(y) = \psi_0$ if $y \in (-\varepsilon', \varepsilon')$, where ϕ_0 and ψ_0 are positive constants;
(5) $\phi(x) = 0$ if $|x| \ge \varepsilon$; $\psi(y) \ge 0$ for any y and $\psi(y) = 0$ if $|y| \ge \varepsilon$;
(6) $\|\phi\|_{C^k} \le \delta,\ \|\psi\|_{C^k} \le \delta$;
(7) $\int_0^{\pm\varepsilon} \phi(s)ds = 0$.

We now define the vector field X on Ω_1 by

$$X(x, y, z) = \Big(-\psi(y)\xi'(z)\int_0^x \phi(s)ds,\quad 0,\quad \psi(y)\xi(z)\phi(x)\Big).$$

It is easy to check that X is a divergence free vector field supported on $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times I$.

We define the map h_t on Ω_1 to be the time t map of the flow generated by X and we set $h_t = \text{id}$ on the complement of Ω_1. It is easy to see that h_t is a C^∞ volume preserving diffeomorphism which preserves the y coordinate (the stable direction).

Consider now the coordinate system in Ω_2 originated at $(q, 1/2)$ with x, y, and z-axes to be unstable, stable, and neutral directions respectively. We then switch to the cylindrical coordinate system (r, θ, y), where $x = r\cos\theta$, $y = y$, and $z = r\sin\theta$.

Consider a C^∞ function $\rho : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^+$ satisfying:

(8) $\rho(r) > 0$ if $0.2\varepsilon' \le r \le 0.9\varepsilon$ and $\rho(r) = 0$ if $r \le 0.1\varepsilon'$ or $r \ge \varepsilon$;
(9) $\|\rho\|_{C^k} \le \delta$.

We define now the map $\tilde{h}_\tau$ on Ω_2 by

$$\tilde{h}_\tau(r, \theta, y) = (r,\ \theta + \tau\psi(y)\rho(r),\ y). \tag{3.2}$$

and we set $\tilde{h}_\tau = \text{id}$ on $M \backslash \Omega_e$. It is easy to see that for every τ the map $\tilde{h}_\tau$ is a C^∞ volume preserving diffeomorphism.

Let us set $g = g_{t\tau} = h_t \circ F \circ \tilde{h}_\tau$. For all sufficiently small $t > 0$ and τ, the map $g_{t\tau}$ is C^k close to F and hence, is a partially hyperbolic (in the narrow sense) C^∞ diffeomorphism. It preserves the Riemannian volume in M and is ergodic by Lemma 1. It remains to show that $g_{t\tau}$ has non-zero Lyapunov exponents almost everywhere.

Denote by $E^s_{t\tau}(w)$, $E^u_{t\tau}(w)$, and $E^c_{t\tau}(w)$ the stable, unstable, and neutral subspaces at a point $w \in M$ for the map $g_{t\tau}$. It suffices to show that for almost everywhere point $w \in M$ and every vector $v \in E^c_\tau(w)$, the Lyapunov exponent $\chi(w, v) \ne 0$.

Set $\kappa_{t\tau}(w) = Dg_{t\tau}|E^u_{t\tau}(w)$, $w \in M$. By Lemma 2, for all sufficiently small $\tau > 0$,

$$\int_M \log \kappa_{0\tau}(w)\, dw < \log \eta.$$

The subspace $E^u_{t\tau}(w)$ depends continuously on t and τ (for a fixed w; for details see the paper by Burns, Pugh, Shub, and Wilkinson in this volume) and hence, so does $\kappa_{t\tau}$. It follows that for all sufficiently small $\tau > 0$, there is $t > 0$ such that

$$\int_M \log \kappa_{t\tau}(w)\, dw < \log \eta.$$

Denote by $\chi^s_{t\tau}(w)$, $\chi^u_{t\tau}(w)$, and $\chi^c_{t\tau}(w)$ the Lyapunov exponents of $g_{t\tau}$ at the point $w \in M$ in the stable, unstable, and neutral directions respectively (since these directions are on-dimensional the Lyapunov exponents do not depend on the vector). By the ergodicity of $g_{t\tau}$, we have that for almost every $w \in M$,

$$\chi^u_{t\tau}(w) = \lim_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} \kappa_{t\tau}(g^i_{t\tau}(w)).$$

