ASYMPTOTIC EXPANSION OF CORRELATION FUNCTIONS FOR $\mathbb{Z}^d$ COVERS OF HYPERBOLIC FLOWS.

DMITRY DOLGOPYAT, PÉTER NÁNDORI, AND FRANÇOISE PÈNE

Abstract. We establish expansion of every order for the correlation function of sufficiently regular observables of $\mathbb{Z}^d$ extensions of some hyperbolic flows. Our examples include the $\mathbb{Z}^2$ periodic Lorentz gas and geodesic flows on abelian covers of compact manifolds with negative curvature.

1. Introduction

1.1. **Setup.** Let (M,ν,T) be a probability preserving dynamical system. Consider $(\tilde{M},\tilde{\nu},\tilde{T})$– the $\mathbb{Z}^d$-extension of (M,ν,T) by $\kappa : M \to \mathbb{Z}^d$ for a positive integer d. Let $(\Phi_t)_{t\geq 0}$ be the suspension semiflow over (M,ν,T) with roof function $\tau : M \to (0,+\infty)$ and let $(\widetilde{\Phi}_t)_{t\geq 0}$ be the corresponding $\mathbb{Z}^d$ cover. That is, $(\widetilde{\Phi}_t)_{t\geq 0}$ is the semi-flow defined on

$$\widetilde{\Omega} := \{(x,\ell,s) \in M \times \mathbb{Z}^d \times [0,+\infty) \,:\, s \in [0,\tau(x))\}$$

such that $\widetilde{\Phi}_t(x,\ell,s)$ corresponds to $(x,\ell,s+t)$ by identifying (x,ℓ,s) with $(Tx,\ell+\kappa(x),s-\tau(x))$. This semi-flow preserves the restriction $\tilde{\mu}$ on $\widetilde{\Omega}$ of the product measure $\nu\otimes\mathfrak{m}\otimes\mathfrak{l}$, where $\mathfrak{m}$ is the counting measure on $\mathbb{Z}^d$ and $\mathfrak{l}$ is the Lebesgue measure on $[0,+\infty)$.

In the present paper we study the following correlation functions

$$C_t(f,g) := \int_{\widetilde{\Omega}} f.g\circ\widetilde{\Phi}_t \, d\tilde{\mu}\,,$$

as t goes to infinity, for suitable observables f,g. Our goal is to establish expansions of the form

$$C_t(f,g) = \sum_{k=0}^{K} C_k(f,g)\, t^{-\frac{d}{2}-k} + o(t^{-\frac{d}{2}-K})\,. \tag{1.1}$$

More precisely we assume that Φ_t is C^∞ away from singularities, which is a finite (possibly empty) union of positive codimension submanifolds. We say that $\tilde{\Phi}_t$ *admits a complete asymptotic expansion in inverse powers of* t if for f and g which are C^∞ and have compact support which is disjoint from the singularities of $\tilde{\Phi}$, the correlation function $C_t(f,g)$ admits the expansion (1.1) for each $K\in\mathbb{N}$. In this paper we establish a complete asymptotic expansion in inverse powers of t for two classical examples of hyperbolic systems: Lorentz gas and geodesic flows on abelian covers of negatively curved manifolds. In fact, our results are more general. Namely,

- we consider an abstract setup potentially applicable to other hyperbolic flows;
- we allow the support of $\mathfrak{f}$ and $\mathfrak{g}$ to be unbounded (provided they decay sufficiently fast);
- we allow $\mathfrak{f}$ and $\mathfrak{g}$ to take non-zero values on the singularities of the flow. In addition, we allow them to be only Hölder continuous (note that continuity is required in the flow direction as well) with one of them being C^∞ in the flow direction.

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1.2. **Related results.** The correlation function (1.1) has been studied by several authors. The leading term ($K = 0$) for hyperbolic maps (for functions of non-zero integral) is sometimes called mixing, Krickeberg mixing or local mixing. In case of $\mathbb{Z}^d$ extensions as above, it is a consequence of some versions of the local limit theorem. See related results in e.g. [1, 14–16, 29]. Less is known about higher order expansions for maps, but see the recent results in [28]. For flows, the leading term has been studied in e.g. [2, 9, 17, 30]. We also mention that there are other quantities besides the correlation functions whose asymptotic expansions are of interest. In particular, the asymptotic expansions have been obtained (using techniques similar to ones employed in the present paper) for the rate of convergence in the central limit theorem [12] and for the number of periodic orbits in a given homology class [21, 27].

There are several other results for some hyperbolic systems preserving an infinite measure which may not be a $\mathbb{Z}^d$ cover and so the powers may be different from $-\frac{d}{2} - k$. See the leading term in e.g. [10, 25, 26] and expansions in e.g. [20, 23, 24]. We note that the expansions in the above papers are of the form $\phi(t)\tilde{\mu}(f)\tilde{\mu}(g)$ where $\phi(t)$ admits an expansion of the form $\phi(t) = \sum_{k=1}^{K} a_k t^{-\beta_k} + o\left(t^{-\beta_K}\right)$. Thus these expansions do not give the leading term in the case where $\tilde{\mu}(f)\tilde{\mu}(g) = 0$ and they are not suitable for studying the limiting behavior of ergodic sums of zero mean functions. In contrast, our expansion provides the leading term for many observables of zero mean.

1.3. **Layout of the paper.** The rest of the paper is organized as follows. In Section 2, we present some abstract results on expansion of correlation functions for general suspension semi-flows and flows. Theorems 2.1 and 2.2 guarantee that under a list of technical assumptions, expansions of the kind (1.1) hold. The results are proved by a careful study of the twisted transfer operator. One major difference from the case of maps (cf. [28]) is the extra assumption (2.32) (along the lines of [8]). In Section 3 we study billiards and verify the abstract assumptions of Theorem 2.2 for the Lorentz gas obtaining a complete asymptotic expansions in inverse powers of t for that system. In Section 4, we verify the abstract assumptions for geodesic flows on $\mathbb{Z}^d$ covers of compact negatively curved Riemannian manifolds. Some technical computations are presented in the Appendix.

2. Abstract results.

2.1. **Notations.** We will work with symmetric multilinear forms. Let $\mathfrak{S}_m$ be the set of permutations of $\{1, ..., m\}$. We identify the set of symmetric m-linear forms on $\mathbb{C}^{d+1}$ with

$$\mathcal{S}_m := \left\{ A = (A_{i_1,...,i_m})_{(i_1,...,i_m)} \in \mathbb{C}^{\{1,...,d+1\}^m} \ : \ \forall i_1, ..., i_m, \ \ \forall \mathfrak{s} \in \mathfrak{S}_m, \ A_{i_{\mathfrak{s}(1)},...,i_{\mathfrak{s}(m)}} = A_{i_1,...,i_m} \right\}.$$

For any $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_k$, we define $A \otimes B$ as the element C of $\mathcal{S}_{m+k}$ such that

$$\forall i_1, ..., i_{m+k} \in \{1, ..., d+1\}, \quad C_{i_1,...,i_{m+k}} = \frac{1}{(m+k)!} \sum_{\mathfrak{s} \in \mathfrak{S}_{m+k}} A_{i_{\mathfrak{s}(1)},...,i_{\mathfrak{s}(m)}} B_{i_{\mathfrak{s}(m+1)}...,i_{\mathfrak{s}(m+k)}} .$$

Note that $\otimes$ is associative and commutative. For any $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_k$ with $k \leq m$, we define $A * B$ as the element $C \in \mathcal{S}_{m-k}$ such that

$$\forall i_1, ..., i_{m-k} \in \{1, ..., d+1\}, \quad C_{i_1,,...,i_{m-k}} = \sum_{i_{m-k+1},...,i_m \in \{1,...,d+1\}} A_{i_1,...,i_m} B_{i_{m-k+1},...,i_m}.$$

Note that when $k = m = 1$, $A * B$ is simply the scalar product $A.B$. For any C^m-smooth function $F : \mathbb{C}^{d+1} \to \mathbb{C}$, we write $F^{(m)}$ for its differential of order m, which is identified with a

m-linear form on $\mathbb{C}^{d+1}$. We write $A^{\otimes k}$ for the product $A \otimes ... \otimes A$. With these notations, Taylor expansions of F at 0 are simply written

$$\sum_{k=0}^{m} \frac{1}{k!} F^{(k)}(0) * x^{\otimes k}\,.$$

It is also worth noting that $A * (B \otimes C) = (A * B) * C$, for every $A \in \mathcal{S}_m$, $B \in \mathcal{S}_k$ and $C \in \mathcal{S}_\ell$ with $m \geq k + \ell$.

For any $\nu \otimes \mathfrak{l}$-integrable function $h_0 : M \times \mathbb{R} \to \mathbb{C}$, we set

$$\hat{h}_0(x, \xi) := \int_{\mathbb{R}} e^{i\xi s} h_0(x, s)\, ds\,,$$

(this quantity is well defined for ν-a.e. x).

Notations $\lambda_0^{(k)}$, $a_0^{(k)}$, $\Pi_0^{(k)}$ stand for the k-th derivatives of λ, a and Π at 0.
We write P for the Perron-Frobenius operator of T with respect to ν, which is defined by:

$$\forall f, g \in L^2(\nu), \quad \int_M Pf.g\, d\nu = \int_M f.g \circ T\, d\nu. \tag{2.1}$$

We also consider the family $(P_{\theta,\xi})_{\theta \in [-\pi,\pi]^d, \xi \in \mathbb{R}}$ of operators given by

$$P_{\theta,\xi}(f) := P\left(e^{i\,\theta\cdot\kappa} e^{i\,\xi\tau} f\right)\,. \tag{2.2}$$

To simplify notations, we write $\nu(h) := \int_M h\, d\nu$.

Let Σ be a $(d+1)$-dimensional positive symmetric matrix. We will denote by $\Psi = \Psi_\Sigma$ the $(d+1)$-dimensional centered Gaussian density with covariance matrix Σ:

$$\Psi(s) = \Psi_\Sigma(s) := \frac{e^{-\frac{1}{2}\Sigma^{-1} * s^{\otimes 2}}}{(2\pi)^{\frac{d+1}{2}} \sqrt{\det \Sigma}}\,. \tag{2.3}$$

In particular, $\Psi^{(k)}$ is the differential of Ψ of order k. Let

$$a_s := e^{-\frac{1}{2}\Sigma * s^{\otimes 2}} \tag{2.4}$$

be the Fourier transform of Ψ. Given a non-negative integer α and a real number γ, we define

$$h_{\alpha,\gamma} : \mathbb{R}^2 \to \mathcal{S}_m, \quad h_{\alpha,\gamma}(s, z) = z^\gamma \Psi^{(\alpha)}\left(\mathbf{0}, s/\sqrt{z/\nu(\tau)}\right) \tag{2.5}$$

where $\mathbf{0}$ denotes the origin in $\mathbb{R}^d$.

We will use the notations

$$\kappa_n := \sum_{k=0}^{n-1} \kappa \circ T^k \quad \text{and} \quad \tau_n := \sum_{k=0}^{n-1} \tau \circ T^k\,.$$

Note that with this notation, we have

$$\widetilde{\Phi}_t(x, \ell, s) = (T^n x, \ell + \kappa_n(x), s + t - \tau_n(x))\,, \quad \text{with } n \text{ s.t. } \tau_n(x) \leq s + t < \tau_{n+1}(x)\,.$$

It will be also useful to consider the suspension flow $(\Phi_t)_{t\geq 0}$ over (M, ν, T) with roof function τ which is defined on $\Omega := \{(x, s) \in M \times [0, +\infty) : s \in [0, \tau(x))\}$ and preserves the measure μ which is the restriction of the product measure $\nu \otimes \mathfrak{l}$ to Ω. Note that μ is a finite measure but not necessarily a probability measure.

2.2. **A general result under spectral assumptions.**

Theorem 2.1. *Assume τ uniformly bounded from above and below. Let Σ be a $(d+1)$-dimensional positive symmetric matrix. Let K and J be two positive integers such that $3 \leq J \leq K+3$. Let $\mathcal{B}$ be a Banach space of complex valued functions $f : M \to \mathbb{C}$ such that $\mathcal{B} \hookrightarrow L^1(M,\nu)$ and $\mathbf{1}_M \in \mathcal{B}$. Assume that $(P_{\theta,\xi})_{\theta\in[-\pi,\pi]^d,\xi\in\mathbb{R}}$ is a family of linear continuous operators on $\mathcal{B}$ such that there exist constants $b \in (0,\pi]$, $C>0$, $\vartheta \in (0,1)$, $\beta>0$ and three functions $\lambda_\cdot : [-b,b]^{d+1} \to \mathbb{C}$ (assumed to be C^{K+3}-smooth) and $\Pi_\cdot, R_\cdot : [-b,b]^{d+1} \to \mathcal{L}(\mathcal{B},\mathcal{B})$ (assumed to be C^{K+1}-smooth) such that $\Pi_0 = \mathbb{E}_\nu[\cdot]\mathbf{1}_M$, and $\tilde\lambda_{\theta,\xi} := \lambda_{\theta,\xi}e^{-i\xi\nu(\tau)}$ satisfies*

$$\forall k < J, \quad \tilde\lambda_0^{(k)} = a_0^{(k)}, \tag{2.6}$$

and, in $\mathcal{L}(\mathcal{B},\mathcal{B})$,

$$\forall s \in [-b,b]^{d+1}, \quad P_s = \lambda_s\Pi_s + R_s, \quad \Pi_s R_s = R_s\Pi_s = 0, \quad \Pi_s^2 = \Pi_s\,, \tag{2.7}$$

$$\sup_{s\in[-b,b]^{d+1}} \|R_s^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} + \sup_{\theta\in[-\pi,\pi]^d\setminus[-b,b]^d,\ |\xi|\leq b} \|P_{\theta,\xi}^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \leq C\vartheta^k\,. \tag{2.8}$$

Let $f,g : \widetilde\Omega \to \mathbb{C}$ be two functions. We assume that there exist two families $(f_\ell)_{\ell\in\mathbb{Z}^d}$ and $(g_\ell)_{\ell\in\mathbb{Z}^d}$ of functions defined on $M\times\mathbb{R} \to \mathbb{C}$ and vanishing outside $\widetilde\Omega_0 := \widetilde\Omega \cup \left(M \times \left[-\frac{\inf\tau}{10}, 0\right]\right)$ such that

$$\forall h \in \{f,g\} \quad \forall (x,\ell,s) \in \widetilde\Omega, \quad h(x,\ell,s) = h_\ell(x,s) + h_{\ell+\kappa(x)}(Tx, s-\tau(x))\,.$$

We assume moreover that one of these families is made of functions continuous in the last variable and that [1]

$$\int_{\mathbb{R}} \sum_{\ell\in\mathbb{Z}^d} (1+|\ell|^K)(\|f_\ell(\cdot,u)\|_{\mathcal{B}} + \|g_\ell(\cdot,u)\|_{\mathcal{B}'})\,du < \infty\,, \tag{2.9}$$

$$\exists p_0,q_0 \in [1,+\infty]\ s.t.\ \frac{1}{p_0}+\frac{1}{q_0} = 1 \quad and \quad \sum_{\ell,\ell'\in\mathbb{Z}^d} \|f_\ell\|_{L^{p_0}(\nu\otimes\mathfrak{l})}\,\|g_{\ell'}\|_{L^{q_0}(\nu\otimes\mathfrak{l})} < \infty\,, \tag{2.10}$$

$$\sup_{\xi\in\mathbb{R}} \sum_{\ell,\ell'\in\mathbb{Z}^d} \|\hat f_\ell(\cdot,-\xi)\|_{\mathcal{B}}\|\hat g_{\ell'}(\cdot,\xi)\|_{\mathcal{B}'} < \infty \tag{2.11}$$

Assume furthermore that $\hat f_\ell(\cdot,\xi) \in \mathbb{B}$ for every $\ell\in\mathbb{Z}^d$ and $\xi\in\mathbb{R}$, where $\mathbb{B}$ is a Banach space such that

$$\sup_{\theta\in[-\pi,\pi]^d} \|P_{\theta,\xi}^n\|_{\mathcal{L}(\mathbb{B},L^1)} \leq C|\xi|^\alpha e^{-n\delta|\xi|^{-\alpha}} \tag{2.12}$$

for some suitable positive C,δ,α and

$$\forall\gamma>0, \quad \sum_{\ell,\ell'\in\mathbb{Z}^d} \left(\|\hat f_\ell(\cdot,-\xi)\|_{\mathbb{B}}\,\|\hat g_{\ell'}(\cdot,\xi)\|_\infty\right) = O(|\xi|^{-\gamma})\,. \tag{2.13}$$

Then

$$C_t(f,g) = \sum_{p=0}^{\lfloor\frac{K}{2}\rfloor} \tilde C_p(f,g)\left(\frac{t}{\nu(\tau)}\right)^{-\frac{d}{2}-p} + o\left(t^{-\frac{K+d}{2}}\right)\,, \tag{2.14}$$

as $t\to+\infty$ where

$$\begin{aligned}\tilde C_p(f,g) \ :=\ & \sum \frac{1}{q!}\int_{\mathbb{R}} \partial_2^q h_{m+j+r,k-\frac{m+j+d+r+1}{2}}(s\sqrt{\nu(\tau)},1)(-s)^q\,ds \\ & *\frac{i^{m+j}}{r!m!}\left(\sum_{\ell,\ell'}\int_{\mathbb{R}^2} \nu\left(g_{\ell'}(\cdot,v)\left(\Pi_0^{(m)}(f_\ell(\cdot,u))\right)\right)\otimes(\ell'-\ell,u-v)^{\otimes r}dudv\otimes A_{j,k}\right)\end{aligned} \tag{2.15}$$

[1] The notation $\|G\|_{\mathcal{B}'}$ means here $\|G\|_{\mathcal{B}'} := \sup_{F\in\mathcal{B},\ \|F\|_{\mathcal{B}}=1} |\mathbb{E}_\nu[G.F]|$.

where the first sum is taken over the nonnegative integers m, j, r, q, k satisfying

$$m + j + r + q - 2k = 2p \quad \text{and } j \geq kJ$$

and $\partial_2^q h_{\alpha,\gamma}$ denotes the derivative of order q with respect to the second variable of $h_{\alpha,\gamma}$ (defined by (2.5)) and $A_{j,k} \in \mathcal{S}_j$ is given by (A.2) of Appendix A for $k > 0$, $A_{0,0} = 1$ and $A_{j,0} = 0$ for $j > 0$.

Proof of Theorem 2.1. **Step 1: Fourier transform.**

Notice that

$$C_t(f,g) = \sum_{\ell,\ell'} \sum_{n\geq 0} \int_{M\times\mathbb{R}} f_\ell(x,s)\, g_{\ell'}\left(T^n x, s+t-\tau_n(x)\right) \mathbf{1}_{\{\kappa_n(x)=\ell'-\ell\}}\ d(\nu\otimes \mathrm{l})(x,s)\,, \tag{2.16}$$

due to the dominated convergence theorem, (2.10) and the fact that the sum over n is compactly supported, as explained below. Indeed $g_{\ell'}(T^n x, s+t-\tau_n(x)) \neq 0$ implies that

$$-\frac{\inf\tau}{10} \leq s+t-\tau_n(x) < \tau(T^n x), \text{ i.e. } \tau_n(x) - \frac{\inf\tau}{10} - s \leq t < \tau_{n+1}(x) - s \text{ with } -\frac{\inf\tau}{10} \leq s < \tau(x)$$

and so the sum over n in (2.16) is in fact is supported in $\{t_-, t_-+1, ..., t_+\}$, where

$$t_- = \lceil t/\sup\tau\rceil - 2, \quad t_+ = \lfloor t/\inf\tau\rfloor + 2\,.$$

Note that

$$\mathbf{1}_{\{\kappa_n(x)=\ell'-\ell\}} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-i\,\theta\cdot(\ell'-\ell)} e^{i\,\theta\cdot\kappa_n}\, d\theta\,. \tag{2.17}$$

Moreover, for every $x \in M$ and every positive integer n,

$$h_{\ell,\ell',x,n}(\cdot) := \int_{\mathbb{R}} f_\ell(x,s)\, g_{\ell'}\left(T^n x, s+\cdot\right)\, ds$$

is the convolution of $f_\ell(x,-\cdot)$ with $g_{\ell'}(T^n x,\cdot)$. Due to (2.10), for ν-a.e. x and any choice of ℓ, ℓ', n, this $h_{\ell,\ell',x,n}(\cdot)$ well defined. Furthermore, it is continuous (since $f_\ell(x,\cdot)$ or $g_{\ell'}(T^n x,\cdot)$ is continuous) with compact support and its Fourier transform is

$$\hat{f}_\ell(x,-\cdot)\hat{g}_{\ell'}(T^n x,\cdot) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}).$$

Consequently, $h_{\ell,\ell',x,n}$ is equal to its inverse Fourier transform, that is

$$h_{\ell,\ell',x,n}(t-\tau_n(x)) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi(t-\tau_n(x))}\hat{f}_\ell(x,-\xi)\hat{g}_{\ell'}(T^n x,\xi)\, d\xi\,.$$

Combining this with (2.16) and with (2.17), we obtain

$$\begin{aligned} &C_t(f,g) \\ &= \frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'}\sum_{n\geq 0}\int_M \left(\int_{[-\pi,\pi]^d\times\mathbb{R}} e^{-i\xi t}\hat{f}_\ell(x,-\xi)\, e^{-i\theta\cdot(\ell'-\ell)} e^{i\theta\cdot\kappa_n(x)} e^{i\xi\tau_n(x)}\hat{g}_{\ell'}\left(T^n x,\xi\right)\, d\theta d\xi\right) d\nu(x) \qquad (2.18)\\ &= \frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_M \left(\int_{[-\pi,\pi]^d\times\mathbb{R}} e^{-i\xi t} e^{-i\theta\cdot(\ell'-\ell)} P^n_{\theta,\xi}\left(\hat{f}_\ell(\cdot,-\xi)\right)\hat{g}_{\ell'}\left(\cdot,\xi\right)\, d\theta d\xi\right) d\nu\,, \qquad (2.19)\end{aligned}$$

where we used the fact that $P^n(e^{i\theta\cdot\kappa_n + i\xi\tau_n}F) = P^n_{\theta,\xi}F$. We split $(2\pi)^{d+1}C_t(f,g) = I_1 + I_2$ where I_1 stands the contribution of $\xi \in [-b,b]$ and I_2 stands the contribution of $|\xi| > b$.

Step 2: Reduction to the integration over a compact domain.

Here we prove that $|I_2| = o\left(t^{-\frac{K+d}{2}}\right)$. Observe that

$$|I_2| \le \sum_{\ell,\ell'} \sum_{n=t_-}^{t_+} \int_{[-\pi,\pi]^d\times([-\infty,-b]\cup[b,\infty])} \int_M |P^n_{\theta,\xi}\left(\hat f_\ell(\cdot,-\xi)\right) \hat g_{\ell'}(\cdot,\xi)|\, d\nu d\theta d\xi$$

$$\le C' t \int_{[-\pi,\pi]^d\times([-\infty,-b]\cup[b,\infty])} \left(\sup_{n\in[t_-,t_+]} \sum_{\ell,\ell'} \left\| P^n_{\theta,\xi}\left(\hat f_\ell(\cdot,-\xi)\right)\right\|_1 \|\hat g_{\ell'}(\cdot,\xi)\|_\infty \right) d\theta d\xi\,.$$

Now due to (2.12), we have

$$|I_2| \le C'' t \int_{[-\pi,\pi]^d} \int_{b<|\xi|} |\xi|^\alpha e^{-\delta t|\xi|^{-\alpha}} \sum_{\ell,\ell'} \left\|\hat f_\ell(\cdot,-\xi)\right\|_{\mathbb{B}} \|\hat g_{\ell'}(\cdot,\xi)\|_\infty\, d\xi d\theta\,.$$

We apply (2.13) to see that for any $\gamma>0$ there is $C''_\gamma>0$ such that

$$|I_2| \le C''_\gamma t \int_{b<|\xi|} e^{-\delta t|\xi|^{-\alpha}} |\xi|^{\alpha-\gamma} d\xi \le C''_\gamma t^{2+\frac{1-\gamma}{\alpha}} \int_{\mathbb{R}} e^{-\delta|u|^{-\alpha}} |u|^{\alpha-\gamma}\, du.$$

Choosing γ large, we get $|I_2| = o\left(t^{-\frac{K+d}{2}}\right)$. In the remaining part of the proof, we compute I_1.

