PREVALENCE OF RAPID MIXING IN HYPERBOLIC FLOWS.

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Abstract. We provide necessary and sufficient conditions for a suspension flow over a subshift of a finite type to mix faster than any power of time. Then we show that these conditions are satisfied if the flow have two periodic orbits such that the ratio of the periods can not be well approximated by rationals.

1. Introduction. Bowen-Ruelle conjecture states that every mixing Axiom A flow is also exponentially mixing.This is equivalent to the corresponding property of suspension flows over subshifts of finite type. However it turns out that if the roof function is locally constant no such flow can be exponentially mixing. Moreover, given any function of time one can produce a flow whose mixing rate is not faster than this function. In this paper we prove that nonetheless most flows have quite fast decay of correlations. This is done by showing that given the lengths of two periodic orbits the correlations can not decay slower than in a locally constant roof function flow.

We begin with describing necessary and sufficient conditions for rapid mixing. For this we need some notation. Let (Σ, σ) the one-sided subshift of a finite type with transition matrix Q. (All theorems we prove are valid also for two-sided subshifts but we consider one-sided case to simplify formulae.) A word means an admissible sequence of finite length. We assume that σ is topologically mixing that is some power of Q is strictly positive. Consider the distance $d_\theta(\omega, \varpi) = \theta^k$, where $k = \max\{j : \omega_i = \varpi_i \text{ for } i \leq j\}$, $\theta < 1$. Let $C_\theta(\Sigma)$ be the space of functions Lipschitzian with respect to d endowed with the norm $\|h\|_\theta = \max(\|h\|_0, L(h))$ where $L(h)$ is the Lipschitz constant of h. If $\tau \in C_\theta(\Sigma)$ is positive the suspension flow with the roof function τ is defined on

$$\Sigma^\tau = \{(\omega, t) : 0 \leq t \leq \tau(\omega)\} / \left((\omega, \tau(\omega)) \sim (\sigma(\omega), 0)\right)$$

by $S^t(\omega, s) = (\omega, t + s)$ subject to the identification above. We will denote points of Σ^τ by q. We write $\tau_1 \sim \tau_2$ if τ_1 is cohomologous to τ_2 that is $\tau_2(\omega) = \tau_1(\omega) + T(\omega) - T(\sigma\omega)$. We refer the reader to [PP] for the background about the basic notions of thermodynamic formalism such as Gibbs measures, Ruelle-Perron-Frobenius theorem, Livshic's theorem etc. Let us introduce the set of test functions.

DEFINITION. The space $D(\Sigma^\tau)$ consist of functions of the form $D(\Sigma^\tau) = \{A(\omega, t) = 1_{\mathcal{C}}(\omega) a(t)$, where $\mathcal{C}$ is a cylinder in Σ and $a(t)$ is a C^∞ function with a compact support$\}$. It is assumed in this definition that $\operatorname{supp} a(t)$ lies strictly inside the interval $[0, \min_{\mathcal{C}} \tau(\omega)]$.

On Σ^τ we have the distance $d_\theta((\omega, t), (\omega', t')) = d_\theta(\omega, \omega') + |t - t'|$. Let $F(q)$ be a $d_\theta-$Lipschitz function and μ_F be its Gibbs measure. For $A, B \in C(\Sigma^\tau)$ denote $R_{A,B}(F, t) = \mu_F(A(q)B(S^t q)) - \mu_F(A)\mu_F(B)$.

DEFINITION. $\{S^t\} \in RM(F)$ ($\{S^t\}$ is rapidly mixing with respect to F) if
1) for $A, B \in D(\Sigma^\tau)$ $R_{A,B}(F, t)$ belongs to Schwartz space;
2)the map $(A, B) \to R_{A,B}$ is continuous in the sense that given natural n there are $N(n)$, $C(n)$, $K(n)$ such that

$$|R_{A,B}(F, t) t^n| \leq C(n) \|a\|_{C^N} \|b\|_{C^N} K^{l(\mathcal{C}(A)) + l(\mathcal{C}(B))}$$

where $A(\omega, t) = 1_{\mathcal{C}(A)}(\omega) a(t)$, $B(\omega, t) = 1_{\mathcal{C}(B)}(\omega) b(t)$ and $l(\mathcal{C})$ denotes the length of $\mathcal{C}$. $\{S^t\}$ is called rapidly mixing ($\{S^t\} \in RM$) if it is rapidly mixing with respect to any Lipschitz F.

DEFINITION. Let $\{S^t\}$ be a semiflow on a metric space space M. We write $\{S^t\} \in TPM(q_1, q_2, t_0, \alpha_0)$ if for any r and $t \geq \max\left(\left(\frac{1}{r}\right)^{\alpha_0}, t_0\right)$

$$S^t \mathcal{B}(q_1, r) \bigcap \mathcal{B}(q_2, r) \neq \emptyset$$

where $\mathcal{B}(q,r)$ is the ball of radius r about q. $\{S^t\}$ is called topologically power mixing ($\{S^t\} \in TPM$) if there are t_0, α_0 such that $\forall q_1, q_2$ $\{S^t\} \in TPM(q_1, q_2, t_0, \alpha_0)$.

We write $h_n(\omega) = \sum\limits_{j=0}^{n-1} h(\sigma^j \omega)$. Given $f \in C_\theta(\Sigma)$ denote by $\mathcal{L}_f$ the transfer operator on $C_\theta(\Sigma)$

$$\mathcal{L}_f h(\omega) = \sum_{\sigma\varpi=\omega} e^{f(\varpi)} h(\varpi)$$

and and by $\mathcal{V}_b$ the operator

$$(\mathcal{V}_b h)(\omega) = e^{ib\tau(\omega)} h(\sigma\omega).$$

Below we give several conditions equivalent to rapid mixing. Our result is a quantitative version of the similar statement for mixing. Let us recall this theorem to the reader.