By the Birkhorff ergodic theorem, we get

$$\chi^u_{t\tau}(w) = \int_M \log \kappa_{t\tau}(w)\, dw < \log \eta.$$

Since $E^s_{t\tau}(w) = E^s_{00}(w) = E^s_F(w)$ for every t and τ, we conclude that $\chi^s_{t\tau}(w) = -\log \eta$ for almost every $w \in M$. Since $g_{t\tau}$ is volume preserving,

$$\chi^s_{t\tau}(w) + \chi^u_{t\tau}(w) + \chi^c_{t\tau}(w) = 0$$

for almost every $w \in M$. It follows that $\chi^c_{t\tau}(w) \neq 0$ for almost every $w \in M$ and hence, $g_{t\tau}$ has non-zero Lyapunov exponents almost everywhere. This completes the proof of Main Proposition. □

IV. Ergodicity of the Map $g_{t\tau}$.

Lemma 1. *For every sufficiently small t and τ the map $g_{t\tau}$ is ergodic.*

Proof. Consider a partially hyperbolic (in the narrow sense) diffeomorphism f of a compact Riemannian manifold M preserving the Riemannian volume. Two points $x, y \in M$ are called *accessible (with respect to f* if they can be joined by a piecewise differentiable piecewise nonsingular path which consists of segments tangent to either E^u or E^s. The diffeomrphism f satisfies the *essential accessibility property* if almost any two points in M (with respect to the Riemannian volume) are accessible. We will show that the map $g_{t\tau}$ has the essential accessibility property. The ergodicity of the map will then follow from the result by Pugh and Shub (see [PS]; see also the paper by Burns, Pugh, Shub, and Wilkinson in this volume).

Given a point $w \in M$, denote by $\mathcal{A}(w)$ the set of points $z \in M$ such that w and z are accessible. Set $I_p = \{p\} \times I$.

Sublemma 1.1. *For every $z \in (0,1)$,*

$$\mathcal{A}(p,z) \supset I_p. \tag{4.1}$$

Proof of Sublemma 1.1. We use the coordinate system (x,y,z) in Ω_1 described above. Since the map h_t presrves the center leaf I_p, we have that

$$h_t(0,0,z) = (h_t^1(0,0,z),\, h_t^2(0,0,z),\, h_t^3(0,0,z)) = (0,0,h_t^3(0,0,z)),\ z \in (0,1).$$

It suffices to show that for every $z \in (0,1)$,

$$\mathcal{A}(p,z) \supset \{(p,a) : a \in [(h_t^{-\ell})^3(p,z), z]\}, \tag{4.2}$$

where ℓ is chosen by (3.1). In fact, since accessibility is a transtive relation and $h_t^{-n}(p,z) \to (p,0)$ for any $z \in (0,1)$, (4.2) implies that $\mathcal{A}(p,z) \supset \{(p,a) : a \in (0,z]\}$. Since this holds true for all $z \in (0,1)$ and accessibility is a reflective relation, we obtain (4.1).

Now we proceed with the proof of (4.2).

Let $q_1 \in V_{t\tau}^u(p)$ and $q_2 \in V_{t\tau}^s(p)$ be two points constructed in Section III. The intersection $V_{t\tau}^s(q_1) \cap V_{t\tau}^u(q_2)$ is not empty and consists of a single point q_3. We will prove that for any $z_0 \in (0,1)$, there exist $z_i \in (0,1)$, $i = 1,2,3,4$ such that

$$(q_1,z_1) \in V_{t\tau}^u((p,z_0)), \quad (q_3,z_3) \in V_{t\tau}^s((q_1,z_1)),$$
$$(q_2,z_2) \in V_{t\tau}^u((q_3,z_3)), \quad (p,z_4) \in V_{t\tau}^s((q_2,z_2))$$

and

$$z_4 \le (h_t^{-\ell})^3(p,z_0). \tag{4.3}$$

This means that $(p,z_4) \in \mathcal{A}(p,z_0)$. By continuity, we conclude that

$$\{(p,a) : a \in [z_4,z_0]\} \subset \mathcal{A}(p,z_0)$$

and (4.2) follows.