Step 3: Expansion of the leading eigenvalue and eigenprojector.

First, we use (2.7), (2.8) and (2.11) to write

$$C_t(f,g) \simeq \frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'} \sum_n \int_{[-b,b]^{d+1}} e^{-i\xi t} e^{-i\theta\cdot(\ell'-\ell)} \lambda^n_{\theta,\xi} \nu\left(\Pi_{\theta,\xi}\left(\hat f_\ell(\cdot,-\xi)\right)\hat g_{\ell'}(\cdot,\xi)\right)\, d(\theta,\xi)\,,$$

where $\simeq$ means that the difference between the LHS and the RHS is $o\left(t^{-\frac{K+d}{2}}\right)$.

Now the change of variables $(\theta,\xi)\mapsto(\theta,\xi)/\sqrt{n}$ gives

$$C_t(f,g) \simeq \frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'} \sum_n n^{-\frac{d+1}{2}} \mathcal{I}(\ell,\ell',n)$$

where

$$\mathcal{I}(\ell,\ell',n) = \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} e^{-i\frac{\xi}{\sqrt{n}}t} e^{-i\theta\cdot\frac{\ell'-\ell}{\sqrt{n}}} \lambda^n_{(\theta,\xi)/\sqrt{n}} \nu\left(\Pi_{(\theta,\xi)/\sqrt{n}}\left(\hat f_\ell(\cdot,-\frac{\xi}{\sqrt{n}})\right)\, \hat g_{\ell'}\left(\cdot,\frac{\xi}{\sqrt{n}}\right)\right)\, d\theta\, d\xi\,.$$

Next with an error $o\left(t^{-\frac{K+d}{2}}\right)$, we can replace $\mathcal{I}(\ell,\ell',n)$ in the last sum by

$$\int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} e^{-i\frac{\xi}{\sqrt{n}}t} e^{-i\theta\cdot\frac{\ell'-\ell}{\sqrt{n}}} \lambda^n_{(\theta,\xi)/\sqrt{n}} \sum_{m=0}^{K+1} \frac{1}{m!} \nu\left(\Pi_0^{(m)}\left(\hat f_\ell\left(\cdot,-\frac{\xi}{\sqrt{n}}\right)\right)\hat g_{\ell'}\left(\cdot,\frac{\xi}{\sqrt{n}}\right)\right) * \frac{(\theta,\xi)^{\otimes m}}{n^{\frac{m}{2}}}\, d\theta\, d\xi\,. \tag{2.20}$$

Indeed, for every $u\in\mathbb{R}^{d+1}$, there exist $\omega\in[0,1]$ and $x_u=\omega u$ such that

$$\Pi_u(\cdot) = \sum_{m=0}^{K} \frac{1}{m!}\Pi_0^{(m)}(\cdot) * u^{\otimes m} + \frac{1}{(K+1)!}\Pi^{(K+1)}_{x_u}(\cdot) * u^{\otimes(K+1)}.$$

Denote

$$E_n := \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} \left|\lambda^n_{s/\sqrt{n}}\right| \left\|\Pi^{(K+1)}_{x_{s/\sqrt{n}}} - \Pi_0^{(K+1)}\right\| |s|^{K+1}\, ds\,.$$

Then $\lim_{n\to+\infty} E_n = 0$ by the Lebesgue dominated convergence theorem. Therefore

$$\lim_{t\to+\infty} t^{\frac{K+d}{2}} \sum_{n=t_-}^{t_+} n^{-\frac{d+1}{2}} \frac{E_n}{n^{\frac{K+1}{2}}} = 0\,,$$

justifying the replacement of Π by its jet.

Recalling elementary identities $a^n_{s/\sqrt{n}} = a_s$ and $a_s/a_{s/\sqrt{2}} = a_{s/\sqrt{2}}$, Lemma A.1 gives

$$\left| \tilde{\lambda}^n_{s/\sqrt{n}} - a_s \sum_{k=0}^{\lfloor (K+1)/(J-2)\rfloor} \sum_{j=kJ}^{K+1+2k} n^k A_{j,k} * (s/\sqrt{n})^{\otimes j} \right| \le a_{s/\sqrt{2}}\, n^{-\frac{K+1}{2}} (1+|s|^{K_0})\eta(s/\sqrt{n})\,,$$

with $\lim_{t\to 0}\eta(t) = 0$ and $\sup_{[-b,b]^d} |\eta| < \infty$. Let

$$E'_n := \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} a_{s/\sqrt{2}}(1+|s|^{K_0})\eta(s/\sqrt{n})\, ds\,.$$

Since the Lebesgue dominated convergence theorem gives $\lim_{n\to\infty} E'_n = 0$, the same argument as above shows that the error term arising from replacing in (2.20) $\tilde{\lambda}^n_{s/\sqrt{n}}$ by the above sum is negligible. Since $\tilde{\lambda}_{\theta,\xi} = \lambda_{\theta,\xi} e^{-i\xi\nu(\tau)}$, we conclude

$$C_t(f,g) \simeq \frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'} \sum_n n^{-\frac{d+1}{2}} \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} e^{-i\xi\frac{t-n\nu(\tau)}{\sqrt{n}}} e^{-i\theta.\frac{\ell'-\ell}{\sqrt{n}}} a_{(\theta,\xi)}$$

$$\sum_{m=0}^{K+1} \frac{1}{m!}\nu\left(\hat{g}_{\ell'}\left(\cdot,\frac{\xi}{\sqrt{n}}\right)\Pi_0^{(m)}\left(\hat{f}_\ell\left(\cdot,-\frac{\xi}{\sqrt{n}}\right)\right)\right) * \frac{(\theta,\xi)^{\otimes m}}{n^{\frac{m}{2}}} \left(\sum_{k=0}^{\lfloor (K+1)/(J-2)\rfloor} \sum_{j=kJ}^{(K+1)+2k} n^k A_{j,k} * \frac{(\theta,\xi)^{\otimes j}}{n^{\frac{j}{2}}}\right) d\theta\, d\xi.$$

Step 4. Integrating by parts.

Note that $\forall A \in \mathcal{S}_j, \forall B \in \mathcal{S}_m$ and $s \in \mathbb{C}^{d+1}$, $(B * s^{\otimes m})(A * s^{\otimes j}) = (A\otimes B) * s^{\otimes(m+j)}$. We claim that

$$\begin{aligned}
&\frac{1}{(2\pi)^{d+1}} \int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}} e^{-i\xi\frac{t-n\nu(\tau)}{\sqrt{n}} - i\theta.\frac{\ell'-\ell}{\sqrt{n}}} a_{(\theta,\xi)} \nu\left(\hat{g}_{\ell'}\left(\cdot,\frac{\xi}{\sqrt{n}}\right)\left(\Pi_0^{(m)}\left(\hat{f}_\ell\left(\cdot,-\frac{\xi}{\sqrt{n}}\right)\right)\otimes A_{j,k}\right)\right) \\
&\qquad\qquad * (\theta,\xi)^{\otimes(m+j)}\, d\theta d\xi \\
&= i^{m+j} \int_{\mathbb{R}^2} \Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}}, \frac{t-n\,\nu(\tau)+u-v}{\sqrt{n}}\right) * \nu\left(\Pi_0^{(m)}(f_\ell(\cdot,u)) g_{\ell'}(\cdot,v)\otimes A_{j,k}\right) du dv \\
&\qquad\qquad + o\left(\rho^n \sup_{\xi\in\mathbb{R}} \left\|\hat{f}_\ell(\cdot,\xi)\right\|_{\mathcal{B}} \|\hat{g}_{\ell'}(\cdot,\xi)\|_{\mathcal{B}'}\right)
\end{aligned} \tag{2.21}$$

where Ψ is defined by (2.3) and $\rho < 1$. Note that the integration in the second line of (2.21) is over a compact set since f_ℓ and $g_{\ell'}$ vanish outside of a compact set.

To prove (2.21), we first note that, due to (2.11) by making an exponentially small error we can replace the integration in the first line to $\mathbb{R}^{d+1}$. Second, we observe that $\Pi_0^{(m)}\hat{f}_\ell = \widehat{f_{m,\ell}}$ where $f_{m,l} = \Pi_0^{(m)} f_\ell$ and that $\hat{h}(\xi/\sqrt{n}) = (\widehat{\sqrt{n}h(\sqrt{n}\cdot)})(\xi)$. Third, since a is the Fourier transform of Ψ, it follows that

$$(\theta,\xi) \mapsto (-i)^{\sum_{j=1}^{d+1} k_j}\, \theta_1^{k_1}\dots\theta_d^{k_d}\xi^{k_{d+1}} a_{(\theta,\xi)} \text{ is the Fourier transform of } s \mapsto \frac{\partial^{\sum_{j=1}^{d+1} k_j}}{(\partial s_1)^{k_1}\cdots(\partial s_{d+1})^{k_{d+1}}}\Psi.$$

Fourth, we use the inversion formula for the Fourier transform. To take the inverse Fourier transform with respect to ξ we note that we have a triple product, which is a Fourier transform of the triple convolution of the form

$$i^{m+j}\int_{\mathbb{R}^2}\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\,\nu(\tau)}{\sqrt{n}}-t_1-t_2\right)*nf_{m,\ell}(\cdot,-\sqrt{n}t_1)g_{\ell'}(\cdot,\sqrt{n}t_2)dt_1dt_2.$$

Making the change of variables $u=-\sqrt{n}t_1$, $v=\sqrt{n}t_2$ we obtain (2.21).

Formula (2.21) implies that

$$C_t(f,g)\simeq\sum_{m=0}^{K+1}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\frac{i^{m+j}}{m!}\sum_{\ell,\ell'}\sum_n n^{-\frac{m+j+d+1-2k}{2}} \tag{2.22}$$

$$\int_{[-\frac{\inf\tau}{10},\sup\tau)^2}\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\,\nu(\tau)+u-v}{\sqrt{n}}\right)*\nu\left(\Pi_0^{(m)}(f_\ell(\cdot,u))g_{\ell'}(\cdot,v)\otimes A_{j,k}\right)\,dudv$$

Step 5: Simplifying the argument of Ψ.

Note that there exist $a_0,a_0',c_{m+j},c_{m+j}'>0$ such that, for every $\ell',\ell\in\mathbb{Z}^2$ and every $u,v\in(-\frac{\inf\tau}{10},\sup\tau)$,

$$\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\,\nu(\tau)+u-v}{\sqrt{n}}\right)\le c_{m+j}e^{-\frac{a_0}{n}\left((\ell'-\ell)^2+(t-n\nu(\tau)+u-v)^2\right)}\le c_{m+j}'e^{-\frac{a_0'}{n}(t-n\nu(\tau))^2}\,. \tag{2.23}$$

Combining this estimate with Lemma A.3 (with $\alpha=0$), we obtain that

$$\sup_{u,v\in(-\frac{\inf\tau}{10},\sup\tau)}\sum_{n=t_-}^{t_+}n^{-\frac{m+j+d+1-2k}{2}}\left|\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\,\nu(\tau)+u-v}{\sqrt{n}}\right)\right|=O\left(t^{-\frac{m+j+d-2k}{2}}\right)\,.$$

Therefore, the terms of (2.22) corresponding to (m,k,j) with $m+j-2k>K$ are in $o\left(t^{-\frac{K+d}{2}}\right)$ and so the third summation in (2.22) can be replaced by $\sum_{j=kJ}^{K-m+2k}$. The constraint $K-m+2k\ge kJ$ implies that we can replace the second summation in (2.22) by $\sum_{k=0}^{\lfloor K/(J-2)\rfloor}$.

Next let $p=K-m-j+2k$. We claim that we can replace $\Psi^{(m+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\,\nu(\tau)-u-v}{\sqrt{n}}\right)$ in (2.22) by

$$\sum_{r=0}^{p}\frac{1}{r!\,n^{\frac{r}{2}}}\Psi^{(m+j+r)}\left(0,\frac{t-n\,\nu(\tau)}{\sqrt{n}}\right)*(\ell'-\ell,u-v)^{\otimes r}\,.$$

Indeed by Taylor's theorem, we just need to verify that for

$$\lim_{t\to+\infty}t^{\frac{K+d}{2}}\sum_{\ell,\ell'}\int_{\mathbb{R}^2}\|f_\ell(\cdot,u)\|_{\mathcal{B}}\|g_{\ell'}(\cdot,v)\|_{\mathcal{B}'}|(\ell'-\ell,u-v)|^p\sum_{n=t_-}^{t_+}n^{-\frac{m+j+d+1-2k+p}{2}} \tag{2.24}$$

$$\sup_{x\in(0,1)}\left|\Psi^{(m+j+p)}\left(x\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\,\nu(\tau)+x(u-v)}{\sqrt{n}}\right)-\Psi^{(m+j+p)}\left(0,\frac{t-n\,\nu(\tau)}{\sqrt{n}}\right)\right|\,dudv$$

$$=\ 0\,.$$

By (2.23) and Lemma A.3

$$\sum_{n=t_-}^{t_+}n^{-\frac{m+j+d+1-2k-p}{2}}\sup_{x\in(0,1)}\left|\Psi^{(m+j+p)}\left(x\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\,\nu(\tau)+x(u-v)}{\sqrt{n}}\right)\right|$$

$$\le c_{m+j+p}'\sum_{n=t_-}^{t_+}n^{-\frac{m+j+d+1-2k+p}{2}}e^{-\frac{a_0'}{n}(t-n\,\nu(\tau))^2}=O(t^{-\frac{m+j+d-2k+p}{2}})$$

uniformly in $\ell, \ell' \in \mathbb{Z}^d$ and $u, v \in (-\frac{\inf \tau}{10}, \sup \tau)$. This combined with (2.9) shows that the LHS of (2.24) is dominated by an integrable function, so (2.24) follows by the dominated convergence theorem.

Therefore

$$C_t(f,g) \simeq \sum_{\ell,\ell'} \sum_{m=0}^{K+1} \sum_{k=0}^{\lfloor K/(J-2)\rfloor} \sum_{j=kJ}^{K-m+2k} \sum_{r=0}^{K-m-j+2k} \frac{i^{m+j}}{r!m!} \sum_{n=t_-}^{t_+} n^{-\frac{m+j+d+r+1-2k}{2}} \Psi^{(m+j+r)}\left(0, \frac{t-n\nu(\tau)}{\sqrt{n}}\right) * \int_{\mathbb{R}^2} \left(\nu\left(g_{\ell'}(\cdot,v)\left(\Pi_0^{(m)}(f_\ell(\cdot,u))\right)\right) \otimes (\ell'-\ell, +u-v)^{\otimes r} du dv \otimes A_{j,k}\right). \tag{2.25}$$

Step 6: Summing over n.

Performing the summation over n and using Lemma A.3 we obtain

$$C_t(f,g) \simeq \sum_{\ell,\ell'} \sum_{m=0}^{K+1} \sum_{k=0}^{\lfloor K/(J-2)\rfloor} \sum_{j=kJ}^{K-m+2k} \sum_{r=0}^{K-m-j+2k} \sum_{q=0}^{K+2k-m-j-r} \frac{i^{m+j}(t/\nu(\tau))^{-\frac{m+j+d+r+q-2k}{2}}}{r!m!\, q!\, (\nu(\tau))^{\frac{q+1}{2}}} \tag{2.26}$$

$$\int_{\mathbb{R}} \partial_2^q h_{m+j+r,k-\frac{m+j+d+r+1}{2}}(s,1)(-s)^q\, ds$$

$$* \left(\int_{\mathbb{R}^2} \nu\left(g_{\ell'}(\cdot,v)\left(\Pi_0^{(m)}(f_\ell(\cdot,u))\right)\right) \otimes (\ell'-\ell, u-v)^{\otimes r} du dv \otimes A_{j,k}\right).$$

Therefore $C_t(f,g) \simeq \sum_{p=0}^K \tilde{C}_{p/2}(f,g) \left(\frac{t}{\nu(\tau)}\right)^{-\frac{d+p}{2}}$ where

$$\tilde{C}_{p/2}(f,g) := \sum \frac{1}{q!} \int_{\mathbb{R}} \partial_2^q h_{m+j+r,k-\frac{m+j+d+r+1}{2}}(s\sqrt{\nu(\tau)},1)(-s)^q\, ds \tag{2.27}$$

$$* \frac{i^{m+j}}{r!m!} \left(\sum_{\ell,\ell'} \int_{\mathbb{R}^2} \nu\left(g_{\ell'}(\cdot,v)\left(\Pi_0^{(m)}(f_\ell(\cdot,u))\right)\right) \otimes (\ell'-\ell, u-v)^{\otimes r} du dv \otimes A_{j,k}\right),$$

and the first sum is taken over the nonnegative integers m, j, r, q, k satisfying $m+j+r+q-2k = p$. Applying Lemma A.4 with $b = m+j+r$, we see that $\tilde{C}_{p/2} = 0$ if p is an odd integer. This concludes the proof of Theorem 2.1. □

2.3. **A general result for hyperbolic systems.** Here we consider extensions of systems with good spectral properties.

Theorem 2.2. *Assume τ and κ uniformly bounded, and that $\inf \tau > 0$. Let Σ be a $(d+1)$-dimensional positive symmetric matrix. Let K, J be two integers such that, $3 \le J \le L = K+3$. Let $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be a complex Banach space of functions $f : M \to \mathbb{C}$ such that $\mathcal{V} \hookrightarrow L^\infty(\nu)$. Assume that (M, ν, T) is an extension, by $\mathfrak{p} : M \to \bar{\Delta}$, of a dynamical system $(\bar{\Delta}, \bar{\nu}, \bar{T})$ with Perron-Frobenius operator $\bar{P}$ and that there exists a Banach space $\mathcal{B}$ of complex functions $f : \bar{\Delta} \to \mathbb{C}$ such that $\mathcal{B} \hookrightarrow L^1(\bar{\Delta}, \bar{\nu})$ and $\mathbf{1}_{\bar{\Delta}} \in \mathcal{B}$. Assume moreover that the following conditions hold true:*

- *there exist a positive integer m_0 and a $\bar{\nu}$-centered bounded function $\bar{\kappa} : \bar{\Delta} \to \mathbb{Z}^d$ such that $\bar{\kappa} \circ \mathfrak{p} = \kappa \circ T^{m_0}$,*
- *there exist $\beta_0 \ge 0$, a function $\bar{\tau} : \bar{\Delta} \to \mathbb{R}$ and a function $\chi : M \to \mathbb{R}$ s.t. $\tau = \bar{\tau} \circ \mathfrak{p} + \chi - \chi \circ T$ and for every $\xi \in \mathbb{R}$, we have $e^{i\xi\chi} \in \mathcal{V}$ with $\left\|e^{i\xi\chi}\right\|_{\mathcal{V}} = O\left(|\xi|^{\beta_0}\right)$ and $(\bar{\tau}_{m_0})^q e^{-i\xi\bar{\tau}_{m_0}} \in \mathcal{B}$ for every $q \le L$.*
- *$(\bar{P}_{\theta,\xi} : \bar{f} \mapsto \bar{P}(e^{i\theta\cdot\bar{\kappa}} e^{i\xi\bar{\tau}} \bar{f}))_{(\theta,\xi)\in[-\pi,\pi]^d\times\mathbb{R}}$ is a family of linear continous operators on $\mathcal{B}$ such that*

$$\sup_{\theta,\xi,n} \|\bar{P}^n_{\theta,\xi}\| < \infty, \tag{2.28}$$

and there exist constants $b \in (0,\pi]$, $C>0$, $\vartheta \in (0,1)$, $\beta>0$ *and three functions* $\lambda_\cdot : [-b,b]^{d+1} \to \mathbb{C}$ *and* $\Pi_\cdot, R_\cdot : [-b,b]^{d+1} \to \mathcal{L}(\mathcal{B},\mathcal{B})$ *(assumed to be* C^L*-smooth) such that*

$$\tilde\lambda_{\theta,\xi} := \lambda_{\theta,\xi} e^{-i\xi\nu(\tau)} = 1 - \frac{1}{2}\Sigma*(\theta,\xi)^{\otimes 2} + o(|(\theta,\xi)|^2), \quad \text{as } (\theta,\xi)\to 0\,, \tag{2.29}$$

$\lambda_0 = 1$ *and* $\Pi_0 = \mathbb{E}_{\bar\nu}[\cdot]\mathbf{1}_{\bar\Delta}$ *and such that, in* $\mathcal{L}(\mathcal{B},\mathcal{B})$,

$$\forall s \in [-b,b]^{d+1}, \quad \bar P_s = \lambda_s \Pi_s + R_s, \quad \Pi_s R_s = R_s \Pi_s = 0, \quad \Pi_s^2 = \Pi_s\,, \tag{2.30}$$

$$\forall k \in \mathbb{N} \sup_{m=0,\dots,L} \sup_{s\in[-b,b]^{d+1}} \|(R_s^k)^{(m)}\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} + \sup_{\theta\in[-\pi,\pi]^d\setminus[-b,b]^d,\ |\xi|\le b} \|\bar P_{\theta,\xi}^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \le C\vartheta^k. \tag{2.31}$$

Furthermore, there is a Banach space $\mathbb{B}$ *such that*

$$\exists C,\delta,\alpha>0, \quad \sup_{\theta\in[-\pi,\pi]^d} \|\bar P_{\theta,\xi}^n\|_{\mathcal{L}(\mathbb{B},L^1)} \le C|\xi|^\alpha e^{-n\delta|\xi|^{-\alpha}}\,, \tag{2.32}$$

and $\forall k<J$, $\tilde\lambda_0^{(k)} = a_0^{(k)}$ *where* a_s *is given by* (2.4).

- *there exist* $C_0>0$ *and* $\vartheta\in(0,1)$ *and continuous linear maps* $\mathbf{\Pi}_n : \mathcal{V} \to \mathcal{B}\cap\mathbb{B}$, *such that, for every* $f\in\mathcal{V}$ *and every integer* $n\ge m_0$ *and for any* $\theta\in[-\pi,\pi]^d, \xi\in\mathbb{R}$ *and for any non-negative integer* $j=0,\dots,L$,

$$\|f\circ T^n - \mathbf{\Pi}_n(f)\circ\mathfrak{p}\|_\infty \le C_0\|f\|_{\mathcal{V}}\,\vartheta^n\,, \tag{2.33}$$

$$\left\|\bar P_{\theta,\xi}^{2n}(e^{-i\theta.\bar\kappa_{n-m_0}-i\xi.\bar\tau_n}\mathbf{\Pi}_n f)\right\|_{\mathbb{B}} \le C_0(1+|\xi|)\|f\|_{\mathcal{V}}\,, \tag{2.34}$$

$$\left\|\frac{\partial^j}{\partial(\theta,\xi)^j}(\bar P_{\theta,\xi}^{2n}(e^{-i\theta.\bar\kappa_{n-m_0}-i\xi.\bar\tau_n}\mathbf{\Pi}_n f))\right\|_{\mathcal{B}} \le C_0 n^j(1+|\xi|)\|f\|_{\mathcal{V}}\,, \tag{2.35}$$

$$\left\|\frac{\partial^j}{\partial(\theta,\xi)^j}(\mathbf{\Pi}_n(f)e^{i\theta\cdot\bar\kappa_{n-m_0}+i\xi.\bar\tau_n})\right\|_{\mathcal{B}'} \le C_0 n^j\|f\|_{\mathcal{V}}\,, \tag{2.36}$$

with $\bar\kappa_n := \sum_{k=0}^{n-1}\bar\kappa\circ\bar T^k$ *and* $\bar\tau_n := \sum_{k=0}^{n-1}\bar\tau\circ\bar T^k$.