Proposition 1. *The following conditions are equivalent*

(i) $\exists F$ *such that* $(\Sigma^\tau, \{S^t\}, \mu_F)$ *is mixing;*

(ii) $\forall F$ $(\Sigma^\tau, \{S^t\}, \mu_F)$ *is mixing;*

(iii) $\exists F$ *such that* $(\Sigma^\tau, \{S^t\}, \mu_F)$ *is weak mixing;*

(iv) $\forall F$ $(\Sigma^\tau, \{S^t\}, \mu_F)$ *is weak mixing;*

(v) $\{S^t\}$ *has no Lipschitz continuous eigenfunction;*

(vi) $\{S^t\}$ *is topologically mixing;*

(vii) $\exists f$ *such that* $\mathcal{L}_f 1 = 1$ *and for all real* $b \neq 0$ $1 - \mathcal{L}_{f+ib\tau}$ *is invertible;*

(viii) $\forall f$ *such that* $\mathcal{L}_f 1 = 1$ *for all real* $b \neq 0$ $1 - \mathcal{L}_{f+ib\tau}$ *is invertible;*

(ix) $\exists f$ *such that* $\mathcal{L}_f 1 = 1$ *and for all real* $b \neq 0$ $\mathcal{L}_{f+ib\tau}$ *is not conjugated to* $\mathcal{L}_f$*;*

(x) $\forall f$ *such that* $\mathcal{L}_f 1 = 1$ *for all real* $b \neq 0$ $\mathcal{L}_{f+ib\tau}$ *is not conjugated to* $\mathcal{L}_f$*;*

(xi) *for all real* $b \neq 0$ *the operator* $\mathcal{V}_b$ *has no Lipschitz eigenfunction with eigenvalue 1.*

To introduce our conditions we need two definitions.

Definition. Let $\mathcal{L} : C_\theta(\Sigma) \to C_\theta(\Sigma)$ be a linear operator. h is called $(\lambda, N, \epsilon)-$approximate eigenfunction if $1 \leq \|h\|_\theta \leq N$ and

$$\forall \omega \ |(\mathcal{L}h - \lambda h)(\omega)| \leq \epsilon.$$

Definition. $H \in C_\theta(\Sigma^\tau)$ is called (λ, N, ϵ) approximate eigenfunction for $\{S^t\}$ if $1 \leq \|h\|_\theta \leq N$, $H(t,q) = H(S^t q)$ is differentiable function of t and $\forall q, t$

$$|(\partial_t - \lambda) H(t,q)| \leq \epsilon.$$

THEOREM 1. *The following conditions are equivalent:*
(i) $\{S^t\} \in RM$;
(ii) $\exists F : \{S^t\} \in RM(F)$;
(iii) $\{S^t\} \in TPM$;
(iv) $\exists q_1, q_2, : \ \{S^t\} \in TPM(q_1, q_2, t_0, \alpha_0)$;
(v) *given real* $f \in C_\theta(\Sigma)$ *such that* $\mathcal{L}_f 1 = 1$ *there are constants* α_1, C_1 *such that for real* b $|b| > 1$ $\|(1 - \mathcal{L}_{f+ib\tau})^{-1}\| \le C_1 |b|^{\alpha_1}$;
(vi) (v) *holds for some* f;
(vii) *given real* $f \in C_\theta(\Sigma)$ *such that* $\mathcal{L}_f 1 = 1$ *there is a constant* α_2 *such that for* ξ *satisfying* $|\Im\xi| > 1$, $|\Re\xi| \le |\Im\xi|^{-\alpha_2}$ *the operator* $1 - \mathcal{L}_{f+\xi\tau}$ *is invertible;*
(viii) (vii) *holds for some* f.
The theorem above may be considered as a quantitative version of the complex Ruelle-Perron-Frobenius theorem ([P1]). The equivalence of (i) $-$ (ii) to (vii) $-$ (viii) can be regarded as Paley-Wiener theorem for suspension flows.
Set $n(\beta, b) = [\beta \ln |b|]$.
THEOREM 2. The following conditions are equivalent
(i) $\{S^t\} \notin TPM$;
(ii) There is a function f satisfying $\mathcal{L}_f 1 = 1$ and a constant α_3 such that given α there are a constant β and sequences $b_k \to \infty$, $k \to \infty$ and $G_k(\omega)$ such that $|G_k| = 1$, $\|G_k\|_\theta \le |b_k|^{\alpha_3}$ and if

$$\mathcal{M}_k h = \frac{1}{G_k} \mathcal{L}_{f+ib_k\tau}^{n(\beta,b_k)}(G_k h) \tag{1}$$

then $\|\mathcal{M}_k - \mathcal{L}_f^{n(\beta,b_k)}\|_0 \le |b_k|^{-\alpha}$;
(iii) (ii) holds for all f satisfying $\mathcal{L}_f 1 = 1$;
(iv) There exists α_4 such that for any α $\{S^t\}$ has $(ib_k, |b_k|^{\alpha_4}, |b_k|^{-\alpha})-$approximate eigenfunction of absolute value 1 for some sequence $b_k \to \infty$, $k \to \infty$.
(v) There exists α_5 such that given α there exist a constant β and a sequence $b_k \to \infty$, $k \to \infty$ such that $\mathcal{V}_{b_k}^{n(\beta,b_k)}$ has $(1, |b_k|^{\alpha_5}, |b_k|^{-\alpha})-$approximate eigenfunction of absolute value 1;
Usually it is hard to verify any of the conditions above. However there is quite simple sufficient condition.
EXAMPLE.([R2], [P3]) Let (Σ, σ) be the full shift on two symbols