Since $g_{t\tau}$ preserves the xz-plane, we have that $V_F^{uc}((p,z_0)) = V_F^{uc}((p,z_0))$. Hence, there is a unique $z_1 \in (0,1)$ such that $(q_1,z_1) \in V_{t\tau}^u((p,z_0))$. Notice that

$$g_{t\tau}^{-n}(p,z_0) = (p, h_t^{-n}((p,z_0)), \quad g_{t\tau}^{-n}(q_1,z_1) = l(A^{-n}q_1, z_1)$$

for $n \le \ell$. This is true because the points $A^{-n}q_1$, $n = 0,1,\dots,\ell$ lie outside the ε-neighborhood of I_p, where the perturbation map $h_t = \mathrm{id}$. Similarly, since the points $A^{-n}q_1$, $n > \ell$ lie inside the ε'-neighborhood of I_p, and the third component of h_t depends only on the z-coordinate, we have

$$g_{t\tau}^{-n}(q_1,z_1) = \big(A^{-n}q_1, h_t^{-n+l}z_1\big).$$

Since $d(g_{t\tau}^{-n}((p,z_0)), g_{t\tau}^{-n}((q_1,z_1))) \to 0$ as $n \to \infty$, we have $d(h_t^{-n}((p,z_0)), h_t^{-n+l}((p,z_1))) \to 0$ as $n \to \infty$. It follows that $z_1 = (h_t^{-\ell})^3((p,z_0))$.

By the construction of the map h_t (that is $h_t = \mathrm{id}$ outside Ω_1 the sets $A^{-n_1}V_{t\tau}^s(p')$ and $A^{n_2}V_{t\tau}^u(p')$ are pieces of horizontal lines. This means that $z_2 = z_3 = z_1$.

Since the third component of h_t is non-decreasing from (q_2,z_2) to (p,z_4) along $V_{t\tau}^s(p)$, we conclude that $z_4 \le z_3 = z_1 = (h_t^{-\ell})^3(p,z_0)$ and thus (4.3) holds. ☐

The essential accessibility property follows from Sublemma 1.1 and the following statement.

Sublemma 1.2. *(see* [NT]*). Assume that any two points in I_p are accessible. Then the map $g_{t\tau}$ satisfies the essential accessibility property.*

Proof of Sublemma 1.2. It is easy to see that for any two points $x, y \in M$ which do not lie on the boundary of M one can find points $x', y' \in I_p$ such that the pairs (x, x') and (y, y') are accessible. By Sublemma 1.1 the points x', y' are accessible. Since accessibility is a transitive relation the result follows. □

V. Hyperbolicity of the Map $g_{0\tau}$.

Lemma 2. *For any sufficiently small $\tau > 0$,*

$$\int_M \log \kappa_{0\tau}(w) dw < \log \eta. \tag{5.1}$$

Proof. Our approach is an elaboration of an arguments in [SW].

For any $w \in M$, we introduce the coordinate system in $T_w M$ associated with the splitting $E_F^u(w) \oplus E_F^s(w) \oplus E_F^c(w)$. Given $\tau \geq 0$ and $w \in M$, there exists a unique number $\alpha_\tau(w)$ such that the vector $v_\tau(w) = (1, 0, \alpha_\tau(w))^\perp$ lies in $E_{0\tau}^u(w)$, (where $\perp$ denote the transpose). Since the map $\tilde{h}_\tau$ preserves the y coordinate, by the definition of the function $\alpha_\tau(w)$, one can write the vector $Dg_{0\tau}(w)v_\tau(w)$ in the form

$$Dg_{0\tau}(w)v_\tau(w) = \left(\bar{\kappa}_\tau(w), 0, \bar{\kappa}_\tau(w)\alpha_\tau(g_{t0}(w))\right)^\perp \tag{5.2}$$

for some $\bar{\kappa}_\tau(w) > 1$. Sinse the expanding rate of $Dg_{0\tau}(w)$ along its unstable direction is $\kappa_{0\tau}(w)$ we obtain that

$$\kappa_{0\tau}(w) = \bar{\kappa}_\tau(w) \frac{\sqrt{1 + \alpha_\tau(g_{0\tau}(w))^2}}{\sqrt{1 + \alpha_\tau(w)^2}}.$$