Let $f,g : \widetilde\Omega \to \mathbb{C}$ *such that*

$$\forall h\in\{f,g\} \quad \forall (x,\ell,s)\in\widetilde\Omega, \quad h(x,\ell,s) = h_\ell(x,s) + h_{\ell+\kappa(x)}(Tx, s-\tau(x))\,, \tag{2.37}$$

where $(f_\ell)_{\ell\in\mathbb{Z}^d}$ *and* $(g_\ell)_{\ell\in\mathbb{Z}^d}$ *are two families of functions defined on* $M\times\mathbb{R}\to\mathbb{C}$ *and vanishing outside* $\widetilde\Omega_0 := \widetilde\Omega \cup \left(M\times\left[-\frac{\inf\tau}{10},0\right]\right)$. *We assume moreover that one of these families is made of functions continuous in the last variable and that there exists* β_0 *such that* $\xi\mapsto e^{i\xi.\chi}\hat f_\ell(\cdot,\xi)$ *and* $\xi\mapsto e^{i\xi.\chi}\hat g_\ell(\cdot,\xi)$ *are* C^L *from* $\mathbb{R}$ *to* $\mathcal{V}$ *and for every* $k=0,\dots,L$,

$$\sup_{|\xi|\le b}\sum_{\ell\in\mathbb{Z}^d}\left(\left\|\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi.\chi}\hat f_\ell(\cdot,\xi)\right)\right\|_{\mathcal{V}} + \left\|\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi.\chi}\hat g_\ell(\cdot,\xi)\right)\right\|_{\mathcal{V}}\right) < \infty\,, \tag{2.38}$$

$$\sum_\ell \int_{\mathbb{R}} (1+|\ell|)^K \left(\|f_\ell(\cdot,u)\|_{\mathcal{V}} + \|g_\ell(\cdot,u)\|_{\mathcal{V}}\right) du < \infty\,, \tag{2.39}$$

$$\forall\gamma>0, \quad \sum_{\ell,\ell'}\left(\|e^{i\xi.\chi}\hat f_\ell(\cdot,-\xi)\|_{\mathcal{V}}\,\|e^{-i\xi.\chi}\hat g_{\ell'}(\cdot,\xi)\|_{\mathcal{V}}\right) = O(|\xi|^{-\gamma})\,. \tag{2.40}$$

$$\sum_{\ell\in\mathbb{Z}^d}\|f_\ell\|_\infty<\infty \quad \text{or} \quad \sum_{\ell\in\mathbb{Z}^d}\|g_\ell\|_\infty<\infty\,, \tag{2.41}$$

Then

$$C_t(f,g) = \sum_{p=0}^{\lfloor\frac{K}{2}\rfloor}\tilde C_p(f,g)\left(\frac{t}{\nu(\tau)}\right)^{-\frac{d}{2}+p} + o\left(t^{-\frac{K+d}{2}}\right)\,,$$

as $t \to +\infty$, where

$$\tilde{C}_p(f,g) = \sum \frac{1}{q!} \frac{1}{(\nu(\tau))^{\frac{q+1}{2}}} \int_{\mathbb{R}} \partial_2^q h_{m+j+r,k-\frac{m+j+d+r+1}{2}}(s,1)(-s)^q \, ds \tag{2.42}$$
$$* \frac{i^{m+j}}{r!m!} \left(\sum_{\ell,\ell'} \int_{\mathbb{R}^2} \mathcal{B}_m \left(f_\ell(\cdot,u), g_{\ell'}(\cdot,v) \right) \otimes (\ell'-\ell, u-v)^{\otimes r} \, du dv \otimes A_{j,k} \right),$$

where the first sum is taken over the nonnegative integers m, j, r, q, k satisfying

$$m + j + r + q - 2k = 2p \quad \text{and } j \geq kJ,$$

h is defined in (2.5)*, $A_{j,k}$ for $k > 0$ are the multilinear forms given by equation* (A.2) *from Appendix A, $A_{0,0} = 1$ and $A_{j,0} = 0$ for $j > 0$ and $\mathcal{B}_m : \mathcal{V} \times \mathcal{V} \to \mathcal{S}_m$ are bilinear forms defined in* (2.43) *below.*

To define $\mathcal{B}_m$ we need the following preliminary lemma, the proof of which is given at the end of this section, after the proof of Theorem 2.2.

Lemma 2.3. *Under the assumptions of Theorem 2.2, let $u, v : M \times ([-\pi,\pi]^d] \times \mathbb{R}) \to \mathbb{C}$ be two functions such that $(\theta,\xi) \mapsto e^{-i\xi\chi} u(\cdot,\theta,\xi)$ and $(\theta,\xi) \mapsto e^{-i\xi\chi} v(\cdot,\theta,\xi)$ are L times differentiable at 0 as functions from $[-\pi,\pi]^d \times \mathbb{R}$ to $\mathcal{V}$.*
Then, for every integer $N = 0, ..., L$, the quantity

$$\mathcal{A}_N(u,v) := \lim_{n\to+\infty} \left(\mathbb{E}_\nu \left[u(\cdot,-\theta,-\xi) e^{i\theta\cdot\kappa_n + i\xi\tau_n} v(T^n(\cdot),\theta,\xi) \right] \lambda_{\theta,\xi}^{-n} \right)^{(N)}_{|(\theta,\xi)=0}$$

is well defined and satisfies

$$|\mathcal{A}_N(u,v)| = O\left(\|u\|_{\mathcal{W},+} \, \|v\|_{\mathcal{W},-} \right).$$

Moreover for each $\bar{L} \in \mathbb{N}$ we have

$$\left| \mathcal{A}_N(u,v) - \left(\mathbb{E}_\nu \left[u(\cdot,-\theta,-\xi) e^{e^{i\theta\cdot\kappa_n + i\xi\tau_n}} v(\bar{T}^n(\cdot),\theta,\xi) \right] \lambda_{\theta,\xi}^{-n} \right)^{(N)}_{|(\theta,\xi)=0} \right| = O\left(\|u\|_{\mathcal{W},+} \, \|v\|_{\mathcal{W},-} n^{-\bar{L}} \right)$$

with

$$\|u\|_{\mathcal{W},\pm} := \sum_{m=0}^{L} \left\| \left(e^{-i\xi\chi} u(\cdot,\theta,\xi) \right)^{(m)}_{|(\theta,\xi)=0} \right\|_{\mathcal{V}} < \infty.$$

We let $\mathcal{B}_m$ to be the restriction of $\mathcal{A}_m$ on the space of functions depending on neither θ nor ξ. Thus

$$\mathcal{B}_m(F,G) := \lim_{n\to+\infty} \left(\mathbb{E}_\nu \left[F(\cdot) e^{i\theta\cdot\kappa_n(\cdot) + i\xi(\tau_n(\cdot) - n\nu(\tau))} G(T^n(\cdot)) \right] \tilde{\lambda}_{\theta,\xi}^{-n} \right)^{(m)}_{|(\theta,\xi)=0}. \tag{2.43}$$

Observe that (2.42) has the same form as (2.15) with $\nu\left(G\Pi_0^{(m)}(F) \right)$ replaced by $\mathcal{B}_m(F,G)$. In fact these two quantities coincide under the assumptions of Theorem 2.1. More precisely, suppose that $(M,\nu,T) = (\bar{\Delta},\bar{\nu},\bar{T})$. Then, for $(\theta,\xi) \in [-b,b]^{d+1}$,

$$\lim_{n\to+\infty} \left(\mathbb{E}_\nu \left[F(\cdot) e^{i\theta\cdot\kappa_n(\cdot) + i\xi(\tau_n(\cdot) - n\nu(\tau))} G(T^n(\cdot)) \right] \tilde{\lambda}_{\theta,\xi}^{-n} \right) = \lim_{n\to+\infty} \left(\mathbb{E}_\nu \left[\left(P_{\theta,\xi}^n F \right) G \right] \lambda_{\theta,\xi}^{-n} \right)$$
$$= \lim_{n\to+\infty} \nu\left(G \left[\Pi_{\theta,\xi} F + \lambda_{\theta,\xi}^{-n} R_{\theta,\xi}^n F \right] \right) = \nu(G\Pi_{\theta,\xi}(F)).$$

In particular, in this case $\mathcal{B}_0(F,G) = \nu(G\Pi_0(F))$. A similar argument shows that

$$\mathcal{B}_m(F,G) = \nu(G\Pi_0^{(m)}(F)),$$

see the proof of Lemma 2.3 for details.

We also note that due to mixing of T we have

$$\mathcal{B}_0(F,G) = \nu(F)\nu(G). \tag{2.44}$$

Let us mention that $\mathcal{B}_m(F,G)$ for $m \le 3$ as well as $\lambda_0^{(k)}$ for $k \le 4$ have been computed in [28] in the case of the Sinai billiard with finite horizon with κ_n instead of $(\kappa_n, \tau_n - n\nu(\tau))$ (see Lemma 4.3 and Propositions A.3 and A.4 therein) but the formulas can be extended to the present context since (κ,τ) is dynamically Lipschitz and since the reversibility property stated in [28, Lemma 4.3] also holds for (κ,τ).

Proof of Theorem 2.2. We note that the proof of Theorem 2.2 is in many places similar to the proof of Theorem 2.1 so below we mostly concentrate on the places requiring significant modifications. We note that we could have presented Theorem 2.2 without discussing Theorem 2.1 first, however, since the formulas are quite cumbersome in the present setting we prefer to discuss the argument in the simpler setup of Theorem 2.1 first.

Decreasing the value of b if necessary, we can assume that

$$\forall s \in [-b,b]^{d+1},\ \vartheta^{\frac{1}{10L(d+1)}} \le |\lambda_s| \le a_{s/\sqrt{2}}\,, \tag{2.45}$$

where ϑ is given by (2.31). Let $k_t := \lceil (L + \frac{L+1+d}{2}) \log t / |\log\vartheta| \rceil$.

We consider $F_t, G_t : \bar\Delta \times \mathbb{Z}^d \times \mathbb{R} \to \mathbb{C}$ given by

$$\forall \ell \in \mathbb{Z}^d, \forall \xi \in \mathbb{R}, \quad F_t(\cdot,\ell,\xi) := \mathbf{\Pi}_{k_t}(e^{-i\xi\chi(\cdot)}\hat f_\ell(\cdot,\xi)) \quad \text{and} \quad G_t(\cdot,\ell,\xi) := \mathbf{\Pi}_{k_t}(e^{-i\xi\chi(\cdot)}\hat g_\ell(\cdot,\xi))\,.$$

As in (2.18), using (2.39) and (2.41), $C_t(f,g)$ is equal to

$$\frac{1}{(2\pi)^{d+1}} \sum_{\ell,\ell'} \sum_{n=t_-}^{t_+} \int_M \left(\int_{[-\pi,\pi]^d\times\mathbb{R}} e^{-i\xi t} \hat f_\ell(x,-\xi)\, e^{-i\theta\cdot(\ell'-\ell)} e^{i\theta\cdot\kappa_n(x)} e^{i\xi\tau_n(x)} \hat g_{\ell'}\left(T^n x,\xi\right)\, d\theta d\xi \right) d\nu(x)\,. \tag{2.46}$$

In order to apply the spectral method, as in the proof of Theorem 2.1, we want to reduce the integration over M in (2.46) to integration over $\bar\Delta$. Namely

$$\begin{aligned}
&\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\, e^{i\theta\cdot\kappa_n} e^{i\xi\tau_n} \hat g_{\ell'}\left(T^n\cdot,\xi\right)\right] \\
=\ &\mathbb{E}_\nu\left[e^{i\xi\chi\circ T^{k_t}}\, \hat f_\ell(T^{k_t}(\cdot),-\xi)\, e^{i\theta\cdot\bar\kappa_n\circ\bar T^{k_t-m_0}\circ\mathfrak{p}} e^{i\xi\bar\tau_n\circ\bar T^{k_t}\circ\mathfrak{p}} e^{-i\xi\chi\circ T^{k_t+n}} \hat g_{\ell'}\left(T^{k_t+n}\cdot,\xi\right)\right] \\
=\ &\mathbb{E}_\nu\Big[e^{i\xi\chi\circ T^{k_t}}\, \hat f_\ell(T^{k_t}(\cdot),-\xi) e^{-i\theta.\bar\kappa_{k_t-m_0}\circ\mathfrak{p} - i\xi.\bar\tau_{k_t}\circ\mathfrak{p}}\, e^{i\theta\cdot\bar\kappa_n\circ\mathfrak{p}} e^{i\xi\bar\tau_n\circ\mathfrak{p}} \\
&\qquad e^{i\theta.\bar\kappa_{k_t-m_0}\circ\bar T^n\circ\mathfrak{p} + i\xi.\bar\tau_{k_t}\circ\bar T^n\circ\mathfrak{p}} e^{-i\xi\chi\circ T^{k_t+n}} \hat g_{\ell'}\left(T^{k_t+n}\cdot,\xi\right)\Big] \\
=\ &\mathbb{E}_{\bar\nu}\Big[F_t(\cdot,\ell,-\xi) e^{-i\theta.\bar\kappa_{k_t-m_0} - i\xi.\bar\tau_{k_t}}\, e^{i\theta\cdot\bar\kappa_n} e^{i\xi\bar\tau_n} \\
&\qquad e^{i\theta.\bar\kappa_{k_t-m_0}\circ\bar T^n + i\xi.\bar\tau_{k_t}\circ\bar T^n} G_t\left(\bar T^n(\cdot),\ell',\xi\right)\Big] + O\left(\vartheta^{k_t} d_{\ell,\ell'}(\xi)\right),
\end{aligned} \tag{2.47}$$

with $d_{\ell,\ell'}(\xi) := \left(\|e^{i\xi.\chi}\hat f_\ell(\cdot,-\xi)\|_{\mathcal{V}}\, \|e^{-i\xi.\chi}\hat g_{\ell'}(\cdot,\xi)\|_{\mathcal{V}}\right)$ where we used

- the T-invariance of ν and the definitions of $\bar\kappa$ and $\bar\tau$ in the first equation,
- the identities $\bar\kappa_n\circ\bar T^{k_t-m_0} = \bar\kappa_n - \bar\kappa_{k_t-m_0} + \bar\kappa_{k_t-m_0}\circ\bar T^n$ and $\bar\tau_n\circ\bar T^{k_t} = \bar\tau_n - \bar\tau_{k_t} + \bar\tau_{k_t}\circ\bar T^n$ in the second one,
- (2.33) and $\mathcal{V} \hookrightarrow L^\infty(\nu)$ in the last one.

Now using the properties of Perron-Frobenius operator given by (2.1) and (2.2) we obtain

$$\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\, e^{i\theta\cdot\kappa_n} e^{i\xi\tau_n} \hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]$$

$$= \mathbb{E}_{\bar\nu}\left[\bar P^n_{\theta,\xi}(\bar F_{t,-\theta}(\cdot,\ell,-\xi))\bar G_{t,\theta}(\cdot,\ell',\xi)\right] + O\left(\vartheta^{k_t} d_{\ell,\ell'}(\xi)\right), \tag{2.48}$$

where

$$\bar F_{t,-\theta}(x,\ell,-\xi) := F_t(x,\ell,-\xi)e^{-i\theta\bar\kappa_{k_t-m_0}(x)}e^{-i\xi\bar\tau_{k_t}(x)}$$
$$\bar G_{t,\theta}(x,\ell',\xi) := G_t(x,\ell',\xi)e^{i\theta\bar\kappa_{k_t-m_0}(x)}e^{i\xi\bar\tau_{k_t}(x)}.$$

Due to (2.38) and (2.40), substituting (2.48) into (2.46) yields

$$\begin{aligned} C_t(f,g) &= \frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_{[-\pi,\pi]^d\times\mathbb{R}}\Big(e^{-i\xi t}\,e^{-i\theta\cdot(\ell'-\ell)} \\ &\qquad \mathbb{E}_{\bar\nu}\left[\bar P^{n-2k_t}_{\theta,\xi}\left(\bar P^{2k_t}_{\theta,\xi}\bar F_{t,-\theta}(\cdot,\ell,-\xi)\right)\bar G_{t,\theta}(\cdot,\ell',\xi)\right]\Big)\,d\theta d\xi + O(\vartheta^{k_t}). \end{aligned} \tag{2.49}$$

Note that (2.49) is the analogue of (2.19) (with (M,ν), $P^n_{\theta,\xi}$, $\hat f_\ell(\cdot,-\xi)$ and $\hat g_{\ell'}(\cdot,\xi)$ being replaced by $(\bar\Delta,\bar\nu)$, $\bar P^{n-2k_t}_{\theta,\xi}$, $\bar P^{2k_t}_{\theta,\xi}\bar F_{t,-\theta}(\cdot,\ell,-\xi)$ and $\bar G_{t,\theta}(\cdot,\ell',\xi)$, respectively).

Due to (2.34) and (2.35)

$$\|\bar P^{2k_t}_{\theta,\xi}\bar F_{t,-\theta}(\cdot,\ell,-\xi)\|_{\mathcal{B}} + \|\bar P^{2k_t}_{\theta,\xi}\bar F_{t,-\theta}(\cdot,\ell,-\xi)\|_{\mathbb{B}} \le 2C_0(1+|\xi|)\|e^{i\xi\chi(\cdot)}\hat f_\ell(\cdot,-\xi))\|_{\mathcal{V}}.$$

Next, we estimate

$$\begin{aligned} \|\bar G_{t,\theta}(\cdot,\ell,\xi)\|_{\mathcal{B}'} &\le \|\bar G_{t,\theta}(\cdot,\ell',\xi)\|_\infty \\ &\le \|e^{-i\xi\chi(\cdot)}\hat g_{\ell'}(\cdot,\xi)\|_\infty + \|e^{-i\xi\chi\circ T^n}\hat g_{\ell'}(T^n(\cdot),\xi) - \mathbf{\Pi}_{k_t}(e^{-i\xi\chi(\cdot)}\hat g_{\ell'}(\cdot,\xi))\circ\mathfrak{p}\|_\infty \\ &\le (1+C_0)\|e^{-i\xi\chi(\cdot)}\hat g_{\ell'}(\cdot,\xi)\|_{\mathcal{V}}, \end{aligned}$$

where we used the fact that L^∞ is continuously embedded into $\mathcal{B}'$ in the first line, the definition of G_t and the triangle inequality in the second one and (2.33) and $\mathcal{V}\hookrightarrow L^\infty(\nu)$ in the third one. Therefore, due to (2.40),

$$\forall\gamma>0,\quad \sum_{\ell,\ell'\in\mathbb{Z}^d}\|\bar P^{2k_t}_{\theta,\xi}\bar F_{t,-\theta}(\cdot,\ell,-\xi)\|_{\mathbb{B}}\|\bar G_{t,\theta}(\cdot,\ell',\xi)\|_\infty = O(|\xi|^{-\gamma}).$$

Hence, proceeding as in Step 2 of the proof of Theorem 2.1 we obtain that

$$\begin{aligned} C_t(f,g) &\simeq \frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_{[-b,b]^{d+1}} e^{-i\xi t}\,e^{-i\theta\cdot(\ell'-\ell)} \\ &\qquad \mathbb{E}_{\bar\nu}\left[\bar P^{n-2k_t}_{\theta,\xi}\left(\bar P^{2k_t}_{\theta,\xi}\bar F_{t,-\theta}(\cdot,\ell,-\xi)\right)\bar G_{t,\theta}(\cdot,\ell',\xi)\right]\,d\theta d\xi\,. \end{aligned} \tag{2.50}$$

Using (2.48) again we obtain

$$\begin{aligned} C_t(f,g) &\simeq \frac{1}{(2\pi)^{d+1}}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\int_{[-b,b]^{d+1}} e^{-i\xi t}\,e^{-i\theta\cdot(\ell'-\ell)} \\ &\qquad \mathbb{E}_{\nu}\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\,d\theta d\xi\,. \end{aligned} \tag{2.51}$$

Moreover, for every $(\theta,\xi)\in[-b,b]^{d+1}$ and every integer n satisfying $t_-\le n\le t_+$, using Taylor expansion, the following holds true

$$\begin{aligned}&\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda_{\theta,\xi}^{-n}\\ =\;&\sum_{N=0}^{L-1}\frac{1}{N!}\left(\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda_{\theta,\xi}^{-n}\right)^{(N)}_{|(\theta,\xi)=0}*(\theta,\xi)^{\otimes N}\\ &+O\left(\sup_{u\in[0,1],(\theta',\xi')=(u\theta,u\xi)}\left(\frac{\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi')\,e^{i\theta'\cdot\kappa_n}e^{i\xi'\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi'\right)\right]}{\lambda_{\theta,\xi}^{n}}\right)^{(L)}_{|(\theta',\xi')}|(\theta,\xi)|^L\right).\end{aligned}\tag{2.52}$$

Let us study the derivatives involved in this formula. First, since $\mathbf{\Pi}_{k_t}$ is linear and continuous, for every $m=0,...,L$, we have

$$\left(\mathbf{\Pi}_{k_t}\left(e^{-i\xi\chi}\hat h_\ell(\cdot,\theta,\xi)\right)\right)^{(m)}_{|(\theta,\xi)}=\mathbf{\Pi}_{k_t}\left(\left(e^{-i\xi\chi}\hat h_\ell(\cdot,\theta,\xi)\right)^{(m)}_{|(\theta,\xi)}\right).\tag{2.53}$$

Using (2.53) and (2.47) we obtain the following analogue of (2.48),

$$\left|\left(\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda_{\theta,\xi}^{-n}\right)^{(L)}_{(\theta,\xi)}\right|=$$

$$\left(\mathbb{E}_{\bar\nu}\left[\bar P_{\theta,\xi}^{n-2k_t}\left(\bar P_{\theta,\xi}^{2k_t}\left(\bar F_{t,-\theta}(\cdot,\ell,-\xi)\right)\right)\bar G_{t,\theta}(\cdot,\ell',\xi)\right]\lambda_{\theta,\xi}^{-n}\right)^{(L)}_{(\theta,\xi)}+O\left(\vartheta^{k_t}n^L\widetilde d_{\ell,\ell'}(\xi)\left|\lambda_{\theta,\xi}^{-n}\right|\right)\tag{2.54}$$

with $\widetilde d_{\ell,\ell'}(\xi):=\sup_{m,m'=0,...,L}\left(\left\|\frac{\partial^m}{\partial\xi^m}\left(e^{i\xi\cdot\chi}\hat f_\ell(\cdot,-\xi)\right)\right\|_{\mathcal V}\left\|\frac{\partial^{m'}}{\partial\xi^{m'}}\left(e^{-i\xi\cdot\chi}\hat g_{\ell'}(\cdot,\xi)\right)\right\|_{\mathcal V}\right)$.

Using (2.30), (2.35), (2.36), we find that the first term of (2.54) is bounded from above by

$$C_0^2(1+|\xi|)\sup_{m=0,...,L}k_t^m\widetilde d_{\ell,\ell'}(\xi)\left\|\left((R_{\theta,\xi}^{n-2k_t}/\lambda_{\theta,\xi}^n)+\lambda_{\theta,\xi}^{-2k_t}\right)\Pi_{\theta,\xi}\right)^{(L-m)}_{(\theta,\xi)}\right\|_{\mathcal L(\mathcal B,\mathcal B)},$$

which is in $O\left(k_t^L\widetilde d_{\ell,\ell'}(\xi)\left(\frac{\vartheta^{n-2k}}{\vartheta^{\frac{n}{10}}}+\vartheta^{-\frac{k_t}{5L(d+1)}}\right)\right)$. This observation, combined with (2.52), (2.54) and our choice of k_t yields

$$\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda_{\theta,\xi}^{-n}\tag{2.55}$$

$$\begin{aligned}=\;&\sum_{N=0}^{L-1}\frac{1}{N!}\left(\mathbb{E}_\nu\left[\hat f_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat g_{\ell'}\left(T^n\cdot,\xi\right)\right]\lambda_{\theta,\xi}^{-n}\right)^{(N)}_{|(\theta,\xi)=0}*(\theta,\xi)^{\otimes N}+O\left(n^{\frac25}\widetilde d_{\ell,\ell'}(\xi)|(\theta,\xi)|^L\right)\\ &+O\left(n^{-\frac{L+1+d}{2}}\widetilde d_{\ell,\ell'}(\xi)\left|\lambda_{\theta,\xi}^{-n}\right|\right),\end{aligned}\tag{2.56}$$

for $(\theta,\xi)\in[-b,b]^{d+1}$.