$$\tau(x) = \begin{cases} l_1 & \text{if } x_0 = 0, \\ l_2 & \text{if } x_0 = 1 \end{cases}$$

and $f(x) \equiv \ln 2$. One readily sees that $\mathcal{L}_{f+ib\tau}1 = \frac{1}{2}\left(e^{ibl_1} + e^{ibl_2}\right) \cdot 1$. Moreover it is shown in above-mentioned papers that

$$\|(1 - \mathcal{L}_{f+ib\tau})^{-1}\| \leq \max(1 - \varepsilon, \frac{2}{2 - (e^{ibl_1} + e^{ibl_2})}).$$

Conditions of Theorem 1 are satisfied if $\frac{l_1}{l_2}$ is a Diophantine number that is there are C_2, α_6 such that $\left|\frac{l_1}{l_2} - \frac{m_1}{m_2}\right| \geq C_2 m_2^{-\alpha_6}$ for all integers m_1, m_2. This condition works also in a more general set up.

THEOREM 3. *Assume that $\{S^t\}$ has two periodic points $\omega_1 = \sigma\omega_1$ and $\omega_2 = \sigma\omega_2$ with periods $l_1 = \tau(\omega_1)$, $l_2 = \tau(\omega_2)$, such that $\frac{l_1}{l_2}$ is a Diophantine number then $\{S^t\}$ is rapidly mixing.*

This statement is proven in Section 13.

COROLLARY 1. *Let $\{\tau_s\}$ be continuous 1-parameter family of roof functions and $\{S^t_s\}$ be corresponding suspension flows. Assume that $\{\{S^t_s\}\} \bigcap RM = \emptyset$. Then there exist a function $\Gamma(s)$ such that all flows $\{S_s^{t/\Gamma(s)}\}$ are Holder conjugated to each other.*

PROOF OF COROLLARY 1: Let $(\omega_1, 0)$ be periodic point $\omega_1 = \sigma^n\omega_1$. Set $\Gamma(s) = (\tau_s)_n(\omega_1)$. Then all flows $\{S_s^{t/\Gamma(s)}\}$ have the periodic point $(\omega_1, 0)$ of period 1. Therefore by Theorem 3 all flows $S_s^{t/\Gamma(s)}$ have only periodic points of non-Diophantine length. However any continuous function of the parameter which assumes only non-Diophantine values is a constant. Hence all flows $\{S_s^{t/\Gamma(s)}\}$ have the same length spectrum so the statement follows by Livshic's theorem. ■

COROLLARY 2. *In a generic 1-parameter family $\{\tau_s\}$* $\mathrm{mes}\{s : \{S^t_s\} \notin RM\} = 0$.

The counterparts of these results are valid also for smooth flows. The proofs are very similar. Let $\{S^t\}$ be a flow with basic hyperbolic set set Λ. We say that S^t is rapidly mixing on Λ with respect to Holder continuous potential F ($\{S^t\} \in RM(\Lambda, F)$) if the map $(A, B) \to R_{A,B}(F, t)$ is continuous from $C^\infty(M) \times C^\infty(M)$ to the Schwartz space.

THEOREM 4. *The following conditions are equivalent*

(i) $\{S^t\} \in RM(\Lambda)$;

(ii) $\exists F : \{S^t\} \in RM(\Lambda, F)$;

(iii) $\{S^t\} \in TPM(\Lambda)$;

THEOREM 5. *Assume that $\{S^t\}$ has two periodic points γ_1 and γ_2 in Λ such that the ratio of their periods is a Diophantine number then $\{S^t\}$ is rapidly mixing on Λ.*

COROLLARY 3. *Let $\{\{S^t_s\}\}$ be continuous 1-parameter family of the flows such that $\{S^t_s\}$ has a basic hyperbolic set Λ_s near Λ_0. Then if no $\{S^t_s\}$ is rapidly mixing on Λ_s there is a function $\Gamma(s)$ such that $S_s^{t/\Gamma(s)}|_{\Lambda_s}$ is Holder conjugated to $S^t_0|_{\Lambda_0}$.*

COROLLARY 4. *For any $k \geq 1$ in a generic 1-parameter C^k family $\{\{S^t_s\}\}$* $\mathrm{mes}\{s : \{S^t_s\} \notin RM(\Lambda_s)\} = 0$.

2. Scheme of the proof of Theorems 1 and 2. From now on we write simply (i) – (viii) for conditions (i) – (viii) of Theorem 1 and (2.i) – (2.v) for conditions (i) – (v) of Theorem 2. (ix) – (xii) will mean the converses of (2.ii) – (2.v) respectively.