Since $E_{0\tau}^u(w)$ is close to $E_{00}^u(w)$ the function $\alpha_\tau(w)$ is uniformly bounded. Using the fact that the map $g_{0\tau}$ preserves the Riemannian volume we find that

$$L_\tau = \int_M \log \kappa_{0\tau}(w)\, dw = \int_M \log \bar{\kappa}_\tau(w)\, dw. \tag{5.3}$$

Consider the map $\tilde{h}_\tau$. Since it preserves the y-coordinate using (3.2), we can write that

$$\tilde{h}_\tau(x, y, z) = (r \cos\sigma, y, r \sin\sigma),$$

where $\sigma = \sigma(\tau, r, \theta, y) = \theta + \tau\psi(y)\rho(r)$. Therefore, the differential

$$D\tilde{h}_\tau : E_F^u(w) \oplus E_F^c(w) \to E_F^u(g_{0\tau}(w)) \oplus E_F^c(g_{0\tau}(w))$$

can be written in the matrix form

$$D\tilde{h}_\tau(w) = \begin{pmatrix} A(\tau, w)\, B(\tau, w) \\ C(\tau, w)\, D(\tau, w) \end{pmatrix} = \begin{pmatrix} r_x \cos\sigma - r\sigma_x \sin\sigma & r_y \cos\sigma - r\sigma_y \sin\sigma \\ r_x \sin\sigma + r\sigma_x \cos\sigma & r_y \sin\sigma + r\sigma_y \cos\sigma \end{pmatrix},$$

where

$$r_x = \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad r_z = \frac{\partial r}{\partial z} = \frac{y}{r} = \sin\theta,$$

$$\sigma_x = \frac{\partial \sigma}{\partial x} = \frac{-z}{r^2} + \frac{z}{r}\tau\tilde{\rho}_r(y,r) = \frac{\sin\theta}{r} + \tau\tilde{\rho}_r(y,r)\cos\theta,$$

$$\sigma_z = \frac{\partial \sigma}{\partial z} = \frac{x}{r^2} + \frac{x}{r}\tau\tilde{\rho}_r(y,r) = \frac{\cos\theta}{r} + \tau\tilde{\rho}_r(y,r)\sin\theta,$$

and $\tilde{\rho}(y,r) = \psi(y)\rho(r)$. It is easy to check that

$$\begin{aligned}
A &= A(\tau,w) = 1 - \tau r\tilde{\rho}_r \sin\theta\cos\theta - \frac{\tau^2\tilde{\rho}^2}{2} - \tau^2 r\tilde{\rho}\tilde{\rho}_r\cos^2\theta + O(\tau^3),\\
B &= B(\tau,w) = -\tau\tilde{\rho} - \tau r\tilde{\rho}_r\sin^2\theta - \tau^2 r\tilde{\rho}\tilde{\rho}_r\sin\theta\cos\theta + O(\tau^3),\\
C &= C(\tau,w) = \tau\tilde{\rho} + \tau r\tilde{\rho}_r\cos^2\theta - \tau^2 r\tilde{\rho}\tilde{\rho}_r\sin\theta\cos\theta + O(\tau^3),\\
D &= D(\tau,w) = 1 + \tau r\tilde{\rho}_r \sin\theta\cos\theta - \frac{\tau^2\tilde{\rho}^2}{2} - \tau^2 r\tilde{\rho}\tilde{\rho}_r\sin^2\theta + O(\tau^3).
\end{aligned} \tag{5.4}$$