Now we apply Lemma 2.3 to conclude that (2.55) is equal to

$$\sum_{N=0}^{L-1}\frac{1}{N!}\mathcal A_N\left(\hat f_\ell,\hat g_{\ell'}\right)*(\xi,\theta)^{\otimes N}+O\left(\widetilde d_{\ell,\ell'}(\xi)\left(n^{-\frac{K+d+1}{2}}+n^{\frac25}|(\theta,\xi)|^L+n^{-\frac{L+1+d}{2}}\left|\lambda_{\theta,\xi}^{-n}\right|\right)\right).\tag{2.57}$$

Recalling the notation $a_s:=e^{-\frac12\Sigma*s^{\otimes2}}$ and Lemma A.1, we have

$$\begin{aligned}\lambda_s^n\;&=\;e^{in\xi\nu(\tau)}a_{s\sqrt n}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}n^kA_{j,k}*s^{\otimes j}\\ &+\;O\left(a_{s\sqrt n/\sqrt2}\,n^{-\frac{K+1}{2}}(1+|s\sqrt n|^{K_0})\eta(s)\right),\end{aligned}\tag{2.58}$$

where $\lim_{s\to 0}\eta(s)=0$. Note that the modulus of the dominating term of (2.57) is bounded by $O\left(\widetilde{d}_{\ell,\ell'}(\xi)\right)$ uniformly in $(\theta,\xi)\in[-b,b]^{d+1}$ and that the modulus of λ_s^n in (2.58) is bounded by $O(a_{s\sqrt{n}/\sqrt{2}})$ (the first one follows from Lemma 2.3, the second one follows from (2.45)). Thus multiplying (2.57) and (2.58) we conclude

$$\begin{aligned}
&\mathbb{E}_\nu\left[\hat{f}_\ell(\cdot,-\xi)\,e^{i\theta\cdot\kappa_n}e^{i\xi\tau_n}\hat{g}_{\ell'}\left(T^n\cdot,\xi\right)\right] \qquad (2.59)\\
=&\sum_{N=0}^{L-1}\sum_{k=0}^{\lfloor (K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\frac{e^{in\xi\nu(\tau)}a_{s\sqrt{n}}n^k}{N!}\left(\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)\otimes A_{j,k}\right)*s^{\otimes(N+j)}\\
+&\;O\left(|\lambda_s^n|\widetilde{d}_{\ell,\ell'}(\xi)\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|s|^L+n^{-\frac{L+1+d}{2}}\left|\lambda_s^{-n}\right|\right)\right)\\
+&\;O\left(\sum_{N=0}^{L-1}\frac{1}{N!}\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)*s^{\otimes N}a_{s\sqrt{n}/\sqrt{2}}\,n^{-\frac{K+1}{2}}(1+|s\sqrt{n}|^{K_0})\eta(s)\right)
\end{aligned}$$

where $s=(\theta,\xi)$. This leads to the following error term

$$\begin{aligned}
&O\left(\widetilde{d}_{\ell,\ell'}(\xi)\left(a_{s\sqrt{n}/\sqrt{2}}\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|s|^L\right)+n^{-\frac{L+1+d}{2}}\right)\right)\\
+&\;O\left(\widetilde{d}_{\ell,\ell'}(\xi)a_{s\sqrt{n}/\sqrt{2}}\,n^{-\frac{K+1}{2}}(1+|s\sqrt{n}|^{K_0})\eta(s)\right) \qquad (2.60)\\
=&\;O\left(\widetilde{d}_{\ell,\ell'}(\xi)\left(n^{-\frac{L+1+d}{2}}+a_{s\sqrt{n}/\sqrt{2}}\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|s|^L+n^{-\frac{K+1}{2}}\left(1+|s\sqrt{n}|^{K_0}\right)\eta(s)\right)\right)\right),
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}}a_{s\sqrt{n}/\sqrt{2}}\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}}|s|^L+n^{-\frac{K+1}{2}}(1+|s\sqrt{n}|^{K_0})\eta(s)\right)ds\\
=&\;n^{-\frac{d+1}{2}}\int_{\mathbb{R}^{d+1}}a_{s/\sqrt{2}}\left(n^{-\frac{K+d+1}{2}}+n^{\frac{2}{5}-\frac{L}{2}}|s|^L+n^{-\frac{K+1}{2}}(1+|s|^{K_0})\eta(s/\sqrt{n})\right)ds\\
=&\;o\left(n^{-\frac{K+2+d}{2}}\right).
\end{aligned}$$

Therefore (2.38), (2.51) and (2.59), (2.60) imply

$$C_t(f,g)\simeq\frac{1}{(2\pi)^{d+1}}\sum_{N=0}^{L-1}\frac{1}{N!}\sum_{k=0}^{\lfloor (K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\sum_{\ell,\ell'}\sum_{n=t_-}^{t_+}\mathcal{I}_{\ell,\ell',n}^{N,k,j}, \qquad (2.61)$$

where

$$\mathcal{I}_{\ell,\ell',n}^{N,k,j}=n^k\int_{[-b,b]^{d+1}}e^{-i\xi(t-n\nu(\tau))}\,e^{-i\theta\cdot(\ell'-\ell)}\left(\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)\otimes A_{j,k}\right)*(\theta,\xi)^{\otimes(N+j)}a_{\sqrt{n}(\theta,\xi)}\,d\theta d\xi\,.$$

By changing variables, we see that

$$\mathcal{I}_{\ell,\ell',n}^{N,k,j}=n^{-\frac{d+1+N+j-2k}{2}}\int_{[-b\sqrt{n},b\sqrt{n}]^{d+1}}\left(\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)\otimes A_{j,k}\right)*e^{-i\frac{\xi(t-n\nu(\tau))}{\sqrt{n}}}\,e^{-i\frac{\theta\cdot(\ell'-\ell)}{\sqrt{n}}}(\theta,\xi)^{\otimes(N+j)}a_{\theta,\xi}\,d\theta d\xi.$$

At first sight, this expression looks simpler than (2.21) since $\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)$ does not depend on ξ and so no convolution is involved when taking the inverse Fourier transform. Namely we obtain

$$\mathcal{I}_{\ell,\ell',n}^{N,k,j}\approx(2\pi)^{d+1}n^{-\frac{d+1+N+j-2k}{2}}i^{N+j}\Psi^{(N+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\nu(\tau)}{\sqrt{n}}\right)*\left(\mathcal{A}_N\left(\hat{f}_\ell,\hat{g}_{\ell'}\right)\otimes A_{j,k}\right), \qquad (2.62)$$

where $\mathcal{I}\approx\mathcal{I}'$ means that (2.61) holds for $\mathcal{I}$ and $\mathcal{I}'$ at the same time (i.e. the difference obtained when substituting $\mathcal{I}$ and $\mathcal{I}'$ to (2.61) is in $o\left(t^{-\frac{K+d}{2}}\right)$). Now recall the definition $\mathcal{B}_N$ from (2.43).

Note that the difference between $\mathcal{A}_N$ and $\mathcal{B}_N$ is that the latter one is defined for function that do not depend on ξ. Thus

$$\mathcal{A}_N\left(\hat{f}_\ell, \hat{g}_{\ell'}\right) = \sum_{m_1+m_2+m_3=N} \frac{N!}{m_1!m_2!m_3!}(-1)^{m_1}\mathcal{B}_{m_2}\left((\hat{f}(.,\ell,\xi))^{(m_1)}_{|\xi=0}, (\hat{g}(.,\ell,\xi))^{(m_3)}_{|\xi=0}\right). \quad (2.63)$$

Note that

$$(\hat{f}(x,\ell,\xi))^{(m_1)}_{|\xi=0}(\hat{g}(y,\ell,\xi))^{(m_3)}_{|\xi=0} = \int_{\mathbb{R}^2}(iu)^{m_1}(iv)^{m_3}f(x,\ell,u)g(y,\ell,v)dudv.$$

Thus (2.63) is equal to

$$\sum_{m_1+m_2+m_3=N} \frac{N!}{m_1!m_2!m_3!}\int_{\mathbb{R}^2}(0,-iu)^{\otimes m_1}\otimes(0,iv)^{\otimes m_3}\otimes\mathcal{B}_{m_2}\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)\,dudv.$$

Now using the binomial theorem, we find that (2.63) is equal to

$$\sum_{m=0}^{N}\frac{N!}{m!(N-m)!}\int_{\mathbb{R}^2}(0,i(v-u))^{\otimes N-m}\otimes\mathcal{B}_m\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv.$$

Substituting this into (2.62) and using (2.61) and the identity $(-1)^{N-m}i^{N+N-m}=i^m$, we find

$$C_t(f,g)\simeq\sum_{N=0}^{L-1}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\sum_{\ell,\ell'}\sum_{m=0}^{N}\sum_{n=t_-}^{t_+}\frac{1}{m!(N-m)!}i^{m+j}n^{-\frac{d+1+N+j-2k}{2}}$$
$$\Psi^{(N+j)}\left(\frac{\ell'-\ell}{\sqrt{n}},\frac{t-n\nu(\tau)}{\sqrt{n}}\right)*\left(\int_{\mathbb{R}^2}(0,u-v)^{\otimes N-m}\otimes\mathcal{B}_m\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv\otimes A_{j,k}\right).$$

Now proceeding as in Step 5 of the proof of Theorem 2.1 we find

$$C_t(f,g)\simeq\sum_{N=0}^{L-1}\sum_{k=0}^{\lfloor(K+1)/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\sum_{\ell,\ell'}\sum_{m=0}^{N}\sum_{r=0}^{K-N-j+2k}\sum_{n=t_-}^{t_+}\frac{i^{m+j}}{m!(N-m)!r!n^{\frac{d+1+N+j+r-2k}{2}}}$$
$$\Psi^{(N+j+r)}\left(0,\frac{t-n\nu(\tau)}{\sqrt{n}}\right)*(\ell'-\ell)^{\otimes r}\left(\int_{\mathbb{R}^2}(0,u-v)^{\otimes N-m}\otimes\mathcal{B}_m\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv\otimes A_{j,k}\right).$$

Performing summation over n as in Step 6 of the proof of Theorem 2.1 (using again Lemma A.3), we derive

$$C_t(f,g)\simeq\sum_{N=0}^{K}\sum_{k=0}^{\lfloor K/(J-2)\rfloor}\sum_{j=kJ}^{K+1+2k}\sum_{\ell,\ell'}\sum_{m=0}^{N}\sum_{r=0}^{K-N-j+2k}\sum_{q=0}^{K+2k-N-j-r}\frac{1}{m!(N-m)!r!q!}i^{m+j}$$
$$\frac{(t/\nu(\tau))^{-\frac{d+N+j+r+q-2k}{2}}}{(\nu(\tau))^{\frac{q+1}{2}}}\int_{\mathbb{R}}\partial_2^q h_{N+j+r,k-\frac{N+j+d+r+1}{2}}(s,1)(-s)^q\,ds$$
$$*(\ell'-\ell)^{\otimes r}\left(\int_{\mathbb{R}^2}(0,u-v)^{\otimes N-m}\otimes\mathcal{B}_m\left(f(\cdot,\ell,u),g(\cdot,\ell,v)\right)dudv\otimes A_{j,k}\right).$$

We will set $R=N-m+r$. The binomial theorem tells us that, m,j,k being fixed, for every $R=0,...,K-m-j+2k$, the following identity holds true

$$\sum_{(r,N):\,N-m+r=R}\frac{R!}{(N-m)!r!}(\ell'-\ell)^{\otimes r}\otimes(0,u-v)^{\otimes N-m}=(\ell'-\ell,u-v)^{\otimes R}.$$

We conclude that

$$C_t(f,g)\simeq\sum_{\ell,\ell'}\sum_{m=0}^{K}\sum_{k=0}^{\lfloor K/(J-2)\rfloor}\sum_{j=kJ}^{K-m+2k}\sum_{R=0}^{K-m-j+2k}\sum_{q=0}^{K+2k-m-j-R}\frac{i^{m+j}(t/\nu(\tau))^{-\frac{m+j+d+R+q-2k}{2}}}{R!m!\,q!\,(\nu(\tau))^{\frac{q+1}{2}}}$$

$$\int_{\mathbb{R}} \partial_2^q h_{m+j+R,k-\frac{m+j+d+R+1}{2}}(s,1)(-s)^q\,ds$$
$$* \left(\int_{\mathbb{R}^2} \mathcal{B}_m\left(f_\ell(\cdot,u), g_{\ell'}(\cdot,v)\right) \otimes (\ell'-\ell, u-v)^{\otimes R}\, du dv \otimes A_{j,k} \right).$$

This implies the theorem. □

Proof of Lemma 2.3. Let $N \in \{0,...,L\}$ be fixed. Let us prove that, for every N,

$$\left(\mathcal{A}_{N,n}(u,v) := \left(\mathbb{E}_\nu \left[u(\cdot,-\theta,-\xi) e^{i\theta\cdot\kappa_n + i\xi\tau_n} v(T^n(\cdot),\theta,\xi) \right] \lambda_{(\theta,\xi)}^{-n} \right)^{(N)}_{|(\theta,\xi)=0} \right)_n$$

is a Cauchy sequence. Observe that (2.47) is valid with k_t replaced by any integer k such that $m_0 \le k \le n$. That is, for such k we have

$$\begin{aligned} \mathcal{A}_{N,n}(u,v) &= \Big(\mathbb{E}_\nu \Big[\left(e^{i\xi\chi\circ T^k} u(T^k(\cdot),-\theta,-\xi) e^{-i\theta\bar\kappa_{k-m_0}\circ\mathfrak{p} - i\xi\bar\tau_k\circ\mathfrak{p}} \right) e^{i\theta\cdot\bar\kappa_n\circ\mathfrak{p} + i\xi\bar\tau_n\circ\mathfrak{p}} \\ &\qquad e^{i\theta\bar\kappa_{k-m_0}\circ\bar T^n\circ\mathfrak{p} + i\xi\bar\tau_k\circ\bar T^n\circ\mathfrak{p}} e^{-i\xi\chi\circ T^{n+k}} v(T^{n+k}(\cdot),\theta,\xi) \Big] \lambda_{(\theta,\xi)}^{-n} \Big)^{(N)}_{|(\theta,\xi)=0}. \end{aligned}$$

Thus, we obtain

$$\mathcal{A}_{N,n}(u,v) = \widetilde{\mathcal{A}}_{N,n}\left(\tilde U_k, \tilde V_k\right), \tag{2.64}$$

where

$$\widetilde{\mathcal{A}}_{N,n}(U,V) = \left(\mathbb{E}_\nu \left[U(\cdot,-\theta,-\xi) e^{i\theta\cdot\bar\kappa_n\circ\mathfrak{p} + i\xi\bar\tau_n\circ\mathfrak{p}} V(T^n(\cdot),\theta,\xi) \right] \lambda_{(\theta,\xi)}^{-n} \right)^{(N)}_{|(\theta,\xi)=0},$$
$$\tilde U_k(\cdot,\theta,\xi) := (e^{-i\xi\chi} u(\cdot,\theta,\xi)) \circ T^k . e^{i(\theta\cdot\bar\kappa_{k-m_0} + \xi\bar\tau_k)\circ\mathfrak{p}},$$

and

$$\tilde V_k(\cdot,\theta,\xi) := (e^{-i\xi\cdot\chi} v(\cdot,\theta,\xi)) \circ T^k . e^{i(\theta\cdot\bar\kappa_{k-m_0} + \xi\cdot\bar\tau_k)\circ\mathfrak{p}}.$$

Recall (2.33) and denote

$$U_k(\cdot,\theta,\xi) := \mathbf{\Pi}_k(e^{-i\xi\chi} u(\cdot,\theta,\xi)) . e^{i(\theta\cdot\bar\kappa_{k-m_0} + \xi\bar\tau_k)} \quad \text{and} \quad V_k(\cdot,\theta,\xi) := \mathbf{\Pi}_k(e^{-i\xi\chi} v(\cdot,\theta,\xi)) . e^{i(\theta\cdot\bar\kappa_{k-m_0} + \xi\bar\tau_k)}.$$

Since $\mathbf{\Pi}_k$ is linear and continuous and since $(\theta,\xi) \mapsto e^{-i\xi\cdot\chi} u(\cdot,\theta,\xi)$ is L times differentiable at 0 as a $\mathcal{V}$-valued function, for every $m = 0,...,L$, we have

$$\left(\mathbf{\Pi}_k \left(e^{-i\xi\chi} u(\cdot,\theta,\xi) \right) \right)^{(m)}_{|(\theta,\xi)=0} = \mathbf{\Pi}_k \left(\left(e^{-i\xi\chi} u(\cdot,\theta,\xi) \right)^{(m)}_{|(\theta,\xi)=0} \right). \tag{2.65}$$

Thus

$$\begin{aligned} &\left\| \left(e^{-i\xi\chi\circ T^k} u(T^k(\cdot),\theta,\xi) \right)^{(m)}_{|(\theta,\xi)=0} - \left(\mathbf{\Pi}_k(e^{-i\xi\chi} u(\cdot,\theta,\xi)) \right)^{(m)}_{|(\theta,\xi)=0} \circ \mathfrak{p} \right\|_\infty \\ &\qquad \le C_0 \vartheta^k \left\| \left(e^{-i\xi\chi} u(\cdot,\theta,\xi) \right)^{(m)}_{|(\theta,\xi)=0} \right\|_{\mathcal{V}} \le C_0 \vartheta^k \|u\|_{\mathcal{W},+}, \end{aligned} \tag{2.66}$$

and idem by replacing u by v (and i by $-i$). Next, observe that

$$\left\| \bar\tau_n^m + |\bar\kappa_n|^m \right\|_\infty + \left| \left(\lambda^{-n}\right)^{(m)}_{|(\theta,\xi)=0} \right| = O(n^m). \tag{2.67}$$

Combining (2.65), (2.66), and (2.67) we obtain

$$\begin{aligned} &\mathcal{A}_{N,n}(u,v) - \widetilde{\mathcal{A}}_{N,n}(U_k\circ\mathfrak{p}, V_k\circ\mathfrak{p}) = \widetilde{\mathcal{A}}_{N,n}(\widetilde U_k, \widetilde V_k) - \widetilde{\mathcal{A}}_{N,n}(U_k\circ\mathfrak{p}, V_k\circ\mathfrak{p}) \qquad (2.68) \\ = &\left(\mathbb{E}_\nu \left[e^{i\theta\cdot\kappa_n} e^{i\xi\tau_n} \left(\tilde U_k(\cdot,-\theta,-\xi)\, \tilde V_k(T^n(\cdot),\theta,\xi) - U_k(\mathfrak{p}(\cdot),-\theta,-\xi)\, V_k(\mathfrak{p}(T^n(\cdot)),\theta,\xi) \right) \right] \lambda_{\theta,\xi}^{-n} \right)^{(N)}_{|(\theta,\xi)=0} \\ = &\; O\left(n^N \vartheta^k \|u\|_{\mathcal{W},+} \|v\|_{\mathcal{W},-} \right). \end{aligned}$$

Let $k_n := \lceil \log^2 n \rceil$. Take $n' \in [n, 2n]$. Using (2.68) we obtain

$$\begin{aligned}|\mathcal{A}_{N,n}(u,v) \; &-\mathcal{A}_{N,n'}(u,v)\big| \\ &\le \left|\widetilde{\mathcal{A}}_{N,n}(U_{k_n}\circ\mathfrak{p}, V_{k_n}\circ\mathfrak{p}) - \widetilde{\mathcal{A}}_{N,n'}(U_{k_n}\circ\mathfrak{p}, V_{k_n}\circ\mathfrak{p})\right| + O\left(n^N \|u\|_{\mathcal{W},+}\,\|v\|_{\mathcal{W},-}\vartheta^{k_n}\right).\end{aligned}$$

The main term on the RHS equals to

$$\mathbb{E}_{\bar\nu}\left[\left((\lambda_t^{-n}\bar P_t^{n-2k_n} - \lambda_t^{-n'}\bar P_t^{n'-2k_n})\left(\bar P_t^{2k_n}\left(U_{k_n}(\cdot,-t)\right)\right)V_{k_n}(\cdot,t)\right)^{(N)}_{|t=0}\right]. \tag{2.69}$$

Since $\lambda_t^{-\tilde n}\bar P_t^{\tilde n-2k_n} = \lambda_t^{-2k_n}\Pi_t + \lambda_t^{-\tilde n}R_t^{\tilde n-2k_n}$ we can use the definition of $\mathcal{B}'$ to bound (2.69) by

$$\begin{aligned}&\left\|\left((\lambda_t^{-n}R_t^{n-2k_n} - \lambda_t^{-n'}R_t^{n'-2k_n})\left(\bar P_t^{2k_n}\left(U_{k_n}(\cdot,-t)\right)\right)V_{k_n}(\cdot,t)\right)^{(N)}_{|t=0}\right\|_{L^1(\bar\nu)} \le \\ \le& C_N \max_{n'\in[n,2n],1\le m_1\le N}(\lambda_t^{-n'})^{(m_1)}_{|t=0}\left(\max_{1\le m_2\le N}\left\|(R_t^{n-2k_n})^{(m_2)}_{|t=0}\right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} + \max_{1\le m_2\le N}\left\|(R_t^{n'-2k_n})^{(m_2)}_{|t=0}\right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})}\right) \\ &\times\left\|\max_{1\le m_3\le N}\left(\bar P_t^{2k_n}\left(U_{k_n}(\cdot,-t)\right)\right)^{(m_3)}_{|t=0}\right\|_{\mathcal{B}}\left\|\max_{1\le m_4\le N}V_{k_n}(\cdot,t)^{(m_4)}_{|t=0}\right\|_{\mathcal{B}'}.\end{aligned}$$

Now observe that the max over m_2 is bounded by $O(\vartheta^{n/2})$ by (2.31) and the other terms cannot grow faster than a polynomial in n. In particular, we use (2.35) to bound the max over m_3 and (2.36) to bound the max over m_4. We conclude that (2.69) is exponentially small.

Therefore, for each $\bar L \in \mathbb{N}$ we have

$$\begin{aligned}\sup_{\bar n\ge 0}|\mathcal{A}_{N,n}(u,v) - \mathcal{A}_{N,n+\bar n}(u,v)| \;&\le\; \sum_{p\ge 0}\sup_{\bar n=0,\dots,2^p n}|\mathcal{A}_{N,2^p n}(u,v) - \mathcal{A}_{N,2^p n+\bar n}(u,v)| \\ \le\left(\sum_{p\ge 0}(2^p n)^{-\bar L}\|u\|_{\mathcal{W},+}\,\|v\|_{\mathcal{W},-}\right) \;&=\; O\left(\|u\|_{\mathcal{W},+}\,\|v\|_{\mathcal{W},-}\,n^{-\bar L}\right).\end{aligned}$$

Hence $\mathcal{A}_N(u,v)$ is well defined and satisfies

$$|\mathcal{A}_{N,n}(u,v) - \mathcal{A}_N(u,v)| = O\left(\|u\|_{\mathcal{W},+}\,\|v\|_{\mathcal{W},-}\,n^{-\bar L}\right). \quad \square$$

3. Mixing expansion for the Sinai billiard flow

3.1. **Sinai billiards.** In the plane $\mathbb{R}^2$, we consider a $\mathbb{Z}^2$-periodic locally finite family of scatterers $\{O_i + \ell;\ i = 1, ..., I,\ \ell \in \mathbb{Z}^2\}$. We assume that the sets $O_i + \ell$ are disjoint, open, strictly convex and their boundaries are C^3 smooth with strictly positive curvature.