The plan of the proof of Theorems 1 and 2 is the following. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (v) $\Rightarrow$ (vi), (vii) $\Rightarrow$ (viii), and (ix) $\Rightarrow$ (x). Also (v) $\Rightarrow$ (vii) and (vi) $\Rightarrow$ (viii) since

$$(1-\mathcal{L}_{f+\xi\tau})^{-1} = (1-(1-\mathcal{L}_{f+i\Im\xi\tau})^{-1}(\mathcal{L}_{f+\xi\tau}-\mathcal{L}_{f+i\Im\xi\tau}))^{-1}(1-\mathcal{L}_{f+i\Im\xi\tau})^{-1}$$

and $\|(\mathcal{L}_{f+\xi\tau}-\mathcal{L}_{f+i\Im\xi\tau})\| \leq C_3|\Re\xi|$. In Section 3 we prove that (xi) $\Leftrightarrow$ (xii). In Section 4 we show that (x) $\Rightarrow$ (xii). In Section 5 we demonstrate that (iv) $\Rightarrow$ (xi). Section 6 contains an auxiliary estimate. The implication (xii) $\Rightarrow$ (v) is proven in Sections 7-9. The implication (vi) $\Rightarrow$ (ii) where F and f are related by

$$\bar{F}(x) = f(x) + G(x) - G(\sigma x) + Pr(F)\tau \tag{2}$$

(where $\bar{F}(x) = \int\limits_0^{\tau(x)} F(x,s)\,ds$ and $Pr(F)$ is a constant) is due to Pollicott ([P2], [P3]). For convenience of the reader we provide the proof in Section 10. This gives (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (xi) $\Leftrightarrow$ (xii). Hence (i) is also equivalent to (ii) – (vi) because every F has decomposition (2) (see for example [PP]). Thus we get

$$(\mathrm{ix}) \Rightarrow (\mathrm{x}) \Rightarrow ((\mathrm{i})-(\mathrm{vi}),(\mathrm{xi}),(\mathrm{xii})) \Rightarrow (\mathrm{vii}) \Rightarrow (\mathrm{viii}).$$

In Section 11 we prove that (vii) $\Rightarrow$ (ix) with the same function f. Therefore, also (viii) $\Rightarrow$ (x) which completes the proof of Theorems 1 and 2.

3. Eigenfunctions of $\{S^t\}$ and $\mathcal{V}$. In this section we show that (2.iv) $\Leftrightarrow$ (2.v) or in other words that (xi) $\Leftrightarrow$ (xii).

(2.iv) $\Rightarrow$ (2.v)) : Let $J_k(q)$ be the approximate eigenfunction for $\{S^t\}$. Set $H_k(\omega) = \frac{1}{J_k((\omega,0))}$. Then

$$|((\mathcal{V}^{n(\beta,b_k)}_{b_k} H_k) - H_k)(\omega)| = |e^{ib_k\tau_{n(\beta,b_k)}(\omega)} H_k(\sigma^{n(\beta,b_k)}\omega) - H(\omega)| =$$

$$|e^{ib_k\tau_{n(\beta,b_k)}(\omega)}J_k((\omega,0)) - J_k((\sigma^{n(\beta,b_k)}\omega,0))| =$$
$$|e^{ib_k\tau_{n(\beta,b_k)}(\omega)}J_k((\omega,0)) - J_k(S^{\tau_{n(\beta,b_k)}(\omega)}(\omega,0))| \le$$
$$|b_k|^{-\alpha}\tau_{n(\beta,b_k)}(\omega) \le C_4|b_k|^{-\alpha}n(\beta,b_k). \blacksquare$$

(2.v) $\Rightarrow$ (2.iv)) : Let H_k be the approximate eigenfunction of $\mathcal{V}_{b_k}^{n(\beta,b_k)}$ then if b_k is large enough we can find a branch $\Delta(\omega)$ of

$$\mathrm{Arg}\left(\frac{H_k(\omega)}{H_k(\sigma^{n(\beta,b_k)}\omega)}e^{ib_k\tau_{n(\beta,b_k)}(\omega)}\right)$$

which takes values in $(-\frac{2}{|b_k|^\alpha}, \frac{2}{|b_k|^\alpha})$. Consider the space

$$\tilde{\Sigma}^\tau = \{(\omega,t) : 0 \le t \le \tau_n(\omega)\}/\left((\omega,\tau_n(\omega)) \sim (\sigma^n\omega,0)\right).$$

$\mathbf{Z}_n$ acts on $\tilde{\Sigma}^\tau$ by shifts $T^k(\omega,t) = (\sigma^k\omega, t+\tau_k(\omega))$ and $\tilde{\Sigma}^\tau/\mathbf{Z}_n \cong \Sigma^\tau$. Denote by $\{\tilde{S}^t\}$ the flow $\tilde{S}^t(\omega,s) = (\omega, s+t)$ and by P the natural projection $P : \tilde{\Sigma}^\tau \to \Sigma^\tau$. We may assume without loss of generality that $\tau \ge 1$. Let $\varphi(t)$ be a cutoff C^∞ function such that $\mathrm{supp}\,\varphi \subset [\frac{1}{2},1]$ and $\int_{\frac{1}{2}}^{1}\varphi(s)\,ds = 1$. If $\tilde{q} = (\omega,t) :\ 0 \le t < \tau_{n(b_k)}(\omega)$ set

$$\tilde{J}_k(\tilde{q}) = \frac{1}{H_k(\omega)}\exp\left(i\left[\int_0^t (b_k + \Delta(x)\varphi(s))\,ds\right]\right).$$

$\tilde{J}(\tilde{q})$ can be lifted to $(ib_k, C_5|b_k|^{\alpha_7}, \frac{1}{|b_k|^{-\alpha}})-$approximate eigenfunction for $\{\tilde{S}^t\}$ ($\alpha_7 = \max(1,\alpha_5)$). Let

$$J(q) = \prod_{P\tilde{q}=q}\tilde{J}(\tilde{q}),$$

then J is $(in_k(\beta,b_k)b_k, C_5|b_k|^{\alpha_7}n(\beta,b_k), \frac{n(\beta,b_k)}{|b_k|^{-\alpha}})-$ approximate eigenfunction for $\{S^t\}$. $\blacksquare$

4. Almost conjugacy. In this section we show that (2.v) $\Rightarrow$ (2.iii) or in other words (x) $\Rightarrow$ (xii). This follows immediately from the following statement.