By Sublemma 2.1 below, we have

$$L_\tau = \int_M \log\eta - \log\big(D(\tau,w) - \eta B(\tau,w)\alpha_\tau(g_{0\tau}(w))\big)dw.$$

By Sublemma 2.2, we have

$$\frac{dL_\tau}{d\tau}\Big|_{\tau=0} = 0, \qquad \frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0} < 0.$$

So we can choose τ so small that $L_\tau \neq \log\eta$. □

Sublemma 2.1.

$$L_\tau = \log\eta - \int_M \log\big(D(\tau,w) - \eta B(\tau,w)\alpha_\tau(g_{0\tau}(w))\big)dw.$$

Proof of Sublemma 2.1. Since $g_{0\tau} = h_0 \circ F \circ \tilde{h}_\tau = F \circ \tilde{h}_\tau$, we have that

$$D_\tau(w) = Dg_{0\tau}(w)|E^u_{0\tau}(w) \oplus E^c_{0\tau}(w) = \begin{pmatrix} \eta A(\tau,w), \ \eta B(cw) \\ C(\tau,w), \ D(\tau,w) \end{pmatrix}.$$

By (5.2),

$$D_\tau(w)\begin{pmatrix} 1 \\ \alpha_\tau(w) \end{pmatrix} = \begin{pmatrix} \eta A(\tau,w) + \eta B(\tau,w)\,\alpha_\tau(w) \\ C(\tau,w) + D(\tau,w)\,\alpha_\tau(w) \end{pmatrix} = \begin{pmatrix} \kappa_\tau(w) \\ \kappa_\tau(w)\,\alpha_\tau(g_{0\tau}(w)) \end{pmatrix}. \tag{5.5}$$

Since $\tilde{h}_\tau$ is volume preserving, $AD - BC = 1$ and therefore,

$$A + B\alpha = \frac{1}{D} + \frac{B}{D}(C + D\alpha).$$

Comparing the components in (5.5), we obtain

$$\begin{aligned}\kappa_\tau(w) =&\eta\big(A(\tau,w)+B(\tau,w)\,\alpha_\tau(w)\big)\\ =&\eta\Big(\frac{1}{D(\tau,w)}+\frac{B(\tau,w)}{D(\tau,w)}\big(C(\tau,w)+D(\tau,w)\,\alpha_\tau(w)\big)\Big)\\ =&\eta\Big(\frac{1}{D(\tau,w)}+\frac{B(\tau,w)}{D(\tau,w)}(\kappa_\tau(w)\,\alpha_\tau(g_{0\tau}(w)))\Big).\end{aligned}$$

Solving for $\kappa_\tau(w)$, we get

$$\kappa_\tau(w)=\frac{\eta}{D(\tau,w)-\eta B(\tau,w)\,\alpha_\tau(g_{0\tau}(w))}.$$

The desired result follows from (5.3). □

Sublemma 2.2.

$$\frac{dL_\tau}{d\tau}\Big|_{\tau=0}=0,\qquad \frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0}<0. \tag{5.6}$$

Proof of Sublemma 2.2. In order to simplify notations we set $D'_\tau=\frac{\partial D}{\partial\tau}$, $B'_\tau=\frac{\partial B}{\partial\tau}$, $C'_\tau=\frac{\partial C}{\partial\tau}$, $D''_{\tau\tau}=\frac{\partial^2 D}{\partial\tau^2}$, and $B''_{\tau\tau}=\frac{\partial^2 B}{\partial\tau^2}$. Since the function $\alpha_\tau(w)$ is differentiable over τ (see the paper by Burns, Pugh, Shub, and Wilkinson in this volume) by Sublemma 2.1, we find

$$\frac{dL_\tau}{d\tau}=-\int_M\frac{D'_\tau-\eta B'_\tau\alpha(g_{0\tau}(w))-\eta B\frac{\partial\alpha_\tau(w)}{\partial\tau}(g_{0\tau}(w))}{D(\tau,w)-\eta B(\tau,w)\alpha_\tau(w)(g_{0\tau}(w))}\,dw$$