The dynamics of the Lorentz gas can be described as follows. A point particle of unit speed is flying freely in the interior of $\tilde{\boldsymbol{\mathcal{Q}}} = \mathbb{R}^2 \setminus \cup_{\ell,i}(O_i + \ell)$ and undergoes elastic collisions on $\partial\tilde{\boldsymbol{\mathcal{Q}}}$ (that is, the angle of reflection equals the angle of incidence). Throughout this paper we assume the so-called finite horizon condition, i.e. that the free flight is bounded. The same dynamics on the compact domain is called Sinai billiard. The position of the particle is a point $q \in \tilde{\boldsymbol{\mathcal{Q}}}$ and its velocity is a vector $v \in \mathcal{S}^1$ (as the speed is identically 1). Since collisions happen instantaneously, the pre-collisional and post-collisional data are identified. By convention, we use the post-collisional data, i.e. whenever $q \in \partial\tilde{\boldsymbol{\mathcal{Q}}}$, we assume that v satisfies $\vec n_q.v \ge 0$, where . stands for the scalar product and $\vec n_q$ is the unit vector normal to $\partial\tilde{\boldsymbol{\mathcal{Q}}}$ directed inward $\tilde{\boldsymbol{\mathcal{Q}}}$. The phase space, that is, the set of all possible positions and velocities, will be denoted by $\tilde{\boldsymbol{\Omega}} = \tilde{\boldsymbol{\mathcal{Q}}} \times \mathcal{S}^1$.

The billiard flow is denoted by $\tilde{\boldsymbol{\Phi}}_t : \tilde{\boldsymbol{\Omega}} \to \tilde{\boldsymbol{\Omega}}$, where $t \in \mathbb{R}$. Let $\tilde{\boldsymbol{\mu}}_0$ be the Lebesgue measure on $\tilde{\boldsymbol{\Omega}}$ normalized so that $\tilde{\boldsymbol{\mu}}_0((\tilde{\boldsymbol{\mathcal{Q}}} \cap [0,1]^2) \times S^1) = 1$.

The Sinai billiard is defined analogously on a compact domain. That is, we consider disjoint strictly convex open subsets $\bar{O}_i \subset \mathbb{T}^2$ (corresponding to the canonical projection of O_i), $i = 1, ..., I$, whose boundaries are $\mathcal{C}^3$ smooth with strictly positive curvature. Then we put $\boldsymbol{\mathcal{Q}} = \mathbb{T}^2 \setminus \cup_i O_i$. We define the billiard dynamics $(\boldsymbol{\Omega}, \boldsymbol{\Phi}_t, \boldsymbol{\mu}_0)$ exactly as $(\tilde{\boldsymbol{\Omega}}, \tilde{\boldsymbol{\Phi}}_t, \tilde{\boldsymbol{\mu}}_0)$ except that we use the billiard table $\boldsymbol{\mathcal{Q}}$ instead of $\tilde{\boldsymbol{\mathcal{Q}}}$ and $\boldsymbol{\mu}_0$ is a probability measure.

Next, we represent the flow $\boldsymbol{\Phi}_t$ as a suspension over a map. This map is called the billiard ball map: the Poincaré section of $\boldsymbol{\Phi}_t$ corresponding to the collisions. That is, we define

$$\boldsymbol{M} = \{(q, v) \in \boldsymbol{\Omega} : q \in \partial\boldsymbol{\mathcal{Q}}\} = \{(q, v) \in \boldsymbol{\Omega} : q \in \partial\boldsymbol{\mathcal{Q}}, \vec{n}_q.v \geq 0\}.$$

$\boldsymbol{T} : \boldsymbol{M} \to \boldsymbol{M}$ is defined by $\boldsymbol{T}(x) = \boldsymbol{\Phi}_{\boldsymbol{\tau}}(x)$, where $\boldsymbol{\tau} = \boldsymbol{\tau}(x)$ is the smallest positive number such that $\boldsymbol{\Phi}_{\boldsymbol{\tau}}(x) \in \boldsymbol{M}$. The projection of $\boldsymbol{\mu}_0$ to the Poincaré section is denoted by $\boldsymbol{\nu}$. In fact, $\boldsymbol{\nu}$ has the density $c\vec{n}_q.v dq dv$, where $c = 2|\partial\boldsymbol{\mathcal{Q}}|$ is a normalizing constant such that $\boldsymbol{\nu}$ is a probability measure. Clearly, we can write

$$\boldsymbol{\Omega} = \{(x, t), x \in \boldsymbol{M},\ t \in [0, \boldsymbol{\tau}(x))\}.$$

With this notation, we have $\boldsymbol{\mu}_0 = \frac{1}{\boldsymbol{\nu}(\boldsymbol{\tau})}\boldsymbol{\nu} \otimes \mathfrak{l}$, where $\mathfrak{l}$ is the Lebesgue measure on $[0, +\infty)$. Note that the measure $\boldsymbol{\mu}_0$ is a probability measure unlike μ defined in Section 2.1.

Finally, we define the measure preserving dynamical system $(\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{\nu}})$ analogously to the Lorentz gas. For every $\ell \in \mathbb{Z}^2$, we define the ℓ-cell $\mathcal{C}_\ell$ as the set of the points with last reflection off $\widetilde{\boldsymbol{\mathcal{Q}}}$ took place in the set $\bigcup_{i=1}^{I}(O_i + \ell)$. Identifying $\mathbb{T}^2$ with the unit square $[0, 1)^2 \subset \mathbb{R}^2$, we see that $(\tilde{\boldsymbol{M}}, \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{\nu}})$ is the $\mathbb{Z}^2$-extension of $(\boldsymbol{M}, \boldsymbol{T}, \boldsymbol{\nu})$ by $\boldsymbol{\kappa} : \boldsymbol{M} \to \mathbb{Z}^2$, where $\boldsymbol{\kappa}(x) = \ell$ if $\tilde{\boldsymbol{T}}(x) \in \mathcal{C}_\ell$.

The observable $(\boldsymbol{\kappa}, \boldsymbol{\tau}) : \boldsymbol{M} \to \mathbb{Z}^2 \times \mathbb{R}$ satisfies the central limit theorem (see e.g. [7]). That is, there exists a 3×3 positive definite matrix $\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}$ so that for any $A \subset \mathbb{R}^3$ whose boundary has zero Lebesgue measure

$$\boldsymbol{\nu}\left(x \in \boldsymbol{M} : \frac{(\boldsymbol{\kappa}_n, \boldsymbol{\tau}_n - n\nu(\tau))}{\sqrt{n}} \in A\right) = \int_A \Psi_{\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}}\,,$$

and Ψ is the Gaussian density defined by (2.3). Consequently, the central limit theorem holds for the observable $\boldsymbol{\kappa}$ with a covariance matrix $\Sigma_{\boldsymbol{\kappa}}$, which is obtained from $\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}$ by deleting the last row and the last column.

Denote

$$\|\mathfrak{h}\|_{\mathcal{H}_E^\eta} = \sup_{y \in E} |\mathfrak{h}(y)| + \sup_{y,z \in E,\ y \neq z} \frac{|\mathfrak{h}(y) - \mathfrak{h}(z)|}{d(y, z)^\eta}.$$

We will say that a function $\mathfrak{h} : \tilde{\boldsymbol{\Omega}} \to \mathbb{R}$ is *smooth in the flow direction* if

$$\forall N \geq 0, \quad \sum_\ell \left\| \frac{\partial^N}{\partial s^N}\left(\mathfrak{h} \circ \widetilde{\boldsymbol{\Phi}}_s\right)_{|s=0} \right\|_{\mathcal{H}_{\mathcal{C}_\ell}^\eta} < \infty\,. \tag{3.1}$$

Note that in order for (3.1) to hold, it is sufficient that $\mathfrak{h}$ is C^∞ in the position $q \in \widetilde{\boldsymbol{Q}}$ and satisfies

$$\forall N \geq 0, \quad \sum_\ell \left\| \frac{\partial^N}{\partial q^N}\mathfrak{h} \right\|_{\mathcal{H}_{\mathcal{C}_\ell}^\eta} < \infty\,,$$

$$\forall (q, \vec{v}) \in \partial\widetilde{\boldsymbol{Q}} \times S^1, \quad \frac{\partial^N}{\partial q^N}\mathfrak{h}(q, \vec{v}) = \frac{\partial^N}{\partial q^N}\mathfrak{h}\left(q, \vec{v} - 2(\vec{n}_q.\vec{v})\vec{n}_q\right)\,. \tag{3.2}$$

We say that $\mathfrak{h} : \tilde{\boldsymbol{\Omega}} \to \mathbb{R}$ is *η-Hölder continuous* if it is η-Hölder continuous on $\tilde{Q} \times S^1$ and satisfies (3.2) with $N = 0$.

Now we are ready to formulate the main result of this section.

Theorem 3.1. *Let $\mathfrak{f}, \mathfrak{g} : \tilde{\boldsymbol{\Omega}} \to \mathbb{R}$ be two η-Hölder continuous functions with at least one of them smooth in the flow direction. Assume moreover that there exists an integer $K_0 \geq 1$ such that*

$$\sum_{\ell} (1+|\ell|)^{2K_0} \left(\|\mathfrak{f}\|_{\mathcal{H}^\eta_{\mathcal{C}_\ell}} + \|\mathfrak{g}\|_{\mathcal{H}^\eta_{\mathcal{C}_\ell}} \right) < \infty \,. \tag{3.3}$$

Then there are real numbers $\mathfrak{C}_0(\mathfrak{f},\mathfrak{g}), \mathfrak{C}_1(\mathfrak{f},\mathfrak{g}), ..., \mathfrak{C}_{K_0}(\mathfrak{f},\mathfrak{g})$ so that we have

$$\int_{\tilde{\boldsymbol{\Omega}}} \mathfrak{f}\, \mathfrak{g} \circ \tilde{\Phi}_t d\tilde{\boldsymbol{\mu}}_0 = \sum_{k=0}^{K_0} \mathfrak{C}_k(\mathfrak{f},\mathfrak{g}) t^{-1-k} + o(t^{-1-K_0}) \,, \tag{3.4}$$

as $t \to +\infty$. Furthermore, $\mathfrak{C}_0(\mathfrak{f},\mathfrak{g}) = \mathfrak{c}_0 \int_{\tilde{\boldsymbol{\Omega}}} \mathfrak{f} d\tilde{\boldsymbol{\mu}}_0 \int_{\tilde{\boldsymbol{\Omega}}} \mathfrak{g} d\tilde{\boldsymbol{\mu}}_0$ with

$$\mathfrak{c}_0 = \frac{\boldsymbol{\nu}(\boldsymbol{\tau})}{2\pi\sqrt{\det \Sigma_{\boldsymbol{\kappa}}}} \tag{3.5}$$

and the coefficients $\mathfrak{C}_k$, as functionals over pairs of admissible functions, are bilinear.

We note that the bilinear forms $\mathfrak{C}_k$ are linearly independent. Namely in Appendix B we give examples of parts f_k, g_k such that $\mathfrak{C}_k(f_k,g_k) \neq 0$ while $\mathfrak{C}_j(f_k,g_k) \neq 0$ for all $j < k$.

In the remaining part of Section 3, we derive Theorem 3.1 from Theorem 2.2. However, we will not be applying Theorem 2.2 directly to $(\boldsymbol{M}, \boldsymbol{\nu}, \boldsymbol{T})$, but instead we apply it to the Young tower extension of the Sinai billiard. Thus we first briefly review the Young tower construction in Section 3.2. Then we prove condition (2.32) in Section 3.3 along the lines of [8]. Finally we complete the proof of Theorem 3.1 in Section 3.4. (3.5) is established in Section 3.5.

3.2. **Young towers.** Let $\mathcal{R} \subset \boldsymbol{M}$ be the hyperbolic product set constructed in [31, Section 8]. Furthermore, let (Δ, F) be the corresponding Young tower ("Markov extension"). There is a natural bijection ι between Δ_0, the base of the tower and $\mathcal{R}$. We will denote points of $\mathcal{R}$ by $x = (\gamma^u, \gamma^s)$, which is to be interpreted as $\gamma^u \cap \gamma^s$, where $\gamma^u = \gamma^u(x)$ and $\gamma^s = \gamma^s(x)$ are an unstable and a stable manifold containing x. Points of Δ_0 will be denoted by $\hat{x} = (\hat{\gamma}^u, \hat{\gamma}^s)$. Note that ι can be extended to π, a mapping from Δ to $\boldsymbol{M}$ (this map is in general not one-to-one).

We recall the most important ingredients of the construction of [31]. The base of the tower has the product structure $X = \Delta_0 = \Gamma^u \times \Gamma^s$. The sets of the form $A \times \Gamma^s$, $A \subset \Gamma^u$ are called u-sets if $\iota(A \subset \Gamma^u)$ is compact. Similarly, sets of the form $\Gamma^u \times B$, $B \subset \Gamma^s$ are called s-sets if $\iota(B \subset \Gamma^u)$ is compact. Also, sets of the form $\Gamma^u \times \{\hat{\gamma}^s\}$ are called stable manifolds and sets of the form $\{\hat{\gamma}^u\} \times \Gamma^s$ are unstable manifolds as they are images of (un)stable manifolds (or rather, the intersections of (un)stable manifolds and $\mathcal{R}$) by the map ι^{-1}. Δ_0 has a partition $\Delta_0 = \cup_{k \in \mathbb{Z}_+} \Delta_{0,k}$, where $\Delta_{0,k} = \Gamma^u \times \Gamma^s_k$ are s-sets. The return time to the base on the set $\Delta_{0,k}$ is identically r_k, that is $\Delta = \cup_{k\in\mathbb{Z}_+} \cup_{l=0}^{r_k-1} \Delta_{l,k}$, where $\Delta_{l,k} = \{(\hat{x}, l) : \hat{x} \in \Delta_{0,k}\}$. There is an F-invariant measure ν on Δ so that $\pi_*\nu = \mu$ and F is an isomorphism between $\Delta_{l,k}$ and $\Delta_{l+1,k}$ and $F(\hat{x}, l) = (\hat{x}, l+1)$. Also F is an isomorphism between $\Delta_{r_k-1,k}$ and $F(\Delta_{r_k-1,k})$, the latter being a u-set of Δ_0. Furthermore, if $\hat{x}_1, \hat{x}_2 \in \Delta_{0,k}$ belong to the same (un)stable manifold, so do $F^{r_k}(\hat{x}_1, 0)$ and $F^{r_k}(\hat{x}_2, 0)$. We write $\mathcal{F} = F^{r_k - l}$ on $\Delta_{l,k}$ and $r(\hat{\gamma}^u, \hat{\gamma}^s) = r(\hat{\gamma}^s) = r_k$ for $(\hat{\gamma}^u, \hat{\gamma}^s) \in \Delta_{0,k}$. Define Ξ on Δ by

$$\Xi((\hat{\gamma}^u, \hat{\gamma}^s), l) = ((\hat{\boldsymbol{\gamma}}^u, \hat{\gamma}^s), l) \text{ with a fixed } \hat{\boldsymbol{\gamma}}^u. \tag{3.6}$$

Let $\bar{\Delta} = \Xi(\Delta)$ and $\bar{\nu} = \Xi_*\nu$. There is a well defined $\bar{F} : \bar{\Delta} \to \bar{\Delta}$ such that $\Xi \circ F = \bar{F} \circ \Xi$. The dynamical system $(\bar{\Delta}, \bar{\nu}, \bar{F})$, is an expanding tower, in the sense that it satisfies assumptions (E1)–(E5) below.

Let $(\bar{\Delta}, \bar{\nu}, \bar{F})$ be a probability preserving dynamical system with a partition $(\bar{\Delta}_{l,k})_{k\in I, l=0,...,r_k-1}$ into positive measure subsets, where I is either finite or countable and $r_k = r(\bar{\Delta}_{0,k})$ is a positive integer. We call it *an expanding tower* if

(E1) for every $i \in I$ and $0 \le j < r_i - 1$, F is a measure preserving isomorphism between $\bar\Delta_{j,i}$ and $\bar\Delta_{j+1,i}$.

(E2) for every $i \in I$, $\bar F$ is an isomorphism between $\bar\Delta_{r_i-1,i}$ and

$$\bar X := \bar\Delta_0 := \cup_{i\in I}\bar\Delta_{0,i}.$$

(E3) Let $r(x) = r(\bar\Delta_{0,k})$ if $x \in \bar\Delta_{0,k}$ and $\bar{\mathcal{F}} : \bar X \to \bar X$ be the first return map to the base, i.e. $\bar{\mathcal{F}}(x) = \bar F^{r(x)}(x)$. Let $s(x,y)$, the separation time of $x, y \in X$, be defined as the smallest integer n such that $\bar{\mathcal{F}}^n x \in \bar\Delta_{0,i}$, $\bar{\mathcal{F}}^n y \in \bar\Delta_{0,j}$ with $i \ne j$. As $\bar{\mathcal{F}} : \bar\Delta_{0,i} \to \bar X$ is an isomorphism, it has an inverse. Denote by α the logarithm of the Jacobian of this inverse (w.r.t. the measure $\bar\nu$). Then there are constants $\vartheta_0 < 1$ and $C > 0$ such that for every $x, y \in \bar\Delta_{0,i}$, $|\alpha(x) - \alpha(y)| \le C\vartheta_0^{s(x,y)}$.

(E4) Extend s to $\bar\Delta$ by setting $s(x,y) = 0$ if x, y do not belong to the same $\bar\Delta_{j,i}$ and $s(x,y) = s(\bar F^{-j}x, \bar F^{-j}y)+1$ if $x, y \in \bar\Delta_{j,i}$. $(\bar\Delta, \bar\nu, \bar F)$ is exact (hence ergodic and mixing) with respect to the metric

$$d_\vartheta(x,y) := \vartheta^{s(x,y)}. \tag{3.7}$$

Furthermore, in case of Sinai billiards, we have

(E5) $\bar\nu(x : r(x) > n) \le C\rho^n$ with some $\rho < 1$.

3.3. **Condition** (2.32) **for Sinai billiards.** Given a function $f : \boldsymbol{M} \to \mathbb{C}$, we define $\hat f : \Delta \to \mathbb{C}$ by $\hat f = f \circ \pi$. Now given a function $\hat f : \Delta \to \mathbb{C}$ (which may or may not be a lift-up of a function $f : \boldsymbol{M} \to \mathbb{C}$), we write $X = \Delta_0$ and define

$$\hat f_X : X \to \mathbb{C}, \quad \hat f_X(\hat x) = \sum_{j=0}^{r(\hat x)-1} \hat f(F^j(\hat x)),$$

$$\bar f : \bar\Delta \to \mathbb{C}, \quad \bar f(\hat\gamma^s, l) = \hat f(\hat{\boldsymbol\gamma}^u, \hat\gamma^s, l)\,,$$

$$\bar f_{\bar X} : \bar X \to \mathbb{C}, \quad \bar f_{\bar X}(\hat\gamma^s) = \sum_{j=0}^{r(\hat\gamma^s)-1} \hat f(F^j(\hat{\boldsymbol\gamma}^u, \hat\gamma^s))\,.$$

Fix $\varkappa < 1$ and consider the space of dynamically Lipschitz functions on $\bar X$ (w.r.t. the metric $d_\varkappa$):

$$C_\varkappa(\bar X, \mathbb{C}) = \{f : \bar X \to \mathbb{C} \text{ bounded and } L(f) < \infty\},$$

where

$$L(f) = \inf\{C : \forall x, y \in \bar X : |f(x) - f(y)| \le C\varkappa^{s(x,y)}\}.$$

This space is equipped with the norm

$$\|f\|_\varkappa = L(f) + \|f\|_\infty.$$

Let Q be the Perron-Frobenius-Ruelle operator associated with $\bar{\mathcal{F}}$, i.e.

$$(Qh)(x) = \sum_{y:\bar{\mathcal{F}}y=x} e^{\alpha(y)}h(y)$$

where e^α is the Jacobian defined in (E3). We have for h with $\|h\|_\varkappa < \infty$

$$Qh = \bar\nu(h) + Rh, \tag{3.8}$$

where $\|Rh\|_\varkappa \le \rho\|h\|_\varkappa$ with some $\rho < 1$.

Now we introduce the (signed) temporal distance function D on $\mathcal{R}$ by defining

$$D(x,y) = \sum_{\ell=-\infty}^{\infty} [\boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(x),\gamma^s(x))) - \boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(x),\gamma^s(y))) + \boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(y),\gamma^s(y))) - \boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(y),\gamma^s(x)))], \tag{3.9}$$

where $\boldsymbol{\tau}$ is defined in Section 3.1. Note that there is a lift-up $\hat{\boldsymbol{\tau}} : \Delta \to \mathbb{R}_+$ defined by $\hat{\boldsymbol{\tau}}(\hat{x}) = \boldsymbol{\tau}(\pi(\hat{x}))$ and corresponding functions $\hat{\boldsymbol{\tau}}_X, \bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\tau}}_{\bar{X}}$.

We also define the operators

$$Q_\xi h = Q(e^{i\xi\bar{\boldsymbol{\tau}}_{\bar{X}}} h)\,. \tag{3.10}$$

For real valued functions defined on $\bar{X}$, we will consider the norms

$$\|.\|_\infty, \quad \|.\|_\varkappa, \quad \|.\|_{(\xi)} := \max\{\|.\|_\infty, C_0 L(.)/\xi\}\,,$$

where $\xi \gg 1$ and C_0 is a constant to be specified later.