LEMMA 1. *Assume that* $|G| \equiv 1$ *and* $\|\mathcal{V}_b^n G - e^{i\Theta}G\|_0 \le \epsilon$. *Define* $\mathcal{M}h = G\mathcal{L}_{f+ib\tau}^n(\frac{h}{G})$, *then* $\|(\mathcal{M} - e^{i\Theta}\mathcal{L}_f^n)h\|_0 \le \epsilon\|h\|_0$.

PROOF:

$$|\mathcal{M}-e^{i\Theta}\mathcal{L}_f^n h(\omega)| = \left|\sum_{\sigma^n\varpi=\omega} e^{f_n(\varpi)}\left[\frac{G_k(\sigma^n\varpi)}{G_k(\varpi)}e^{ib_k\tau_n(\varpi)}-e^{i\Theta}\right]h(\varpi)\right| \leq$$

$$\sum_{\sigma^n\varpi=\omega} e^{f_n(\varpi)}\epsilon\|h\|_0 = \epsilon\|h\|_0(\mathcal{L}_f^n 1)(\omega) = \epsilon\|h\|_0.\blacksquare$$

5. Topological power mixing and approximate eigenfunctions. In this section we show that (iv) $\Rightarrow$ (xi). So assume that there are α_0, t_0 so that for $t > \min((\frac{1}{r})^{\alpha_0}, t_0)$ $S^t\mathcal{B}(q_1,r)\bigcap\mathcal{B}(q_2,r) \neq \emptyset$. We prove that for any α_4 $\{S^t\}$ has no approximate eigenvalue of unit absolute value if α and $|b|$ are large enough. Indeed, assume that converse is true. Let $J(q)$ be such an eigenfunction, $J(q_1) = e^{i\Theta_1}$, $J(q_2) = e^{i\Theta_2}$. Let $r = \frac{1}{|b|^\gamma}$ then for $q \in \mathcal{B}(q_1,r)$ $|J(q) - e^{i\Theta_1}| \leq \frac{|b|^{\alpha_4}}{|b|^\gamma}$ and for $q \in \mathcal{B}(q_2,r)$ $|J(q) - e^{i\Theta_2}| \leq \frac{|b|^{\alpha_4}}{|b|^\gamma}$. Also for $q \in S^t\mathcal{B}(q_1,r)$ $|J(q) - e^{i(\Theta_1-bt)}| \leq \frac{t}{|b|^\alpha} + \frac{|b|^{\alpha_4}}{|b|^\gamma}$. So if $t < 2\frac{1}{r^{\alpha_0}} = 2|b|^{\gamma\alpha_0}$ then $|J(q) - e^{i(\Theta_1-bt)}| \leq \frac{2|b|^{\gamma\alpha_0}}{|b|^\alpha} + \frac{|b|^{\alpha_4}}{|b|^\gamma}$. Thus if for any t, $\frac{1}{r^{\alpha_0}} \leq t \leq \frac{2}{r^{\alpha_0}}$ $\exists q \in S^t\mathcal{B}(q_1,r)\bigcap\mathcal{B}(q_2,r)$ then for any such t $|e^{i(\Theta_1+tb)} - e^{i\Theta_2}| \leq 2|b|^{\alpha_4-\gamma} + 2|b|^{\gamma\alpha_0-\alpha}$ which can not be true if $\gamma > \alpha_4$ and $\alpha > \gamma\alpha_0$. $\blacksquare$

6. Apriori bounds. This section contain useful bounds for iterations of complex transfer operators.

PROPOSITION 2. (SEE [PP]) *Let $\mathcal{L}_f 1 = 1$, then*

(a) $\|\mathcal{L}^n_{f+ib\tau}\| \leq 1$;

(b) there exist a constant C_6 such that

$$L(\mathcal{L}^n_{f+ib\tau}h) \leq C_6\left(|b|\|h\|_0 + \theta^n L(h)\right).$$

PROOF: Direct calculation.$\blacksquare$

Consider a new norm on $C_\theta(\Sigma)$

$$\|h\|_{(N)} = \max\left(\|h\|_0, \frac{L(h)}{2C_6N}\right),$$

then $\|\mathcal{L}^n_{f+ib\tau}\|_{(|b|)} \leq 1$.

For the future use we provide an estimate of the resolvent of $\mathcal{L}_f$ in $\|\cdot\|_{(N)}$−norm. By Ruelle-Perron-Frobenius theorem we can decompose $\mathcal{L}_f$ as $\mathcal{L}_f = \mathcal{P} + \mathcal{N}$ where $\mathcal{P}h = \nu_f(h)1$, $\mathcal{N}\mathcal{P} = \mathcal{P}\mathcal{N} = 0$ and $\|\mathcal{N}^k\|_\theta \leq C_7\delta^k$ for some constants $\delta < 1, C_7$.