and therefore,

$$\begin{aligned}\frac{d^2L_\tau}{d\tau^2}=&\int_M\Big(\frac{D'_\tau-\eta B'_\tau\alpha(g_{0\tau}(w))-\eta B(\tau,w)\frac{\partial\alpha_\tau(w)}{\partial\tau}(g_{0\tau}(w))}{D(\tau,w)-\eta B(\tau,w)\alpha_s(g_{0\tau}(w))}\Big)^2\,dw\\ &-\int_M\frac{D''_{\tau\tau}-\eta B''_{\tau\tau}\alpha(g_{0\tau}(w))-\eta B(\tau,w)\frac{\partial^2\alpha_\tau(w)}{\partial\tau^2}(g_{0\tau}(w))-2\eta B'_\tau\frac{\partial\alpha_\tau(w)}{\partial\tau}(g_{0\tau}(w))}{D(\tau,w)-\eta B(\tau,w)\,\alpha_\tau(g_{0\tau}(w))}dw\end{aligned}$$

Note that for all $w\notin\Omega_2$,

$$A(\tau,w)=D(\tau,w)=1,\quad C(\tau,w)=B(\tau,w)=0$$

and for all $w\in M$,

$$A(0,w)=D(0,w)=1,\quad C(0,w)=B(0,w)=0,\quad \alpha_0(w)=0.$$

It follows that

$$\frac{dL_\tau}{d\tau}\Big|_{\tau=0}=\int_{\Omega_2}D'_\tau\,dw, \tag{5.7}$$

and also that

$$\frac{d^2 L_\tau}{d\tau^2}\Big|_{\tau=0} = \int_{\Omega_2} \big[(D'_\tau)^2 - D''_{\tau\tau} + 2\eta B'_\tau \frac{\partial \alpha_\tau(w)}{\partial \tau}(g_{0\tau}(w))\big]_{\tau=0}\, dw. \tag{5.8}$$

By (5.4), we obtain that

$$D'_\tau(0,w) = r\tilde{\rho}_r(r) sin\theta \cos\theta$$

and hence,

$$\int_{\Omega_2} D'_\tau dw = 0.$$

Therefore, (5.7) implies the equality in (5.6).

We now proceed with the inequality in (5.6). Applying Sublemma 2.3 below we obtain that

$$\frac{\partial \alpha}{\partial \tau}(g_{0\tau}(w))\Big|_{\tau=0} = \frac{C'_\tau(0,w)}{\eta} + \sum_{n=1}^{\infty} \frac{C'_\tau(0, g_{00}^{-n}(w))}{\eta^{n+1}}.$$

It follows that

$$2\eta B'_\tau(0,w)\frac{\partial \alpha}{\partial \tau}(g_{0\tau}(w))\Big|_{\tau=0} = 2B'_\tau(0,w)C'_\tau(0,w) + 2B'_\tau(0,w)\sum_{n=1}^{\infty} \frac{C'_\tau(0, g_{00}^{-n}(w))}{\eta^{n}}.$$

First, we evaluate the term

$$\mathcal{F}(w) = D'_\tau(0,w)^2 - D''_{\tau\tau}(0,w) + 2B'_\tau(0,w)C'_\tau(0,w).$$

Using (5.4), we find that

$$\begin{aligned}\mathcal{F}(w) &= \big(r\tilde{\rho}_r \sin\theta\cos\theta\big)^2 + \big(\tilde{\rho}^2 + 2r\tilde{\rho}\tilde{\rho}_r \sin^2\theta\big) - 2\big(\tilde{\rho} + r\tilde{\rho}_r \sin^2\theta\big)\big(\tilde{\rho} + r\tilde{\rho}_r\cos^2\theta\big) \\ &= -\tilde{\rho}^2 - \big(r\tilde{\rho}_r \sin\theta\cos\theta\big)^2 - 2r\tilde{\rho}\tilde{\rho}_r\cos^2\theta.\end{aligned} \tag{5.9}$$