Now, let us consider points $x_m = (\gamma^u(x_m), \gamma^s(x_m)), y_m = (\gamma^u(y_m), \gamma^s(y_m)) \in \mathcal{R}$ which satisfy that $\mathcal{F}^k(\iota^{-1}(x_m)) \in \Delta_{0,1}$, $\mathcal{F}^k(\iota^{-1}(y_m)) \in \Delta_{0,a_k}$ for $k \geq 0$, where

$$a_k = \begin{cases} 2 \text{ if } k = m^2 \text{ or } k = m^2 + m \\ 1 \text{ otherwise.} \end{cases}$$

Let

$$x'_m := \boldsymbol{T}^{r_1(m^2+1)}(x_m) = \iota(\mathcal{F}^{m^2+1}(\iota^{-1}(x_m))) \text{ and } y'_m := \boldsymbol{T}^{r_1 m^2 + r_2}(y_m) = \iota(\mathcal{F}^{m^2+1}(\iota^{-1}(y_m))).$$

Let $\mathcal{Q}_m$ be the solid rectangle with corners x'_m, $[x'_m, y'_m]$, y'_m, $[y'_m, x'_m]$, i.e. the unique topological rectangle inside the convex hull of $\mathcal{R}$ which is bounded by two stable and unstable manifolds, such that two of its corners are x'_m and y'_m. We claim that there are two constants $0 < c_2 < c_1 < 1$ so that $c_2^m < \mu(\mathcal{Q}_m) < c_1^m$ for sufficiently large m. To prove this claim, let $\mathcal{Q}_{0,i}$ denote the smallest topological rectangle containing $\iota(\Delta_{0,i})$ for $i = 1, 2$. Note that $\boldsymbol{T}^{r_1}$ is a $\mathcal{C}^2$ self map of $\mathcal{Q}_{0,i}$. By construction, $\boldsymbol{T}^{jr_1}\mathcal{Q}_m$ is a subset of $\mathcal{Q}_{0,1}$ for $j = 0, 1, ..., m-2$. Now consider a foliation of $\mathcal{Q}_m$ by unstable curves. Each such curve is expanded by a factor $\Lambda > 1$ by the map $\boldsymbol{T}^{r_1}$ and so the upper bound follows. To prove the lower bound, observe that $\boldsymbol{T}^{(m-1)r_1}\mathcal{Q}_m$ intersects both $\mathcal{Q}_{0,1}$ and $\mathcal{Q}_{0,2}$ and so, as we can assume that the distance between $\mathcal{Q}_{0,1}$ and $\mathcal{Q}_{0,2}$ is positive, the length of the image of each unstable curve in our foliation under the map $\boldsymbol{T}^{(m-1)r_1}$ is uniformly bounded from below. Furthermore, the expansion of $\boldsymbol{T}^{r_1}$ on $\mathcal{Q}_{0,1}$ is bounded from above and so the lower bound follows as well. Next, Lemma 5.1 of [18] states that $\mu(\mathcal{Q}_m) = |D(x_m, y_m)|$ (see also [7, §6.11]). Note that $D(x_m, y_m)$ has another representation: it is the unique small number σ so that $\Phi^\sigma Y_1 = Y_5$, where Φ is the billiard flow, $Y_1, ..., Y_5$ are points whose last collisions were at x'_m, $[x'_m, y'_m]$, y'_m, $[y'_m, x'_m]$, x'_m, respectively and the pairs $(Y_1, Y_2), (Y_3, Y_4)$ are on the same stable manifold of Φ while the pairs $(Y_2, Y_3), (Y_4, Y_5)$ are on the same unstable manifold of Φ (see Lemma 6.40 in [7]). We summarize the results of this construction in

Lemma 3.2. *There exist some $a_0 > 0$, and $c \in \mathbb{R}_+$ such that for any $\xi > 3$ there are $x = x(\xi), y = y(\xi) \in \mathcal{R}$ satisfying*

$$\iota^{-1}(\boldsymbol{T}^{r_1 k}(x)) \in \Delta_{0,1} \text{ for all } k = -(\ln \xi)^{3/2}, ..., -1\,, \tag{3.11}$$

$$\iota^{-1}(\boldsymbol{T}^{-r_2}(y)) \in \Delta_{0,2} \text{ and } \iota^{-1}(\boldsymbol{T}^{(k+1)r_1 - r_2}(y)) \in \Delta_{0,1} \text{ for all } k = -(\ln \xi)^{3/2}, ..., -2\,, \tag{3.12}$$

$$\mathcal{F}^k(\iota^{-1}(x)), \mathcal{F}^k(\iota^{-1}(y)) \in \Delta_{0,1} \cup \Delta_{0,2} \text{ for all } k \geq 0\,, \tag{3.13}$$

and

$$|e^{i\xi D(x,y)} - 1| > c\xi^{-a_0}\,. \tag{3.14}$$

Proof. It is sufficient to prove the lemma for ξ large. Indeed, if we can prove the lemma for $\xi > \xi_0$, then we can extend it to any $\xi > 3$ by choosing c small enough unless there is some $\xi' \in [3, \xi_0]$ so that $\xi' D(x,y) = 0 \pmod{2\pi}$ for all x, y. Note that this cannot happen since this would imply $l\xi' D(x,y) = 0 \pmod{2\pi}$ where we can choose $l \in \mathbb{Z}_+$ so that $l\xi' > \xi_0$.

Now given ξ, we choose m so that $c_1^m < \xi^{-1} \le c_1^{m-1}$. Recall that for this m, we have points x'_m, y'_m so that $c_2^m < |D(x'_m, y'_m)| < c_1^m$. We conclude

$$c_2 \xi^{1 - \frac{\ln c_2}{\ln c_1}} \le \xi |D(x'_m, y'_m)| \le 1 .$$

Clearly, (3.11), (3.12) and (3.13) hold for $\xi > \xi_0$ as $m^2 \gg (\ln \xi)^{3/2}$. □

Recall the definition of Q_ξ from (3.10). We have

Lemma 3.3. *There are constants a_1, C_1, C_2 so that for every $\xi > 3$,*

$$\left\| Q_\xi^{C_1 \ln \xi} \right\|_{(\xi)} < 1 - \frac{C_2}{\xi^{a_1}} . \tag{3.15}$$

Proof. Let h satisfy $\|h\|_{(\xi)} = 1$.

First recall that by [6], there exists a constant $C_{0,1}$ such that

$$L(Q_\xi^n h) \le C_{0,1}[\xi \|h\|_\infty + \theta^n L(h)] , \tag{3.16}$$

(see also Proposition 3.7 in [22]). Thus choosing our $C_0 = C_0(C_{0,1})$ small enough in the definition of the norm $\|.\|_{(\xi)}$ and $C_{1,1}$ sufficiently big, we obtain

$$L\left(Q_\xi^{C_{1,1} \ln \xi} h\right) \le \frac{\xi}{2C_0}.$$

In order to prove the lemma, it remains to verify (3.15) for the infinity norm.

This proof is divided into three parts:

Step 1. We show that $\|Q_\xi^{C_{1,2} \ln \xi} h\|_{L^1} < 1 - \frac{C_{2,1}}{\xi^{a_{1,1}}}$ assuming the following hypothesis.

(H): *there is some*

$$u \in \bar{X}_{\le 2} := \{\bar{x} \in \bar{X} : \bar{\mathcal{F}}^n(\bar{x}) \in \bar{\Delta}_{0,1} \cup \bar{\Delta}_{0,2} \text{ for all } n \in \mathbb{N}\}$$

so that

$$|h(u)| < 1 - \frac{C_{2,2}}{\xi^{a_{1,2}}}.$$

Let U denote the $C_{2,2} C_0 \xi^{-a_{1,2}-1}/2$ neighborhood of u (w.r.t the metric $d_\varkappa$) in $\bar{X}$. Since $L(h) \le \xi / C_0$, we have $|h(u')| < 1 - \frac{C_{2,2}}{2\xi^{a_{1,2}}}$ for any $u' \in U$. By the bounded distortion property and by the fact that $u \in \bar{X}_{\le 2}$, we have $\frac{C_{2,3}}{\xi^{a_{1,3}}} \le \bar{\nu}(U)$. Observing that

$$|Q_\xi^n h| \le Q^n |h| \tag{3.17}$$

holds pointwise (by definition of the operators and by induction on n), and using $\|h\|_\infty \le 1$, we derive that for any ℓ

$$\begin{aligned} \int |Q_\xi^\ell h| d\bar{\nu} \le \int Q^\ell |h| d\bar{\nu} = \int |h| d\bar{\nu} &= \int_U |h| d\bar{\nu} + \int_{\bar{X} \setminus U} |h| d\bar{\nu} \\ &\le \left(1 - \frac{C_{2,2}}{2\xi^{a_{1,2}}}\right) \bar{\nu}(U) + 1 - \bar{\nu}(U) \le 1 - \frac{C_{2,4}}{\xi^{a_{1,4}}} , \end{aligned}$$

with $C_{2,4} = C_{2,2} C_{2,3}/2$ and $a_{1,4} = a_{1,2} + a_{1,3}$.

Step 2. Under hypothesis **(H)**, we show that $\left\| Q_\xi^{C_{1,3} \ln \xi} h \right\|_\infty < 1 - \frac{C_{2,5}}{\xi^{a_{1,5}}}$.

For any $u \in \bar{X}$, we have

$$\left|Q_\xi^{C_{1,3}\ln\xi} h\right|(u) = \left|Q_\xi^{(C_{1,3}-C_{1,2})\ln\xi}(Q_\xi^{C_{1,2}\ln\xi} h)\right|(u)$$
$$\leq^{(3.17)} \left(Q^{(C_{1,3}-C_{1,2})\ln\xi}\left|Q_\xi^{C_{1,2}\ln\xi} h\right|\right)(u) \leq \bar{\nu}\left(\left|Q_\xi^{C_{1,2}\ln\xi} h\right|\right) + C\xi\theta^{(C_{1,3}-C_{1,2})\ln\xi},$$

where the last inequality follows from (3.8), (3.16) and (3.17). By Step 1 and by choosing $C_{1,3} - C_{1,2}$ sufficiently large, we see that Step 2 is completed.

Step 3. We show that $\left\|Q_\xi^{C_{1,4}\ln\xi} h\right\|_\infty < 1 - \dfrac{C_{2,5}}{\xi^{a_{1,5}}}$ with $C_{1,4} = 2C_{1,3}$ without assuming **(H)**.

In order to complete Step 3, it suffices to show that there exists some $v \in \bar{X}_{\leq 2}$ that either satisfies **(H)** or satisfies the following:

$$|Q_\xi^n h(v)| < 1 - \frac{C_{2,2}}{\xi^{a_{1,2}}} \text{ with } n = C_{1,3}\ln\xi. \tag{3.18}$$

Indeed, if there is a v satisfying **(H)**, then noting that $\|Q_\xi\|_\infty \leq 1$, the proof in Step 2 applies. On the other hand, if there is a v satisfying (3.18), then since $\|Q_\xi\|_{(\xi)} \leq 1$, we have $\|Q_\xi^n h\|_{(\xi)} \leq 1$ and so we can apply the results of Step 2 for the function h replaced by $Q_\xi^n h$.

For a function $f : \bar{X} \to \mathbb{R}$ and $n \in \mathbb{N}$, we write $f_n(x) = \sum_{j=0}^{n-1} f(\bar{\mathcal{F}}^j x)$.

Recall that for our ξ, Lemma 3.2 gives us $x, y \in \mathcal{R}$ (in fact, with the previous notation $x = x'_m, y = y'_m$ with $m \approx (\ln(1/c_1))^{-1}\ln\xi$). Let us write $(\hat{\gamma}^u(x), \hat{\gamma}^s(x)) = \iota^{-1}(x)$, $(\hat{\gamma}^u(y), \hat{\gamma}^s(y)) = \iota^{-1}(y)$, $v = \bar{\mathcal{F}}^{n/2}(\hat{\gamma}^s(x))$, $w = \bar{\mathcal{F}}^{n/2}(\hat{\gamma}^s(y))$. We will show that in case no point satisfies **(H)**, then either v or w satisfies (3.18). To this end, assume by contradiction that none of them satisfies (3.18).

Writing $h(\bar{x}) = r(\bar{x})e^{i\phi(\bar{x})}$, we have

$$(Q_\xi^n h)(v) = \sum_{u\in\bar{X}:\bar{\mathcal{F}}^n u = v} e^{\alpha_n(u) + i\xi(\bar{\boldsymbol{\tau}}_{\bar{X}})_n(u)} r(u) e^{i\phi(u)}$$
$$= e^{\alpha_n(v'_{-n}) + i\xi(\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v'_{-n})} r(v'_{-n}) e^{i\phi(v'_{-n})} + e^{\alpha_n(v''_{-n}) + i\xi(\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v''_{-n})} r(v''_{-n}) e^{i\phi(v''_{-n})} + \ldots$$

where

$$v'_{-n} = \Xi(\iota^{-1}(\boldsymbol{T}^{-r_1 n/2}(\gamma^u(x), \gamma^s(x)))), v''_{-n} = \Xi(\iota^{-1}(\boldsymbol{T}^{-r_1(n/2-1)-r_2}(\gamma^u(y), \gamma^s(x))))$$

and ... corresponds to all other preimages.

Thus $(Q_\xi^n h)(v)$ is expressed as a weighted sum of the unit vectors $e^{i[\xi(\bar{\boldsymbol{\tau}}_{\bar{X}})_n(u) + \phi(u)]} \in \mathbb{C}$, with weights $e^{\alpha_n(u)} r(u)$. Noting that $\sum_{u\in\bar{X}:\bar{\mathcal{F}}^n u = v} e^{\alpha_n(u)} = 1$ and $|r| \leq 1$, we observe that v can only violate (3.18) if all the unit vectors, whose weights are at least $C_{2,6}/\xi^{a_{1,6}}$ are nearly collinear, i.e. their angle do not differ by more than $C_{2,6}/\xi^{a_{1,6}}$ with $a_{1,6} = a_{1,2}$.

If $r(v'_{-n}) < 1/2$ or $r(v''_{-n}) < 1/2$, then one of these points satisfies (**H**) and so the proof is completed. If $r(v'_{-n}) \geq 1/2$ and $r(v''_{-n}) \geq 1/2$ then we also claim that $e^{\alpha_n(v'_{-n})} > 2C_{2,6}/\xi^{a_{1,6}}$ and $e^{\alpha_n(v''_{-n})} > 2C_{2,6}/\xi^{a_{1,6}}$. Indeed, this holds since $v'_{-n}, v''_{-n} \in \bar{X}_{\leq 2}$ and since α is a Hölder function and so it is bounded from below by a positive number on the compact set $\bar{X}_{\leq 2}$ (and so e^α on the set $\bar{X}_{\leq 2}$ is bounded from below by a number which is bigger than one).

Thus we have derived that

$$|[\xi(\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v'_{-n}) - \xi(\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v''_{-n})] - [\phi(v'_{-n}) - \phi(v''_{-n})]| \leq C_{2,6}/\xi^{a_{1,6}}$$

Repeating the above argument for w, and writing

$$w'_{-n} = \Xi(\iota^{-1}(\boldsymbol{T}^{-r_1(n/2-1)-r_2}(\gamma^u(y), \gamma^s(y)))), w''_{-n} = \Xi(\iota^{-1}(\boldsymbol{T}^{-r_1 n/2}(\gamma^u(x), \gamma^s(y)))),$$

we find

$$|[\xi(\bar{\boldsymbol{\tau}}_{\bar{X}})_n(w'_{-n}) - \xi(\bar{\boldsymbol{\tau}}_{\bar{X}})_n(w''_{-n})] - [\phi(w'_{-n}) - \phi(w''_{-n})]| \leq C_{2,6}/\xi^{a_{1,6}}.$$

By construction, $s(v'_{-n}, w''_{-n}) \geq n/2$ and thus $|\phi(v'_{-n}) - \phi(w''_{-n})| \leq C_{2,6}/\xi^{a_{1,6}}$ assuming that $C_{1,2}$ is sufficiently large. Similarly, we can assume $|\phi(v''_{-n}) - \phi(w'_{-n})| \leq C_{2,6}/\xi^{a_{1,6}}$ and thus with $C_{2,7} = 4C_{2,6}$ and $a_{1,7} = a_{1,6} + 1$,

$$|A| \leq C_{2,7}/\xi^{a_{1,7}} \text{ where } A = (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v'_{-n}) - (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(v''_{-n}) + (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(w'_{-n}) - (\bar{\boldsymbol{\tau}}_{\bar{X}})_n(w''_{-n}). \tag{3.19}$$

Recall (3.6) and (3.9). Using the notations $z = (\gamma^u(z), \gamma^s(z)) \in \mathcal{R}$, $\hat{z} = \iota^{-1}(z) = (\hat{\gamma}^u(z), \hat{\gamma}^s(z))$ and

$$H(z) = \sum_{\ell=0}^{\infty} [\boldsymbol{\tau}(\boldsymbol{T}^\ell(\gamma^u(z), \gamma^s(z))) - \boldsymbol{\tau}(\boldsymbol{T}^\ell(\boldsymbol{\gamma}^u, \gamma^s(z)))],$$

observe that we have

$$\hat{\boldsymbol{\tau}}_X(\hat{\gamma}^u(z), \hat{\gamma}^s(z)) - \bar{\boldsymbol{\tau}}_{\bar{X}}(\hat{\gamma}^s(z)) = H(\gamma^u(z), \gamma^s(z)) - H(\boldsymbol{T}^{r(\hat{\gamma}^s(z))}(\hat{\gamma}^u(z), \hat{\gamma}^s(z))). \tag{3.20}$$

To simplify notation, we write

$$[z_1, z_2] = (\gamma^u(z_1), \gamma^s(z_2)) \tag{3.21}$$

and

$$d_{\ell,f}(z_1, z_2) = f(\boldsymbol{T}^\ell([z_1, z_1]) - f(\boldsymbol{T}^\ell([z_1, z_2])) - f(\boldsymbol{T}^\ell([z_2, z_1])) + f(\boldsymbol{T}^\ell([z_2, z_2])).$$

Recall the dynamical Hölder continuity of $\boldsymbol{\tau}$: there is some C and $\vartheta < 1$ so that if $z_1, z_2 \in \boldsymbol{M}$ are such that $\boldsymbol{T}^\ell(z_1)$ and $\boldsymbol{T}^\ell(z_1)$ stay on the same local unstable manifold for all $\ell \leq L$, then $|\tau(z_1) - \tau(z_2)| < C\vartheta^L$. Likewise, if $\boldsymbol{T}^\ell(z_1)$ and $\boldsymbol{T}^\ell(z_1)$ stay on the same local stable manifold for all $\ell \geq -L$, then $|\tau(z_1) - \tau(z_2)| < C\vartheta^L$.

We have

$$D(x'_m, y'_m) = \sum_{\ell=-\infty}^{\infty} d_{\ell,\boldsymbol{\tau}}(x'_m, y'_m) = S_1 + S_2 + S_3,$$

where

$$\begin{aligned} S_1 = &\sum_{\ell=-\infty}^{-r_1 n/2-1} \boldsymbol{\tau}(\boldsymbol{T}^\ell(x'_m)) - \boldsymbol{\tau}(\boldsymbol{T}^\ell([x'_m, y'_m])) \\ &\sum_{\ell=-\infty}^{-r_1(n/2-1)-r_2-1} -\boldsymbol{\tau}(\boldsymbol{T}^\ell([y'_m, x'_m])) + \boldsymbol{\tau}(\boldsymbol{T}^\ell(y'_m)), \\ S_2 = &\sum_{\ell=-r_1 n/2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell(x'_m)) - \sum_{\ell=-r_1 n/2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell([x'_m, y'_m])) \\ &- \sum_{\ell=-r_1(n/2-1)-r_2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell([y'_m, x'_m])) + \sum_{\ell=-r_1(n/2-1)-r_2}^{r_1(n/2-1)} \boldsymbol{\tau}(\boldsymbol{T}^\ell(y'_m)), \end{aligned}$$

and

$$S_3 = \sum_{\ell=r_1(n/2-1)+1}^{\infty} \boldsymbol{\tau}(\boldsymbol{T}^\ell(x'_m)) - \boldsymbol{\tau}(\boldsymbol{T}^\ell([x'_m, y'_m])) - \boldsymbol{\tau}(\boldsymbol{T}^\ell([y'_m, x'_m])) + \boldsymbol{\tau}(\boldsymbol{T}^\ell(y'_m)).$$

In other words, we rearrange terms in the infinite sum according to the first return to the base in the tower representation. Observe that in view of (3.20),

$$S_2 = \sum_{k=0}^{n-1} \hat{\boldsymbol{\tau}}_X(\bar{\mathcal{F}}^k(v'_{-n})) - \hat{\boldsymbol{\tau}}_X(\bar{\mathcal{F}}^k(v''_{-n})) - \hat{\boldsymbol{\tau}}_X(\bar{\mathcal{F}}^k(w'_{-n})) + \hat{\boldsymbol{\tau}}_X(\bar{\mathcal{F}}^k(w''_{-n})).$$

Next, using (3.19), (3.20) and performing a telescopic sum, we find

$$S_2 = A + d_{0,H}(\boldsymbol{T}^{-r_1 n/2}(x'_m), \boldsymbol{T}^{-r_1(n/2-1)-r_2}(y'_m)) - d_{0,H}(\boldsymbol{T}^{r_1 n/2}(x'_m), \boldsymbol{T}^{r_1(n/2-1)+r_2}(y'_m)).$$

By the dynamical Hölder property of $\boldsymbol{\tau}$, $S_1+(S_2-A)+S_3$ can be made smaller than $C_{2,7}/\xi^{a_{1,7}}$ assuming that $C_{1,2}$ is large enough. Indeed, e.g. both series whose sum defines S_1 are absolutely convergent and are smaller than $C\frac{1}{1-\vartheta}\vartheta^{n/2}$ (the absolute convergence justifies why we can write S_1 as a sum of these two series). Estimating S_3 is even simpler: we can assume $n/2>m$ and so all of the points

$$\boldsymbol{T}^{\ell}(x'_m),\boldsymbol{T}^{\ell}([x'_m,y'_m]),\boldsymbol{T}^{\ell}(y'_m),\boldsymbol{T}^{\ell}([y'_m,x'_m])$$

lie on the same local stable manifold for $\ell>n/2$. Assuming $n/4>m$ as well, the dynamical Hölder continuity of $\boldsymbol{\tau}$ implies $|S_3|\le C\frac{1}{1-\vartheta}\vartheta^{n/4}$. The argument is similar for (S_2-A). Thus we derived that $D(x'_m,y'_m)\le 2C_{2,7}/\xi^{a_{1,7}}$ which is a contradiction with the choice of x'_m and y'_m assuming, as we can, that $a_{1,1}$ is chosen sufficiently big so that $a_{1,7}>a_0$. □

Let the operator $Q_{\theta,\xi}$ be defined by $Q_{\theta,\xi}h=Q(e^{i\theta\cdot\bar{\boldsymbol{\kappa}}_{\bar X}+i\xi\bar\tau_{\bar X}}h)$, where $\boldsymbol{\kappa}:\boldsymbol{M}\to\mathbb{Z}^2$ is defined in Section 3.1. Since $\boldsymbol{\kappa}$ is constant on local stable manifolds, the proof of Lemma 3.3 can be adapted to imply the following generalization (see also Lemma 3.14 in [22] for a similar argument):

$$\sup_{\theta\in[-\pi,\pi]^d}\left\|Q_{\theta,\xi}^{C_1\ln\xi}\right\|_{(\xi)}<1-\frac{C_2}{\xi^{a_1}}. \tag{3.22}$$

Now we revisit the tower $(\bar\Delta,\bar F)$. Recall that a separation time s was defined in (E4). Let

$$\|f\|_{\mathbb{B}}=\|f\|_\infty+\sup\{C:\forall x,y\in\bar\Delta:|f(x)-f(y)|\le C\varkappa^{s(x,y)}\}\,. \tag{3.23}$$

Let us denote by $\bar{\boldsymbol{P}}$ the Perron-Frobenius operator associated with $\bar F$ and let $\bar{\boldsymbol{P}}_{\theta,\xi}$ be defined by $\bar{\boldsymbol{P}}_{\theta,\xi}(f):=\bar{\boldsymbol{P}}\left(e^{i\theta\cdot\bar{\boldsymbol{\kappa}}+i\xi\bar\tau}f\right)$. We conclude this section by

Lemma 3.4. *There are constants C_3,α_2 and δ so that*

$$\sup_{\theta\in[-\pi,\pi]^d}\|\bar{\boldsymbol{P}}^n_{\theta,\xi}\|_{\mathcal{L}(\mathbb{B},L^1)}\le C_3|\xi|^{\alpha_2}e^{-n\delta|\xi|^{-\alpha_2}}\,. \tag{3.24}$$

Proof. This lemma is proved by operator renewal theory. The proof is very similar to Section 4 in [22], based on our Lemma 3.3 (but is easier as we only consider purely imaginary $i\xi$). We do not repeat the proof here. □

3.4. **Proof of Theorem 3.1.** Let $\mathcal{S}_0=\partial\boldsymbol{M}=\{(q,v)\in\boldsymbol{M}:\vec n_q.v=0\}$ be the singularity set, i.e. the collection of points in the phase space corresponding to grazing collisions.

The transformation $\boldsymbol{T}$ defines a C^1 diffeomorphism from $\boldsymbol{M}\setminus(\mathcal{S}_0\cup\boldsymbol{T}^{-1}\mathcal{S}_0)$ to $\boldsymbol{T}\setminus(\mathcal{S}_0\cup\boldsymbol{T}\mathcal{S}_0)$.

Moreover there exist $C_0>0$ and $\theta_0\in(0,1)$ such that the diameter of every connected component of $\boldsymbol{M}\setminus\bigcup_{j=-n}^n\boldsymbol{T}^{-j}\mathcal{S}_0$ is less than $C_0\theta_0^n$. We consider now $\hat s$ is a suitable separation time on Δ. The main difference between s and $\hat s$ is that counts the steps straight up in the tower, i.e. $\hat s((x,l),(y,l))=\hat s((x,0),(y,0))-l$. The exact definition of $\hat s$ is not important for us and can be found in [31].