Lemma 2. *If* $\frac{1+\delta}{2} \leq |\lambda| \leq 1$ *then*

$$\|(\lambda - \mathcal{L}_f^n)^{-1}\|_{(N)} \leq \frac{C_8}{|1-\lambda|}\left(\ln \frac{1}{|1-\lambda|} + \ln N + n\right).$$

Proof: Inequalities

$$\|\cdot\|_\theta \leq \|\cdot\|_{(N)} \leq 2C_6 N \|\cdot\|_\theta$$

imply $\|\mathcal{N}^k\|_{(N)} \leq C_9 \delta^k N$. Now

$$(\lambda - (\mathcal{P})^k)^{-1} = (\lambda - \mathcal{P})^{-1} = \frac{\mathcal{P}}{\lambda - 1} + \frac{1-\mathcal{P}}{\lambda}.$$

Hence

$$\|(1 - \frac{\mathcal{P}^k}{\lambda})^{-1}\|_{(N)} \leq \frac{C_{10}}{|\lambda - 1|}.$$

Take $k = [C_{11}(\ln(N) + \ln \frac{1}{|\lambda - 1|})]$. Then if C_{11} is large enough

$$\|\mathcal{N}^k\|_{(N)} \leq \frac{C_{10}}{2|\lambda - 1|} \leq \frac{1}{2\|(1 - \frac{\mathcal{P}^k}{\lambda})^{-1}\|_{(N)}}.$$

Therefore

$$\|(\lambda - \mathcal{L}_f^{kn})^{-1}\|_{(N)} \leq \frac{C_{12}}{|\lambda - 1|}$$

and we are done. ■

7. Pointwise estimates. In this section we begin with the proof of (xii) $\Rightarrow$ (v). We assume that (v) is false and obtain (2.v). If (v) fails there is a function f, $\mathcal{L}_f 1 = 1$ such that given α there is arbitrary large b such that $\|(1 - \mathcal{L}_{f+ib\tau})^{-1}\|_\theta > |b|^\alpha$. Choose such a function f. In the next three sections we will work only with this weight. We use $\mathcal{L}_b$ as a shorthand for $\mathcal{L}_{f+ib\tau}$. The proof of the above implication consists of two steps. First we show that if (v) is false then given α there are $\beta(\alpha)$ and $b_k \to \infty$ such that $\mathcal{V}_{b_k}^{n(\beta, b_k)}$ has an approximate eigenfunction with approximate eigenvalue $e^{i\Theta}$ and then demonstrate that $e^{i\Theta}$ should be close to 1. The first step in turn, is divided into two parts. In this section we show that if (v) is violated there is a function h, $\|h\|_{(b)} = 1$ whose $\mathcal{L}_b$ iterations remain near the unit sphere for a long time and in the next section we show that some iteration of h give the approximate eigenfunction we need.

LEMMA 3. *If there are α, β such that for any b, —b—¿1 for any h, $\|h\|_{(|b|)} \leq 1$ there exist $\omega_0 \in \Sigma$ and $n : 0 \leq n \leq 3n(\beta, b)$ such that $|(\mathcal{L}_b{}^n(h))(\omega_0)| \leq 1 - \frac{1}{|b|^\alpha}$, then there are constants C_{13}, α' such that $\|(1 - \mathcal{L}_b)^{-1}\|_\theta \leq C_{13}|b|^{\alpha'}$.*

PROOF:

$$|\mathcal{L}_b{}^N h(\omega)| = |(\mathcal{L}_b{}^{N-n}(\mathcal{L}_b{}^n h))(\omega)| \leq (\mathcal{L}_f^{N-n}|\mathcal{L}_b{}^n h|)(\omega).$$

Let ν_f be the Gibbs measure for f (that is $\mathcal{L}_f^* \nu_f = \nu_f$), then by Ruelle-Perron-Frobenius theorem

$$(\mathcal{L}_f^{N-n}|\mathcal{L}_b{}^n h|)(\omega) \leq \nu_f(|\mathcal{L}_b{}^n h|) + C_7 \delta^{N-n} \|\mathcal{L}_b{}^n h\|_\theta$$

By Proposition 2 $\|\mathcal{L}_b{}^n h\|_\theta \leq C_7|b|$. On the other hand $|(\mathcal{L}_b{}^n h)(\omega)| \leq 1 - \frac{1}{2|b|^\alpha}$ for $\omega \in \mathcal{B}(\omega_0, \frac{1}{2C_6|b|^{\alpha+1}})$. Now there are constants C_{14}, α_8 such that $\nu_f(\mathcal{B}(\omega_0, \frac{1}{2C_6|b|^{\alpha+1}})) \geq \frac{C_{14}}{|b|^{\alpha_8}}$. Hence $\nu_f(|\mathcal{L}_b{}^n h|) \leq 1 - \frac{C_{15}}{|b|^{\alpha_9}}$ and

$$\|\mathcal{L}_b{}^N h\|_0 \leq 1 - \frac{C_{15}}{|b|^{\alpha_9}} + C_7 \delta^{N-n}|b|.$$

Take $N = n(\bar{\beta}, b)$ where $\bar{\beta} \gg \beta$, then

$$\|\mathcal{L}_b{}^N h\|_0 \leq 1 - \frac{C_{15}}{2|b|^{\alpha_9}}.$$

Take $\tilde{N} > N$ then

$$\|\mathcal{L}_b{}^{\tilde{N}} h\|_0 \leq \|\mathcal{L}_b{}^N h\|_0 \leq 1 - \frac{C_{15}}{2|b|^{\alpha_9}},$$

whereas

$$L(\mathcal{L}_b{}^{\tilde{N}} h) \leq \left(1 - \frac{C_{15}}{2|b|^\alpha}\right) C_6|b| + C_6 \theta^{\tilde{N}-N}|b|.$$

So if $\tilde{N} = n(\tilde{\beta}, b)$ where $\tilde{\beta} \gg \bar{\beta}$ then

$$\|\mathcal{L}^{\tilde{N}} h\|_{(b)} \leq 1 - \frac{C_{15}}{4|b|^{\alpha_9}}.$$

The estimate of the lemma now follows easily. ∎

8. Construction of approximate eigenfunction. In this section we use Lemma 3 to construct approximate eigenfunctions for $\mathcal{L}_b$.