Recall that $\Omega_2 = B^{uc}(\bar{q}, \varepsilon_0) \times B^s(\bar{q}, \varepsilon_0)$ and $\tilde{\rho}(r) = 0$ if $r \geq \varepsilon$. We have

$$\int_{\Omega_2} 2r\tilde{\rho}\tilde{\rho}_r \cos^2\theta\, dw = \int_{-\varepsilon_0}^{\varepsilon_0} dy \int_0^{2\pi} 2\cos^2\theta\, d\theta \int_0^{\varepsilon} r^2\tilde{\rho}\tilde{\rho}_r\, dr. \tag{5.10}$$

Since $0 = \tilde{\rho}(0) = \tilde{\rho}(\varepsilon)$ (by the definition of the function ρ), we find that

$$\int_0^{\varepsilon} r^2\tilde{\rho}\tilde{\rho}_r\, dr = \frac{1}{2} r^2\tilde{\rho}^2\Big|_0^{\varepsilon} - \int_0^{\varepsilon} r\tilde{\rho}^2\, dr = -\int_0^{\varepsilon} r\tilde{\rho}^2\, dr. \tag{5.11}$$

We also have that

$$\int_0^{2\pi} 2\cos^2\theta\, d\theta = \int_0^{2\pi} d\theta. \tag{5.12}$$

It follows from (5.10) - (5.12) that

$$-\int_{\Omega_2} 2r\tilde{\rho}\tilde{\rho}_r\cos^2\theta\,dw = \int_{\Omega_2} r\tilde{\rho}^2\,dw \le \varepsilon\int_{\Omega_2}\tilde{\rho}^2\,dw. \tag{5.13}$$

Arguing similarly one can show that

$$-\int_{\Omega_2} r\tilde{\rho}_r\sin\theta\cos\theta\,dw = -\frac{1}{8}\int_{\Omega_2}(r\tilde{\rho})^2\,dw \tag{5.14}$$

Thus we conclude using (5.9), (5.13), and (5.14) that

$$\int_{\Omega_2}\mathcal{F}(0,w)\,dw \le -(1-\varepsilon)\int_{\Omega_2}\tilde{\rho}^2\,dw - \frac{1}{8}\int_{\Omega_2}(r\tilde{\rho})^2\,dw < 0. \tag{5.15}$$

We now evaluate the remaining term

$$\mathcal{G}(0,w) = \sum_{n=1}^{\infty}\frac{1}{\eta^i}\int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0,g_{00}^{-n}(w))\,dw.$$

Since the map $g_{00} = F$ preserves the Riemannian volume we obtain that

$$\begin{aligned}\int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0,g_{00}^{-n}(w))\,dw &\le \int_{\Omega_2} B'_\tau(0,w)^2\,dw + \int_{\Omega_2} C'_\tau(0,g_{00}^{-n}(w))^2\,dw\\ &= \int_{\Omega_2} B'_\tau(0,w)^2\,dw + \int_{\Omega_2} C'_\tau(0,w)^2\,dw\end{aligned}$$

Applying (5.4), we find that

$$\begin{aligned}&\int_{\Omega_2} B'_\tau(0,w)^2\,dw + \int_{\Omega_2} C'_\tau(0,w)^2\,dw\\ =&\int_{\Omega_2}(\tilde{\rho} + r\tilde{\rho}_r\sin^2\theta\,dw + \int_{\Omega_2}(\tilde{\rho} + r\tilde{\rho}_r\cos^2\theta\,dw\\ \le& 4\left(\int_{\Omega_2}\tilde{\rho}^2\,dw + \int_{\Omega_2} r^2\tilde{\rho}_r^2\,dw\right).\end{aligned}$$

It follows that for sufficiently large $N > 0$ (which does not depend on ε)

$$\sum_{i=N}^{\infty}\frac{1}{\eta^i}\int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0,g_{00}^{-i}(w))\,dw \le \frac{1}{10}\left(\int_{\Omega_2}\tilde{\rho}^2\,dw + \int_{\Omega_2} r^2\tilde{\rho}_r^2\,dw\right). \tag{5.16}$$