Recall that, by construction of [31], for every $x,y\in\Delta$ in the same unstable manifold, $\pi(x)$ and $\pi(y)$ lie in the same connected component of $\boldsymbol{M}\setminus\bigcup_{j=-\infty}^{\hat s(x,y)}\boldsymbol{T}^{-j}\mathcal{S}_0$, with $\hat s(x,y):=\hat s(\Xi(x),\Xi(y))$.

We will prove that the assumptions of Theorem 2.2 are satisfied with:

- $\Sigma=\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}$
- $K=2K_0$
- $d=2$,
- $(M,\nu,T)=(\Delta,\nu,F)$, $\tau:=\hat{\boldsymbol{\tau}}=\boldsymbol{\tau}\circ\pi$, $\kappa:=\hat{\boldsymbol{\kappa}}=\boldsymbol{\kappa}\circ\pi$,
- $(\bar\Delta,\bar\nu,\bar T)=(\bar\Delta,\bar\nu,\bar F)$, $\mathfrak{p}=\Xi$ and $\bar P=\bar{\boldsymbol{P}}$
- $\mathcal{V}$ the space of functions $f:\Delta\to\mathbb{C}$ such that the following quantity is finite

$$\|f\|_{\mathcal{V}}=\|f\|_\infty+\sup_{\gamma^u;\ x,y\in\gamma^u}\frac{|f(x)-f(y)|}{\varkappa^{\hat s(x,y)}}+\sup_{n\ge0,\ \gamma^s;\ x,y\in\gamma^s}\frac{|f(F^n(x))-f(F^n(y))|}{\varkappa^n}\,,$$

where $\varkappa$ is a fixed real number satisfying

$$\max\left(\theta_0^{1/4}, \theta_0^{\eta}, \vartheta\right) < \varkappa < 1, \tag{3.25}$$

where ϑ is defined in (3.7).

- The space $\mathcal{B}$ is the Young space of complex-valued functions $f : \bar{\Delta} \to \mathbb{C}$ such that $\|f\|_{\mathcal{B}} < \infty$ with $\|\cdot\|_{\mathcal{B}}$ defined by

$$\|f\|_{\mathcal{B}} = \sup_l \|f|_{\bar{\Delta}_l}\|_\infty e^{-l\varepsilon_0} + \sup_l \operatorname{ess\,sup}_{x,y\in\bar{\Delta}_l} \frac{|f(x)-f(y)|}{\varkappa^{\hat{s}(x,y)}} e^{-l\varepsilon_0}\,. \tag{3.26}$$

  with $\varkappa$ as in (3.25) and a suitable ε_0.
- The space $\mathbb{B}$ is the space of complex-valued bounded Lipschitz functions $f : \bar{\Delta} \to \mathbb{C}$ such that $\|f\|_{\mathbb{B}} < \infty$ with $\|\cdot\|_{\mathbb{B}}$ defined in (3.23) for the same choice of $\varkappa$.

In view of (E5),

$$\mathcal{B} \hookrightarrow L^{q_0}(\bar{\nu}) \text{ for some } q_0 \in (1, +\infty) \tag{3.27}$$

provided that ε_0 is small enough.

Observe that, with these notations $(\tilde{\boldsymbol{\Omega}}, \tilde{\boldsymbol{\Phi}}_t, \tilde{\boldsymbol{\mu}}_0)$ can be represented by the suspension semiflow $(\tilde{\Phi}_t)_{t\geq 0}$ (with roof function τ) over the $\mathbb{Z}^2$-extension of (M, ν, T) by τ.

We define

$$\|f\|_{\mathcal{B}_0} = \|f\|_\infty + \inf\{C : \forall x, y \in \bar{\Delta} : |f(x) - f(y)| \leq C\varkappa^{\hat{s}(x,y)}\}.$$

Observe that $\mathcal{B}_0 \subset \mathcal{B} \cap \mathbb{B}$ and that the multiplication by an element of $\mathcal{B}_0$ defines a continuous linear operator on $\mathcal{B}$ and on $\mathbb{B}$.

Since κ is constant on stable manifolds, there exists a $\bar{\nu}$-centered $\mathbb{Z}^2$-valued bounded function $\bar{\kappa} \in \mathbb{B}$ such that $\bar{\kappa} \circ \mathfrak{p} = \kappa$ (therefore $m_0 = 0$).

Moreover, since $\boldsymbol{\tau}$ is 1/2-Hölder on every connected component of $\boldsymbol{M} \setminus (\mathcal{S}_0 \cup T_0^{-1}(\mathcal{S}_0))$ and since $\sqrt{\theta_0} \leq \varkappa$, we have $\tau \in \mathcal{V}$.

Now, on Δ, we define $\chi := \sum_{k\geq 0} \left(\tau \circ F^k - \tau \circ F^k \circ \Xi\right)$. By construction,

$$\tau = \bar{\tau} \circ \mathfrak{p} + \chi - \chi \circ F, \text{ where } \bar{\tau} \circ \Xi(\hat{x}^u, l) = \bar{\boldsymbol{\tau}}(\hat{x}^u, l) = \hat{\boldsymbol{\tau}}(\hat{x}^u, \hat{\boldsymbol{x}}^s, l)\,. \tag{3.28}$$

Next, we claim that $\chi \in \mathcal{V}$ and $\bar{\tau} \in \mathcal{B}_0$.

Indeed, first,

$$\|\chi\|_\infty \leq \sum_{k\geq 0} \|\tau \circ F^k - \tau \circ F^k \circ \Xi\|_\infty \leq \sum_{k\geq 0} \|\tau\|_{\mathcal{V}} \varkappa^k < \infty\,.$$

Second, if $x, y \in \Delta$ are on the same stable manifold, then $\Xi(F^n(x)) = \Xi(F^n(y))$ and so, since τ is 1/2-Hölder, for every nonnegative integer n,

$$|\chi(F^n(x)) - \chi(F^n(y))| \leq \sum_{k\geq 0} \left|\tau(F^{k+n}(x)) - \tau(F^{k+n}(y))\right| \leq C_\tau \sum_{k\geq 0} \left(C_0\theta_0^{k+n}\right)^{\frac{1}{2}} = O(\varkappa^n).$$

Third, if $x, y \in \Delta$ are on the same unstable manifold, then

$$|\tau(F^j(x)) - \tau(F^j(y))| + |\tau(F^j(\Xi(x))) - \tau(F^j(\Xi(y)))| \leq 2C_\tau (C_0\theta_0^{\hat{s}(x,y)-j})^{\frac{1}{2}}$$

and

$$|\tau(F^j(x)) - \tau(F^j(\Xi(x)))| + |\tau(F^j(y)) - \tau(F^j(\Xi(y)))| \leq 2C_\tau (C_0\theta_0^{j})^{\frac{1}{2}}.$$

So, since $\theta_0^{\frac{1}{4}} \leq \varkappa$

$$|\chi(x) - \chi(y)| \leq O\left(\sum_{0\leq k\leq \hat{s}(x,y)/2} \varkappa^{2(\hat{s}(x,y)-k)} + \sum_{k>\hat{s}(x,y)/2} \varkappa^{2k}\right) = O\left(\varkappa^{\hat{s}(x,y)}\right)\,.$$

This shows that $\chi \in \mathcal{V}$. Then clearly $\chi \circ F \in \mathcal{V}$ holds as well. Since $\tau \in \mathcal{V}$, (3.28) implies $\bar{\tau} \circ \mathfrak{p} \in \mathcal{V}$ which in turn gives $\bar{\tau} \in \mathcal{B}_0$.

Observe that $\left\|e^{i\xi.\chi}\right\|_{\mathcal{V}} = O(1+|\xi|)$ and that $(\bar{\tau}_{m_0})^k e^{-i\xi\bar{\tau}_{m_0}} \in \mathcal{B}$ for every k and $m_0 = 1$.

The fact that $(\bar{P}_{\theta,\xi} : \bar{f} \mapsto \bar{P}(e^{i\theta.\bar{\kappa}} e^{i\xi.\bar{\tau}} \bar{f}))_{(\theta,\xi)\in[-\pi,\pi]^d\times\mathbb{R}}$ satisfies (2.28), (2.29), (2.30), (2.31), with $J = 3$ follows from [29, 31] (see also [28]). Condition (2.32) is proved by Lemma 3.4.

For any $f \in \mathcal{V}$ and any nonnegative integer n, we define $\mathbf{\Pi}_n f : \bar{\Delta} \to \mathbb{C}$ by

$$\forall x \in \Delta, \quad (\mathbf{\Pi}_n f) \circ \Xi(x) := \mathbb{E}_\nu[f \circ F^n | \hat{s}(\cdot, x) \geq 2n] \,.$$

Note that $\mathbf{\Pi}_n$ is linear and continuous from $\mathcal{V}$ to $\mathcal{B}_0$ with norm in $O\left(2\varkappa^{-2n}\right)$. By definition of $\mathcal{V}$, if $s(x,y) \geq 2n$, then by considering z in the stable manifold containing x and in the unstable manifold containing y, $F^n(z)$ is in the same unstable manifold as $F^n(y)$ with $\hat{s}(F^n(y), F^n(z)) \geq n$ and so

$$|f(F^n(x)) - f(F^n(y))| \leq |f(F^n(x)) - f(F^n(z))| + |f(F^n(z)) - f(F^n(y))| \leq \|f\|_{\mathcal{V}} \varkappa^n \,.$$

Therefore we have proved that

$$\forall f \in \mathcal{V}, \quad \|f \circ F^n - \mathbf{\Pi}_n(f) \circ \Xi\|_\infty \leq C_0 \|f\|_{\mathcal{V}} \, \varkappa^n \,,$$

and so (2.33) holds for any $\vartheta \geq \varkappa$.

Recall that

$$\bar{P}^{2n}_{\theta,\xi} h(x) = \sum_{z \in \bar{F}^{-2n}(\{x\})} e^{\alpha_{2n}(z) + i\theta.\bar{\kappa}_{2n}(z) + i\xi.\bar{\tau}_{2n}(z)} h(z) \,,$$

with

$$\alpha_l := \sum_{k=0}^{l-1} \alpha \circ \bar{F}^k, \quad \bar{\kappa}_l := \sum_{k=0}^{l-1} \bar{\kappa} \circ \bar{F}^k, \quad \text{and} \quad \bar{\tau}_l := \sum_{k=0}^{l-1} \bar{\tau} \circ \bar{F}^k.$$

By construction of $(\bar{\Delta}, \bar{\nu}, \bar{F})$, for every $x, y \in \bar{\Delta}$ with $\hat{s}(x,y) \geq 1$, there exists a bijection $W_{2n} : \bar{F}^{-2n}(\{x\}) \to \bar{F}^{-2n}(\{y\})$ such that $\hat{s}(z, W_{2n}(z)) \geq 2n$ and so $\mathbf{\Pi}_n f(z) = \mathbf{\Pi}_n f(W_{2n}(z))$. Moreover, since $\alpha, \bar{\kappa}, \bar{\tau} \in \mathcal{B}_0$, for $g \in \{\alpha, \bar{\kappa}, \bar{\tau}\}$ and for any x, y, z as above, we have

$$|g(\bar{F}^k(z)) - g(\bar{F}^k(W_n(z)))| \leq \|g\|_{\mathcal{B}_0} \varkappa^{\hat{s}(x,y)+2n-k} \,.$$

Hence

$$|g_n(\bar{F}^k(z)) - g_n(\bar{F}^k(W_n(z)))| \leq \|g\|_{\mathcal{B}_0} (1-\varkappa)^{-1} \varkappa^{\hat{s}(x,y)+n-k} \,.$$

We conclude that there exists $C_0 > 0$ such that, for every $\theta \in [-\pi,\pi]^d$, $\xi \in \mathbb{R}$ and for every non-negative integer j,

$$\left\| \frac{\partial^j}{\partial(\theta,\xi)^j} (\bar{P}^{2n}_{\theta,\xi}(e^{-i\theta.\bar{\kappa}_n - i\xi.\bar{\tau}_n} \mathbf{\Pi}_n f)) \right\|_{\mathcal{B}_0} \leq \left\| \frac{\partial^j}{\partial(\theta,\xi)^j} \bar{P}^{2n}(e^{i(\theta.\bar{\kappa}_n + \xi.\bar{\tau}_n)\circ\bar{F}^n} \mathbf{\Pi}_n f) \right\|_\infty +$$

$$\sup_{\substack{x,y\in\bar{\Delta},\\ \hat{s}(x,y)\geq 1}} \varkappa^{-\hat{s}(x,y)} \left| \frac{\partial^j}{\partial(\theta,\xi)^j} \sum_{z\in\bar{F}^{-2n}(x)} \left(e^{\alpha_{2n}(z) + (i\theta\bar{\kappa}_n + i\xi\tau_n)\circ\bar{F}^n(z)} - e^{\alpha_{2n}(W_n(z)) + (i\theta\bar{\kappa}_n + i\xi\tau_n)\circ\bar{F}^n(W_n(z))} \right) \mathbf{\Pi}_n f(z) \right|$$

$$\leq C_0 n^j (1+|\xi|) \|f\|_\infty$$

and

$$\begin{aligned} \left\| \frac{\partial^j}{\partial(\theta,\xi)^j} (\mathbf{\Pi}_n(f) e^{i\theta.\bar{\kappa}_{n-m_0} + i\xi.\bar{\tau}_n}) \right\|_{\mathcal{B}'} &\leq \left\| \frac{\partial^j}{\partial(\theta,\xi)^j} (\mathbf{\Pi}_n(f) e^{i\theta.\bar{\kappa}_{n-m_0} + i\xi.\bar{\tau}_n}) \right\|_{L^p(\bar{\mu})} \\ &\leq \left\| \left(\frac{\partial^j}{\partial(\theta,\xi)^j} (\mathbf{\Pi}_n(f) e^{i\theta.\bar{\kappa}_{n-m_0} + i\xi.\bar{\tau}_n}) \right) \right\|_\infty \\ &\leq C_0 n^j \|f\|_\infty \,, \end{aligned}$$

where we used that $\bar\kappa$ and $\bar\tau$ are uniformly bounded and p is such that $\frac{1}{q_0}+\frac{1}{p}=1$ with q_0 defined in (3.27). Therefore we have proved (2.34), (2.35) and (2.36). We define f and g as follows: $f(x,\ell,s)=\mathfrak{f}(q+\ell+s\vec v,\vec v)$ and similarly $g(x,\ell,s)=\mathfrak{g}(q+\ell+s\vec v,\vec v)$ if $\pi(x)=(q,\vec v)$. Note that $(q+\ell+s\vec v,\vec v)=\widetilde{\mathbf{\Phi}}_s(q+\ell,\vec v)$ for $s\in[0,\tau(q,\vec v))$. Let $(\mathfrak{h},h)=(\mathfrak{f},f)$ or $(\mathfrak{g},g)$. We define

$$h_\ell(x,s):=\chi_0(s)\mathfrak{h}\left(\widetilde{\mathbf{\Phi}}_s(q+\ell,\vec v)\right)(1-\chi_0(s-\tau(x)))\,,$$

with $\chi_0:\mathbb{R}\to[0,1]$ a fixed increasing C^∞ function such that $\chi_0(u)=0$ if $u\le-\frac{\min\tau}{10}$ and $\chi_0(u)=1$ if $u\ge0$.
Note that $h_\ell(x,\cdot)$ have support in $\left[-\frac{\min\tau}{10},\tau(x)\right]$, coincide with $h(x,\ell,\cdot)$ in $[0,\tau(x)-\frac{\min\tau}{10}]$, and satisfy (2.37). Let $u\in\mathbb{R}$ be fixed. Then $\|h_\ell(\cdot,u)\|_\infty\le\sup_{|\ell'-\ell|\le\max\tau}\left\|\mathfrak{h}\mathbf{1}_{\mathcal{C}_{\ell'}}\right\|_\infty$. Furthermore, since $\tau\in\mathcal{V}$, $\theta_0^\eta<\varkappa$, and $\mathfrak{h}\circ\widetilde{\mathbf{\Phi}}_s$ is uniformly η-Hölder continuous for $s\in[-\frac{\min\tau}{10},\max\tau]$, we obtain that there exists a uniform constant $\widetilde C>0$ such that

$$\|h_\ell(\cdot,u)\|_{\mathcal{V}}\le\widetilde C\sup_{|\ell'-\ell|\le\max\tau}\|\mathfrak{h}\|_{\mathcal{H}^\eta_{\mathcal{C}_{\ell'}}}\,. \tag{3.29}$$

Thus, (2.41) and (2.39) follow directly from (3.3). Recall that

$$\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi\cdot\chi}\hat h_\ell(x,\xi)\right)=\sum_{m=0}^k\frac{k!}{m!\,(k-m)!}(-i\chi)^m e^{-i\xi\chi}\int_{(-\frac{\min\tau}{10},\tau(x))}(is)^{k-m}e^{i\xi s}h_\ell(x,s)\,ds\,. \tag{3.30}$$

Next, to prove (2.38) it suffices to show that

$$\sum_{\ell\in\mathbb{Z}^d}\left(\left\|\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi\cdot\chi}\hat f_\ell(\cdot,\xi)\right)\right\|_{\mathcal{V}}+\left\|\frac{\partial^k}{\partial\xi^k}\left(e^{-i\xi\cdot\chi}\hat g_\ell(\cdot,\xi)\right)\right\|_{\mathcal{V}}\right)<C(1+|\xi|)\,. \tag{3.31}$$

Observe that $\|e^{-i\xi\chi}\|_{\mathcal{V}}=O(1+|\xi|)$ and the integral in (3.30) is uniformly bounded by $2\max\tau\|h_\ell\|_\infty$. Furthermore, for $x,y\in\gamma^u$ such that $\hat s(x,y)\ge n$ (resp. for $x,y\in F^n(\gamma^s)$) and such that $\tau(x)\le\tau(y)$, we have

$$\begin{aligned}&\left|\int_{(-\frac{\min\tau}{10},\tau(x))}\ldots h_\ell(x,s)\,ds-\int_{(-\frac{\min\tau}{10},\tau(y))}\ldots h_\ell(y,s)\,ds\right|\\&\le\int_{(-\frac{\min\tau}{10},\tau(x))}|\ldots|\;|h_\ell(x,s)-h_\ell(y,s)|\;ds+\int_{\tau(x)}^{\tau(y)}|\ldots|\;|h_\ell(y,s)|\;ds\\&\le\int_{(-\frac{\min\tau}{10},\tau(x))}C\;\|h_\ell(\cdot,s)\|_{\mathcal{V}}\,\varkappa^n\,ds+\|\tau\|_{\mathcal{V}}\,\varkappa^nC\;\|h_\ell(\cdot,s)\|_\infty\;ds\,.\end{aligned}$$

Now (3.31) follows from (3.29) and (3.3).

Assume next that $\mathfrak{h}$ satisfies (3.1), then the functions $h_\ell(x,\cdot)$ are C^∞ and there exists a uniform constant $\widetilde C_0>0$ such that

$$\forall N\in\mathbb{N},\quad\left\|\frac{\partial^N}{\partial s^N}h_\ell(\cdot,s)\right\|_{\mathcal{V}}\le\widetilde C_0\sup_{m=0,\ldots,N}\sup_{|\ell'-\ell|\le\max\tau}\left\|\frac{\partial^m}{\partial s^m}\left(\mathfrak{h}\circ\widetilde{\mathbf{\Phi}}_s\right)_{|s=0}\right\|_{\mathcal{H}^\eta_{\mathcal{C}_{\ell'}}}\,.$$

Moreover, since h_ℓ is C^∞ with compact support, by classical integration by parts, we have

$$\forall N\in\mathbb{N},\quad\hat h_\ell(x,\xi)=(-i)^N\xi^{-N}\int_{\mathbb{R}}e^{i\xi\,s}\frac{\partial^N}{\partial s^N}h_\ell(\cdot,s)\,ds$$

Therefore, since $\chi\in\mathcal{V}$, we have proved that, if $\mathfrak{h}$ satisfies (3.1), we have

$$\forall\gamma>0,\quad\sum_\ell\|e^{-i\xi\,\chi}\hat h_\ell(\cdot,-\xi)\|_{\mathcal{V}}=O(|\xi|^{-\gamma})\,, \tag{3.32}$$

which, combined with (3.31) implies (2.40).

3.5. **Identifying $\mathfrak{C}_0$.** Recall the notations $\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}, \Sigma_{\boldsymbol{\kappa}}$ from Section 3.1 and that here $d=2$.

Let us set $\sigma := \sqrt{\det \Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}} / \det \Sigma_{\boldsymbol{\kappa}}}$. Observe that $\Psi_{\Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}}(0,0,u) = \frac{e^{-\frac{u^2}{2\sigma^2}}}{(2\pi)^{\frac{3}{2}}\sqrt{\det \Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}}}$.

Now the leading term of $C_t(f,g)$ can be obtained by taking $m=j=k=r=q=0$ in (2.42):

$$\begin{aligned}\lim_{t\to\infty} tC_t(f,g) &= \boldsymbol{\nu}(\boldsymbol{\tau})\tilde{C}_0(f,g) \qquad (3.33)\\ &= (\boldsymbol{\nu}(\boldsymbol{\tau}))^{\frac{1}{2}} \int_{\mathbb{R}} \psi\left(0,0,s\sqrt{\boldsymbol{\nu}(\boldsymbol{\tau})}\right) ds \sum_{\ell,\ell'\in\mathbb{Z}^2} \int_{\mathbb{R}^2} \mathcal{B}_0(f_\ell(\cdot,u), g_{\ell'}(\cdot,v))\, dudv\\ &= \frac{\sigma}{2\pi\sqrt{\det \Sigma_{\boldsymbol{\kappa},\boldsymbol{\tau}}}}\tilde{\mu}(f)\tilde{\mu}(g) = \frac{1}{2\pi\sqrt{\det \Sigma_{\boldsymbol{\kappa}}}}\tilde{\mu}(f)\tilde{\mu}(g)\end{aligned}$$

where we used $\mathcal{B}_0(u,v) = \nu(u)\nu(v)$ (see (2.44)).

Recalling that the left hand side of (3.4) is an integral with respect to $\tilde{\boldsymbol{\mu}}_0$ as opposed to $C_t(f,g)$ which is an integral with respect to $\tilde{\mu}$ and using $\tilde{\mu} = \boldsymbol{\nu}(\boldsymbol{\tau})\tilde{\boldsymbol{\mu}}_0$, we obtain (3.5).

4. Geodesic flows

Let Q be a compact Riemannian manifold with strictly negative curvature and $\tilde{Q}$ be a cover of Q with automorphism group $\mathbb{Z}^d$. Then $\tilde{Q}$ can be identified with $Q\times\mathbb{Z}^d$.

The unit tangent bundle of $\tilde{Q}$ is denoted by $\tilde{\Omega}$ and unit tangent bundle of Q is denoted by Ω.

The phase space of the geodesic flow $\tilde{\Phi}$ on $\tilde{Q}$ is $\tilde{\Omega}$ and likewise, the phase space of the geodesic flow Φ on Q is Ω. Thus $\tilde{\Omega}$ is a $\mathbb{Z}^d$ cover of Ω and we denote by Let $\mathfrak{p}$ the covering map. Geodesic flows are Anosov flows and can be represented as a suspension flows over a Poincaré section M such that $T: M\to M$, the first return map to M is Markov (see [4] and [5]). Thus M is a union of rectangles $M = \cup_{k=1}^K \Delta_k$ where Δ_k have product structure $\Delta_k = [\Delta_k^u \times \Delta_k^s]$ where Δ_k^u are u-sets and Δ_k^s are s-sets and $[\cdot,\cdot]$ is defined by (3.21).

Let τ be the first return to M. Choose a copy $\tilde{M}\subset\tilde{\Omega}$ such that $\mathfrak{p}(\tilde{M}) = M$ and $\mathfrak{p}: \tilde{M}\to M$ is one-to-one. As for billiards, we define $\mathcal{C}_\ell$ as the set of points in that $\tilde{\Omega}$ such that the last visit to the Poincaré section was in $\tilde{M}\times\{\ell\}$ for $\ell\in\mathbb{Z}^d$. We denote by $\tilde{\mu}$ the Liouville measure.