LEMMA 4. *Assume that* (v) *fails then given* α *there is* $\beta(\alpha)$ *such that one can find arbitrary large* b *such that*

$$\|(1-\mathcal{L}_b)^{-1}\|_\theta > |b|^\alpha$$

and $\mathcal{V}_b^{n(\beta,b)}$ *has* $(e^{i\Theta}, 2C_6|b|, \frac{1}{|b|^\alpha})-$*approximate eigenfunction of absolute value 1.*

The proof proceeds by a series of lemmas. Take some α', β then by Lemma 3 one can find arbitrary large b such that

$$\|(1-\mathcal{L}_b)^{-1}\|_\theta > |b|^\alpha$$

and there is a function $h(\omega)$ such that

$$\forall n : 0 \le n \le 3n(\beta, b)\ \forall\omega\ |({\mathcal{L}_b}^n h)(\omega)| \ge 1 - \frac{1}{|b|^{\alpha'}}. \tag{3}$$

Write $\bar{h} = {\mathcal{L}_b}^{n(\beta,b)} h$, $\bar{\bar{h}} = {\mathcal{L}_b}^{2n(\beta,b)} h$, $h(\omega) = r(\omega)e^{i\gamma(\omega)}$, $\bar{h}(\omega) = \bar{r}(\omega)e^{i\bar{\gamma}(\omega)}$, $\bar{\bar{h}}(\omega) = \bar{\bar{r}}(\omega)e^{i\bar{\bar{\gamma}}(\omega)}$.

LEMMA 5. $\forall\bar{\alpha}, \beta\exists\alpha'$ *such that if (3) holds then* $\forall(\varpi, \omega) : \omega = \sigma^{n(\beta,b)}\varpi$

$$|\exp(i[b\tau_{n(\beta,b)}(\varpi) + \gamma(\varpi) - \bar{\gamma}(\omega)]) - 1| \le \frac{1}{|b|^{\bar{\alpha}}}, \tag{4}$$

$$|\exp(i[b\tau_{n(\beta,b)}(\varpi) + \bar{\gamma}(\varpi) - \bar{\bar{\gamma}}(\omega)]) - 1| \le \frac{1}{|b|^{\bar{\alpha}}}. \tag{5}$$

PROOF: We show only (4), proof of (5) is the same. By definition

$$\bar{r}(\omega)e^{i\bar{\gamma}(\omega)} = \sum_{\sigma^{n(\beta,b)}\varpi=\omega} e^{(f+ib\tau)_{n(\beta,b)}(\varpi)} r(\varpi)e^{i\gamma(\varpi)}.$$

Thus

$$1 - \bar{r}(\omega) = \sum_{\sigma^{n(\beta,b)}\varpi=\omega} e^{f_{n(\beta,b)}(\omega)} \left(1 - r(\varpi)\exp(i[b\tau_{n(\beta,b)}(\varpi) + \gamma(\varpi) - \bar{\gamma}(\omega)])\right)$$

so that

$$\sum_{\sigma^{n(\beta,b)}\varpi=\omega} e^{f_{n(\beta,b)}(\omega)} \left(1 - \exp(i[b\tau_{n(\beta,b)}(\varpi) + \gamma(\varpi) - \bar{\gamma}(\omega)])\right) = O\left(\frac{1}{|b|^{\alpha'}}\right).$$

Since the real part of each term is positive $\forall(\varpi,\omega)$

$$e^{f_{n(\beta,b)}(\omega)}\Re\left(1-\exp(i[b\tau_{n(\beta,b)}(\varpi)+\gamma(\varpi)-\bar{\gamma}(\omega)])\right)=O\left(\frac{1}{|b|^{\alpha'}}\right).$$

Using the estimate

$$\exp\left[f_{n(\beta,b)}(\omega)\right]\geq\exp\left[-n(\beta,b)\|f\|_0\right]\geq\frac{C_{16}}{|b|^{\beta\|f\|_0}}$$

we get

$$\Re\left(1-\exp(i[b\tau_{n(\beta,b)}(\varpi)+\gamma(\varpi)-\bar{\gamma}(\omega)])\right)=O\left(\frac{1}{|b|^{\alpha'-\beta\|f\|_0}}\right).\blacksquare$$

LEMMA 6. *If β in the previous lemma is large enough then there exists Θ such that $\forall\omega$*

$$|\exp(i[\bar{\bar{\gamma}}(\omega)-\bar{\gamma}(\omega)-\Theta])-1|\leq\frac{4}{|b|^{\bar{\alpha}}}.$$