Note that if $g_{00}^{-n}\Omega_2\cap\Omega_2 = \emptyset$, then $B'_\tau(0,w)C'_\tau(0,g_{00}^{-n}(w)) = 0$ for all w. Hence,

$$\int_{\Omega_2} 2B'_\tau(0,w)C'_\tau(0,g_{00}^{-n}(w))\,dw = 0.$$

We may choose the point q and a small ε such that $g_{00}^{-n}\Omega_2\cap\Omega_2 = F^{-n}\Omega_2\cap\Omega_2 = \emptyset$ for all $n = 1, 2, \ldots, N$. It follows from (5.8), (5.15), and (5.16) that

$$\frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0} = \int_{\Omega_2}\mathcal{F}(0,w)\,dw + \int_{\Omega_2}\mathcal{G}(0,w)\,dw \le -(\frac{9}{10}-\varepsilon)\int_{\Omega_2}\tilde{\rho}^2\,dw - \frac{1}{40}\int_{\Omega_2} r^2\tilde{\rho}_r^2\,dw < 0.$$

The desired result follows. $\square$

Sublemma 2.3.

$$\frac{\partial\alpha}{\partial\tau}(g_{0\tau}(w))\Big|_{\tau=0} = \sum_{n=0}^{\infty}\frac{C'_\tau(0, g_{00}^{-n}(w))}{\eta^{n+1}}.$$

Proof of Sublemma 2.3. Define

$$R(\tau, w, \alpha) = \frac{C(\tau, w) + D(\tau, w)\alpha}{\eta\big(A(\tau, w) + B(\tau, w)\alpha\big)}.$$

Clearly,

$$\alpha_\tau(g_{0\tau}(w)) = R(\tau, w, \alpha_\tau(w)). \tag{5.17}$$

By (5.6), we have

$$\frac{\partial R}{\partial\tau}\Big|_{\tau=0} = \frac{\big(C'_\tau + D'_\tau\alpha\big)\big(A + B\alpha\big) + \big(C + D\alpha\big)\big(A'_\tau + B'_\tau\alpha\big)}{\eta\big(A + B\alpha\big)^2}\Big|_{\tau=0} = \frac{C'_\tau(0, w)}{\eta}.$$

Since $A(0, w), B(0, w), C(0, w)$, and $D(0, w)$ are constant functions over $w = (x, y, z)$ we obtain that

$$\frac{\partial H}{\partial x}\Big|_{\tau=0} = \frac{\partial H}{\partial z}\Big|_{\tau=0} = 0$$

for $H = A, B, C, D$. This implies that

$$\frac{\partial R}{\partial x}\Big|_{\tau=0} = \frac{\partial R}{\partial z}\Big|_{\tau=0} = 0.$$

Since $AD - BC = 1$,

$$\frac{\partial R}{\partial\alpha}\Big|_{\tau=0} = \frac{AD - BC}{\eta(A + B\alpha)^2}\Big|_{\tau=0} = \frac{1}{\eta}.$$

It follows from (5.17) that

$$\frac{\partial\alpha}{\partial\tau}(g_{0\tau}(w))\Big|_{\tau=0} = \frac{C'_\tau(0, w)}{\eta} + \frac{1}{\eta}\cdot\frac{\partial\alpha}{\partial t}(w)\Big|_{\tau=0}.$$

Using (5.17) again, we also obtain that

$$\alpha_\tau(w) = R(\tau, g_{0\tau}^{-1}(w), \alpha_\tau(g_{0\tau}^{-1}(w)))$$

and hence,

$$\frac{\partial\alpha}{\partial\tau}(w)\Big|_{\tau=0} = \frac{C'_\tau(0, g_{0\tau}^{-1}(w))}{\eta} + \frac{1}{\eta}\cdot\frac{\partial\alpha}{\partial\tau}(g_{0\tau}^{-1}(w))\Big|_{\tau=0}.$$

Therefore the result follows by induction. □

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Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

E-mail address: dolgopyat@math.psu.edu, hu@math.psu.edu, pesin@math.psu.edu