Now we have the following analogue of Theorem 3.1

Theorem 4.1. *Let $\mathfrak{f},\mathfrak{g}:\tilde{\Omega}\to\mathbb{R}$ be two η-Hölder continuous functions with at least one of them being smooth in the flow direction. Assume moreover that there exists an integer $K_0\geq 1$ such that* (3.3) *holds. Then there are real numbers $\mathfrak{C}_0(\mathfrak{f},\mathfrak{g}), \mathfrak{C}_1(\mathfrak{f},\mathfrak{g}), ..., \mathfrak{C}_{K_0}(\mathfrak{f},\mathfrak{g})$ so that we have*

$$\int_{\tilde{\Omega}} \mathfrak{f}\,\mathfrak{g}\circ\tilde{\Phi}_t d\tilde{\mu}_0 = \sum_{k=0}^{K_0} \mathfrak{C}_k(\mathfrak{f},\mathfrak{g}) t^{-\frac{d}{2}-k} + o\left(t^{-\frac{d}{2}-K_0}\right), \qquad (4.1)$$

as $t\to+\infty$. Furthermore, $\mathfrak{C}_0(\mathfrak{f},\mathfrak{g}) = \mathfrak{c}_0\int_{\tilde{\Omega}}\mathfrak{f}d\tilde{\mu}_0\int_{\tilde{\Omega}}\mathfrak{g}d\tilde{\mu}_0$ and the coefficients $\mathfrak{C}_k$, as functionals over pairs of admissible functions, are bilinear.

Proof. The proof of Theorem 4.1 is a simplified version of that of Theorem 3.1. Namely, we still apply the abstract Theorem 2.2 to an appropriate symbolic system. This system is now a subshift of finite type that is constructed using a Markov partition $\{\Delta_k\}$. By mixing and by the Perron-Frobenius theorem, there exists r so that for any $i,j=1,...,K$, $T^r(\Delta_i)$ and Δ_j have a non empty intersection. We define the spaces $\mathcal{V},\mathcal{B}$, and $\mathbb{B}$ the same way as in Section 3 with

$$\Delta_0 = M \quad\text{and}\quad \bar{\Delta}_0 = \bigcup_{k=1}^K \Delta_k^u.$$

and with constant height r. Consequently, the norms $\|.\|_{\mathbb{B}}$ and $\|.\|_{\mathcal{B}}$ are equivalent. The assumptions of Theorem 2.2 are verified similarly to Section 3 with additional simplifications coming from the boundedness of the return time and the equivalence of $\mathcal{B}$ and $\mathbb{B}$.

The only point in the proof of Theorem 3.1 where we used the special properties of billiards is in the proof of Lemma 3.2, where we referred to Lemma 6.40 in [7] (which is specific to billiards). It remains to revisit this part of the argument (again, in a simplified version as the alphabet is finite and we do not need to verify conditions (3.11) - (3.13)).

Geodesic flows preserve the natural contact form α on the unit tangent bundle (corresponding to the symplectic structure on the tangent bundle). According to the results of [19] (Lemma B.6), there is some $\varepsilon > 0$ so that for any $z \in Q$ and for any sufficiently small unstable vector $v \in E^u(z)$ and stable vector $w \in E^s(z)$ with the notation $x = \exp_z(v)$, $y = \exp_z(w)$, the temporal distance function $D(x, y)$ (defined as in (3.9)) satisfies

$$D(x, y) = d\alpha(v, w) + O(\|v\|^{\varepsilon}\|w\|^2 + \|v\|^2\|w\|^{\varepsilon}) .$$

Since the contact form is non-degenerate, there is a constant R_0 such that for any z and any $v \in E^u(z)$, we can find some $w \in T_zQ$ such that $\frac{\|v\|\|w\|}{R_0} \geq d\alpha(v, w) \geq R_0\|v\|\|w\|$. Let us decompose w into center unstable and stable components $w = w^{cu} + w^s$. By Lemma B.2 in [19], $d\alpha(v, w^{cu}) = 0$ and so we can assume $w = w^s \in E^s(z)$. We conclude that for fixed z, there are constants δ_0, R_0, so that for any $\delta < \delta_0$ there exist vectors $v \in E^u(z), w \in E^s(z)$ such that $\|v\| = \|w\| = \delta$ and

$$D(x, y) \in \left[\frac{\delta^2}{2R_0}, 2R_0\delta^2\right] .$$

Now we can complete the proof of the analogue of Lemma 3.2 as before by choosing δ in a way that for given ξ, $\delta^2 \approx \xi^{-1}$. □

Appendix A. Some facts about Taylor expansions.

Lemma A.1. *Let a be given by* (2.4) *and a C^{K+3}-smooth function $\tilde{\lambda} : [-b, b]^{d+1} \to \mathbb{C}$ (for some $b > 0$) satisfying* (2.6) *for some $J \leq K + 3$. Denote $\zeta_s = \frac{\tilde{\lambda}_s}{a_s}$, $M = \lfloor (K + 1)/(J - 2) \rfloor$. Then there are $A_{j,k} \in \mathcal{S}_j$ (where $j = 0, ...\lfloor J(K + 1)/(J - 2) \rfloor$, $k = 1, \dots, M$), $K_0 \in \mathbb{N}$ (depending on K and J) and a function $\eta : \mathbb{R}^{d+1} \to [0, +\infty)$ continuous at $\mathbf{0}$, satisfying $\eta(\mathbf{0}) = 0$ such that after, possibly, decreasing the value of b, for every n large enough, every $s \in [-b\sqrt{n}, b\sqrt{n}]^{d+1}$ and every $j = J, ..., K + 3$, we have*

$$\sum_{k=1}^{M} \binom{n}{k} \sum_{j_1,...,j_k \geq J \,:\, j_1+...+j_k=j} \frac{1}{j_1!...j_k!} \left(\zeta_0^{(j_1)} \otimes ... \otimes \zeta_0^{(j_k)}\right) = \sum_{k=1}^{M} n^k A_{j,k} \tag{A.1}$$

and

$$\left| \zeta^n_{s/\sqrt{n}} - 1 - \sum_{k=1}^{M} \sum_{j=kJ}^{K+1+2k} n^k A_{j,k} * \left(\frac{s}{\sqrt{n}}\right)^{\otimes j} \right| \leq \frac{1}{a_{s/\sqrt{2}}} n^{-\frac{K+1}{2}} (1 + |s|^{K_0})\eta(s/\sqrt{n}) . \tag{A.2}$$

Recalling that the first $J - 1$ derivatives of ζ vanish at zero, we see that in case $\tilde{\lambda}$ is C^j (namely, if $j \leq K + 3$), the LHS of (A.1) is simply equal to $\frac{1}{j!}(\zeta^n)^{(j)}_0$.

Proof. Decreasing if necessary the value of b, we may assume that $|\tilde{\lambda}_u| \leq a_{u/\sqrt{2.5}} \leq a_{u/\sqrt{2}}$ and $|\tilde{\lambda}_u - a_u| \leq C|u|^J$ for every $u \in \mathbb{R}^{d+1}$ with $|u| < b$ (the existence of b with these properties follows from our assumptions on J and $\tilde{\lambda}$). Applying Taylor's theorem to the function $x \mapsto x^n$

near 1 we conclude that for every $s \in \mathbb{R}^{d+1}$ with $|s| < b\sqrt{n}$,

$$\left|\zeta^n_{s/\sqrt{n}} - \sum_{k=0}^{M} \binom{n}{k} \left(\zeta\left(\frac{s}{\sqrt{n}}\right) - 1\right)^k\right|$$
$$\leq \binom{n}{M+1} \left|\zeta\left(\frac{s}{\sqrt{n}}\right) - 1\right|^{M+1} (\max(1, |\zeta\left(\frac{s}{\sqrt{n}}\right)|))^{n-M-1}. \tag{A.3}$$

Recall that $|\tilde{\lambda}_{s/\sqrt{n}}| \leq a_{s/\sqrt{1.5\,n}}$. This together with the fact that $a_{s/\sqrt{1.5\,n}}/a_{s/\sqrt{n}} = (a_{s/\sqrt{3n}})^{-1}$ implies that the RHS of (A.3) is bounded by

$$n^{M+1} \left|\zeta(s/\sqrt{n}) - 1\right|^{M+1} (a_{s/\sqrt{3n}})^{-(n-M-1)} = n^{M+1} \left|\tilde{\lambda}(s/\sqrt{n}) - a(s/\sqrt{n})\right|^{M+1} (a_{s/\sqrt{3n}})^{-n-M-1}.$$

Next, we use the identity $(a_{s/\sqrt{3n}})^n = a_{s/\sqrt{3}}$ and the inequality $|\tilde{\lambda}_u - a_u| \leq C|u|^J$ to conclude that the last displayed expression is bounded by

$$C_M n^{M+1} (a_{s/\sqrt{2}})^{-1} \left((s/\sqrt{n})^{J(M+1)}\right),$$

for every s, for every n large enough since $(a_{s/\sqrt{3n}})^{-n-M-1} = \left(a_{s\sqrt{(1+\frac{M+1}{n})/3}}\right)^{-1} \leq (a_{s/\sqrt{2}})^{-1}$ for every n large enough. Now observe that by definition $(2-J)(M+1) < -K-1$ and so $(2-J)(M+1) \leq -K-2$. Thus the last display, and hence (A.3) is bounded by

$$C_M (a_{s/\sqrt{2}})^{-1} n^{-\frac{K+2}{2}} s^{J(M+1)}. \tag{A.4}$$

Clearly, (A.4) can be included in the RHS of (A.2). Thus it remains to compute the sum in the LHS of (A.3).

To do so, we fix some $k = 1, ..., M$. Let $L = K + 1 + 2k - J(k-1)$. Using the elementary estimate $|a^k - b^k| \leq k \max(|a|, |b|)^{k-1}|a-b|$, we find

$$\binom{n}{k} \left|\left(\zeta(s/\sqrt{n}) - 1\right)^k - \left(\sum_{j=J}^{L} \frac{1}{j!} \zeta_0^{(j)} * (s/\sqrt{n})^{\otimes j}\right)^k\right| \tag{A.5}$$
$$\leq \quad n^k k \max\left(|\zeta(s/\sqrt{n}) - 1|, \left|\sum_{j=J}^{L} \frac{1}{j!} \zeta_0^{(j)} * (s/\sqrt{n})^{\otimes j}\right|\right)^{k-1} \tag{A.6}$$
$$\left|\zeta(s/\sqrt{n}) - 1 - \sum_{j=J}^{L} \frac{1}{j!} \zeta_0^{(j)} * (s/\sqrt{n})^{\otimes j}\right|. \tag{A.7}$$

Next by our choice of L

$$L = K + 1 + (2-J)k + J \leq K + 1 + (2-J) + J = K + 3.$$

Recalling that $\tilde{\lambda}/a$ is C^{K+3} smooth and its first $J-1$ derivatives at zero vanish, Taylor's theorem implies that (A.7) is bounded by $(s/\sqrt{n})^L \eta_0(s/\sqrt{n})$, where $\eta_0(0) = 0$ and η is continuous at 0. On the other hand, (A.6) is bounded by $n^k k \left(s/\sqrt{n}\right)^{J(k-1)}$. We conclude that (A.5) is bounded by

$$n^{-\frac{K+1}{2}} s^{K+1+2k} \eta_1(s/\sqrt{n}), \tag{A.8}$$

where $\eta_1 = k\eta_0$. Since $a_{s/\sqrt{2}}$ is bounded from above, (A.8) can be included in the RHS of (A.2). So we have approximated $\zeta^n_{s/\sqrt{n}}$ by

$$\begin{aligned}
& 1+\sum_{k=1}^{M}\binom{n}{k}\left(\sum_{j=J}^{L}\frac{1}{j!}\zeta_0^{(j)} * (s/\sqrt{n})^{\otimes j}\right)^k \\
= \;& 1+\sum_{k=1}^{M}\binom{n}{k}\sum_{j_1,\dots,j_k=J}^{L}\frac{1}{j_1!\dots j_k!}\left(\zeta_0^{(j_1)}\otimes\dots\otimes\zeta_0^{(j_k)}\right) * (s/\sqrt{n})^{\otimes(j_1+\dots+j_k)} \\
= \;& 1+\sum_{k=1}^{M}\binom{n}{k}\sum_{j=kJ}^{K+1+2k}\sum_{j_1,\dots,j_k\geq J\,:\,j_1+\dots+j_k=j}\frac{1}{j_1!\dots j_k!}\left(\zeta_0^{(j_1)}\otimes\dots\otimes\zeta_0^{(j_k)}\right) \\
& *(s/\sqrt{n})^{\otimes j}+O\left(n^{-\frac{K+2}{2}}s^{K+1+2k+1}\right)
\end{aligned}$$

uniformly on $s\in[-b\sqrt{n},b\sqrt{n}]^{d+1}$. Note that the last step above uses the observation that if $j_1,\dots,j_k\geq J$ and $j_1+\dots+j_k\leq K+1+2k$, then necessarily $j_l\leq L$ for all l. Again, the last error term can be included in the right hand side of (A.2) as $a_{s/\sqrt{2}}$ is bounded from above.

Finally, observe that

$$\binom{n}{k}\sum_{j_1,\dots,j_k\geq J\,:\,j_1+\dots+j_k=j}\frac{1}{j_1!\dots j_k!}\left(\zeta_0^{(j_1)}\otimes\dots\otimes\zeta_0^{(j_k)}\right)$$

is a polynomial of degree k in n with values in $\mathcal{S}_j$. This ensures the existence of $A_{j,k}$. □

Lemma A.2. *If $H:\mathbb{R}\to\mathbb{R}$ is in the Schwartz space (i.e. $x^aH^{(b)}(x)$ is bounded for any positive integers a and b), then for any $L\in\mathbb{N}$ there is some constant $c_{H,L}$ such that*

$$\forall t\in\mathbb{R},\forall\eta>0,\quad \left|\sum_{k\in\mathbb{Z}}\eta H(t+k\eta)-\int_{-\infty}^{\infty}H(x)dx\right|<c_{H,L}\eta^L. \tag{A.9}$$

Proof. We can assume without loss of generality that $t\in[0,1)$. Given L,t and η, we choose A_L and B_L so that the above sum for $k\notin[A_L/\eta,B_L/\eta]$ and the above integral as well as the first L derivatives of H for $x\notin(A_L,B_L)$ are less than η^L. Such A_L and B_L exist since H is in the Schwartz space. Now Euler's summation formula (e.g. Theorem 4 in [3] with the notation $f(x)=\eta H(t+x\eta-A_L)$, $m=L$) implies that

$$\begin{aligned}
\sum_{k=-A_L/\eta}^{B_L/\eta}\eta H(t+k\eta)-\int_{A_L}^{B_L}H(x)dx = \;& \frac{1}{(2L+1)!}\int_{A_L}^{B_L}\mathcal{P}_{2L+1}(x/\eta)H^{(2L+1)}(x)dx\eta^{2L+1} \\
& +\sum_{r=1}^{L}\frac{\mathcal{B}_{2r}}{(2r)!}\left[H^{(2r-1)}(B_L)-H^{(2r-1)}(A_L)\right]\eta^{2r} \\
& +\frac{1}{2}\eta[H(B_L)-H(A_L)],
\end{aligned}$$

where $\mathcal{P}_k(x)$ are the periodic Bernoulli polynomials and $\mathcal{B}_k$ are Bernoulli numbers. Now (A.9) follows from the choice of A_L,B_L. □

Observe that (A.9) and the fact that H is in the Schwartz space imply

$$\forall K>0,\quad \forall\varepsilon>0,\quad \sum_{n=t/\nu(\tau)-t^{\frac{1}{2}+\varepsilon}}^{t/\nu(\tau)+t^{\frac{1}{2}+\varepsilon}}H\left(\frac{t-n\nu(\tau)}{\sqrt{t}}\right)=\frac{\sqrt{t}}{\nu(\tau)}\int_{\mathbb{R}}H(x)dx+O(t^{-K}) \tag{A.10}$$

(clearly, the constant in "O" depends on K and ε).

Lemma A.3. *For every $\gamma\in\mathbb{R}$ and $Q\in\mathbb{Z}_+$,*

$$\sum_{n=t_-}^{t_+} n^\gamma \Psi^{(\alpha)}\left(0,\frac{t-n\nu(\tau)}{\sqrt{n}}\right)$$

$$= \left(\frac{t}{\nu(\tau)}\right)^\gamma \sum_{q=0}^{Q} \frac{1}{q!}\frac{t^{-\frac{q-1}{2}}}{\nu(\tau)} \int_{\mathbb{R}} \partial_2^q h_{\alpha,\gamma}(s,1)\,(-s)^q\,ds + O\left(t^{\gamma-\frac{Q}{2}}\right) \tag{A.11}$$

where $h_{\alpha,\gamma}$ is defined by (2.5) *∂_2^q denotes the derivative of order q with respect to the second variable.*

Proof. For ease of notation, we prove the lemma coordinate-wise, i.e. we replace $\Psi^{(\alpha)}(s)$ by $\frac{\partial^\alpha}{\partial s_{j_1}\dots\partial s_{j_\alpha}}\Psi(s)$.

Observe that due to the rapid decay of $\Psi^{(m+j+r)}(0,\cdot)$, we can replace $\sum_{n=t_-}^{t_+}$ by $\sum_{n=t/\nu(\tau)-t^{\frac12+\varepsilon}}^{t/\nu(\tau)+t^{\frac12+\varepsilon}}$, for any $\varepsilon>0$ (here, we can choose e.g., $\varepsilon=1/4$).

Next, observe that by the definition (2.5),

$$\left(\frac{n}{t/\nu(\tau)}\right)^\gamma \Psi^{(\alpha)}\left(0,\frac{t-n\nu(\tau)}{\sqrt{n}}\right) = h_{\alpha,\gamma}\left(\frac{t-n\nu(\tau)}{\sqrt{t}},\frac{n\nu(\tau)}{t}\right).$$

Thus it remains to estimate the sum

$$\sum_{n=t/\nu(\tau)-t^{\frac12+\varepsilon}}^{t/\nu(\tau)+t^{\frac12+\varepsilon}} h_{\alpha,\gamma}\left(\frac{t-n\nu(\tau)}{\sqrt{t}},\frac{n\nu(\tau)}{t}\right). \tag{A.12}$$

Using Taylor expansion, we can rewrite (A.12) as

$$\left[\sum_{n=t/\nu(\tau)-t^{\frac12+\varepsilon}}^{t/\nu(\tau)+t^{\frac12+\varepsilon}} \sum_{q=0}^{Q} \frac{1}{q!}\partial_2^q h_{\alpha,\gamma}\left(\frac{t-n\nu(\tau)}{\sqrt{t}},1\right)\left(-\frac{t-n\nu(\tau)}{t}\right)^q\right] + O\left(t^{-\frac{Q}{2}}\right). \tag{A.13}$$

Indeed, we control the error term using the estimate

$$\sum_{n=t/\nu(\tau)-t^{\frac12+\varepsilon}}^{t/\nu(\tau)+t^{\frac12+\varepsilon}} \sup_{|y-1|<1/2}\left|\partial_2^{Q+1}h_{\alpha,\gamma}\left(\frac{t-n\nu(\tau)}{\sqrt{t}},y\right)\right|\left|\frac{t-n\nu(\tau)}{t}\right|^{Q+1} \;=\; O\left(t^{-\frac{Q}{2}}\right),$$

which can be derived similarly to (A.10). Performing summation over n in (A.13), using (A.10), we obtain that (A.12) (and thus the left hand side of (A.11)) equals to

$$\sum_{q=0}^{Q} \frac{1}{q!}\frac{t^{-\frac{q-1}{2}}}{\nu(\tau)} \int_{\mathbb{R}} \partial_2^q h_{\alpha,\gamma}(s,1)\,(-s)^q\,ds + O\left(t^{-\frac{Q}{2}}\right).$$

This completes the proof of the lemma. □

Lemma A.4. *Let b,q be non-negative integers. The function $s\mapsto \partial_2^q h_{b,\gamma}(s,1)(-s)^q$ is even if $b+q$ is even (and is odd if $b+q$ is odd).*

Proof. The lemma follows since if $P(x)$ is a polynomial with odd (even, resp.) leading term, then $\frac{d}{dx}(P(x)e^{cx^2}) = Q(x)e^{cx^2}$ where $Q(x)$ is a polynomial with even (odd, resp.) leading term. □

Appendix B. Correlation functions of coboundaries

Lemma B.1. *Let $\mathbf{G}^t : \mathbf{M} \to \mathbf{M}$ be a flow preserving a measure $\boldsymbol{\mu}$ (finite or infinite). Let $f, f', g : \mathbf{M} \to \mathbf{M}$ be bounded integrable observables such that $f'(x) = \frac{d}{dt}|_{t=0} f(\mathbf{G}^t x)$. Denote*

$$C_t = \int_{\mathbf{M}} f\left(g \circ \mathbf{G}^t\right) d\mu, \quad C'_t = \int_{\mathbf{M}} f'\left(g \circ \mathbf{G}^t\right) d\mu.$$

Assume that there exist real numbers $\alpha > 0$, $c_0, ..., c_{K-1}, c'_0, ..., c'_K$ satisfying:

$$C_t = t^{-\alpha}\left(\sum_{k=0}^{K-1} c_k t^{-k} + o\left(t^{-(K-1)}\right)\right) \quad \text{and} \quad C'_t = t^{-\alpha}\left(\sum_{k=0}^{K} c'_k t^{-k} + o\left(t^{-K}\right)\right). \tag{B.1}$$

Then $c'_0 = 0$ and $c'_k = -c_{k-1}(\alpha + k - 1)$ for every $k = 1, ..., K-1$.

In particular if $K = 1$ and $c_0 \neq 0$, then $c'_0 = 0$ and

$$C_t(f', g) \sim -c_0 \alpha t^{-\alpha-1} \tag{B.2}$$

We note that the fact that the rate of mixing for coboundaries is faster than for general observables is used, for example, in [11, 13].

Proof. By integration by parts

$$\begin{aligned} C'_t &= \int_{\mathbf{M}} f'\left(g \circ \mathbf{G}^t\right) d\boldsymbol{\mu} = -\int_{\mathbf{M}} f\left(g' \circ \mathbf{G}^t\right) d\boldsymbol{\mu} \\ &= -\int_{\mathbf{M}} f.\frac{\partial}{\partial t}\left(g \circ \mathbf{G}^t\right) d\boldsymbol{\mu} = -\frac{\partial}{\partial t}\int_{\mathbf{M}} f\left(g \circ \mathbf{G}^t\right) d\boldsymbol{\mu} = -\frac{\partial}{\partial t} C_t. \end{aligned}$$

Since $\lim_{t\to+\infty} C_t = 0$

$$C_t = \int_t^{+\infty} C'_s\, ds = \int_t^{+\infty} \sum_{k=0}^{K} c'_k s^{-\alpha-k} + o(s^{-\alpha-K})\, ds\,.$$

It follows that $c'_k = 0$ if $\alpha + k \leq 1$ and

$$C_t = \sum_{k=0}^{K} \frac{c'_k}{-\alpha - k + 1} t^{-\alpha+1-k} + o\left(t^{-\alpha-K+1}\right).$$

The lemma follows by comparing the above expansion with the first equation in (B.1). □

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Department of Mathematics, University of Maryland, College Park, MD 20741, USA
Email address: dmitry@math.umd.edu

Department of Mathematical Sciences, Yeshiva University, New York, NY, 10016, USA
Email address: peter.nandori@yu.edu

Univ Brest, Université de Brest, Institut Universitaire de France, IUF, UMR CNRS 6205, Laboratoire de Mathématiques de Bretagne Atlantique, LMBA, France
Email address: francoise.pene@univ-brest.fr