PROOF: Let $\mathcal{B}$ be a ball of radius $\frac{1}{2C_6b^{\bar{\alpha}+1}}$ in Σ. Then $\exists\Theta_1,\Theta_2$ such that $\forall\varpi\in\mathcal{B}$ $|e^{i\gamma(\varpi)}-e^{i\Theta_1}|\leq\frac{1}{|b|^{\bar{\alpha}}}$, $|e^{i\bar{\gamma}(\varpi)}-e^{i\Theta_2}|\leq\frac{1}{|b|^{\bar{\alpha}}}$. If β is large enough $\forall\omega\exists\varpi\in\mathcal{B}$ such that $\sigma^{n(\beta,b)}\varpi=\omega$ and the statement follows by Lemma 5. $\blacksquare$
Let $\bar{\alpha}>\alpha$. We claim that $H(\omega)=e^{-i\gamma(\omega)}$ is the required eigenfunction. Indeed

$$\frac{\mathcal{V}_b^{n(\beta,b)}H}{H}(\omega)=\exp(i[b\tau_{n(\beta,b)}(\omega)-\bar{\gamma}(\sigma^{n(b,\beta)}\omega)+\bar{\gamma}(\omega)])=$$

$$\exp(i[\Theta+b\tau_{n(\beta,b)}(\omega)-\bar{\bar{\gamma}}(\sigma^{n(b,\beta)}\omega)+\bar{\gamma}(\omega)])\left(1+O\left(\frac{1}{|b|^{\bar{\alpha}}}\right)\right)$$

$$=\exp(i\Theta)\left(1+O\left(\frac{1}{|b|^{\bar{\alpha}}}\right)\right)$$

by Lemma 5. $\blacksquare$

9. Perturbation estimates. Combining the result of the previous section with Lemma 1 we get

LEMMA 7. *If* (v) *fails then given α there exist $\beta(\alpha)$ such that one can find arbitrary large b, a constant Θ and a function H satisfying $|H|\equiv 1$, $\|H\|_{(b)}=1$ such that if $\mathcal{K}$ denotes the operator $\mathcal{K}h=He^{i\Theta}\mathcal{L}_f^{n(\beta,b)}(\frac{h}{H})$ then*

$$\|(1-\mathcal{L}_b)^{-1}\|_\theta\geq|b|^\alpha$$

$$\|({\mathcal{L}_b}^{n(\beta,b)}-\mathcal{K})h\|_0\leq\|h\|_0\frac{1}{|b|^\alpha}.$$

We now show that the last two inequality impose some restrictions on the value of Θ.

PROPOSITION 3. $L(({\mathcal{L}_b}^{n(\beta,b)}-\mathcal{K})h)\leq\frac{1}{|b|^\alpha}L(h)+C_{17}|b|\|h\|_0$.

PROOF: Direct calculation.■

Recall that $\|h\|_{(N)}=\max(\|h\|_0,\frac{L(h)}{C_6N})$. By Proposition 3

$$\|({\mathcal{L}_b}^{n(\beta,b)}-\mathcal{K})h\|_{(N)}\leq\frac{1}{|b|^\alpha}+\frac{C_{17}}{N}|b|.$$

We show that if b is large enough, then $|e^{i\Theta}-1|\leq\frac{1}{|b|^{\alpha^*}}$, $\alpha^*=\frac{\alpha-3}{2}$. Indeed assume that this inequality is false. Then by Lemma 2

$$\|(1-e^{i\Theta}\mathcal{L}_f^{n(\beta,b)})^{-1}\|_{(N)}\leq C_{18}|b|^{\alpha^*}(\ln|b|+\ln N).$$

Now if $N>2C_6|b|$ multiplication by H has norm less than 2 and so

$$\|(1-\mathcal{K})^{-1}\|_{(N)}\leq C_{19}|b|^{\alpha^*}.$$

On the other hand if $N=|b|^{\alpha^*+2}$ then $\|\mathcal{L}_b-\mathcal{K}\|_{(N)}\leq C_{20}|b|^{\alpha^*+1}$. Therefore $\|(1-\mathcal{L}_b)^{-1}\|_{(N)}\leq C_{21}|b|^\alpha$ and

$$\|(1-\mathcal{L}_b)^{-1}\|_\theta\leq 2C_6N\|(1-\mathcal{L}_b)^{-1}\|_{(N)}=C_{22}|b|^{2\alpha^*+2}=C_{22}|b|^{\alpha-1},$$

a contradiction. Hence actually $|e^{i\Theta}-1|\leq\frac{1}{|b|^{\alpha^*(\alpha)}}$. Since $\alpha^*\to\infty$ as $\alpha\to\infty$ the implication (xi) $\Rightarrow$ (v) is proven. ■

10. Decay of correlations.

In this Section we prove that (vi) $\Rightarrow$ (ii). Let $\bar{F}(\omega)=\int\limits_0^{\tau(\omega)}F(\omega,s)\,ds$ and be such that $\bar{F}(\omega)=f(\omega)+G(\omega)-G(\sigma\omega)+Pr(F)\tau$ and $\mathcal{L}_f1=1$. Let $\hat{A}(\omega,\xi)$ be Laplace transform of A $\hat{A}(\omega,\xi)=\int\limits_0^{\tau(\omega)}F(\omega,s)e^{-\xi s}ds$. We use the following expression of the Laplace transform of $R_{A,B}(F,t)$ (see [P2])

$$\hat{R}_{A,B}(F,\xi)=\frac{1}{\nu_f(\tau)}\int\hat{B}(\omega,\xi)\left[(1-\mathcal{L}_{f-\xi\tau})^{-1}\hat{A}(\omega,-\xi)\right]\,d\nu_f+\tilde{R}(\xi)$$

where $\tilde{R}$ is an entire function bounded in any strip $|\Re\xi|\leq M$.

We already noticed that (vi) $\Rightarrow$ (viii) and moreover for Σ^τ Lipschitz A, B we have the following inequality in the region $|b| > 1$, $a > -|b|^{-\alpha_2}$

$$\hat{R}_{A,B}(F, a+ib) \leq C_{23}\|A\|_1\|B\|_1|b|^{\alpha_{10}}.$$

Now consider the case when $A, B \in D(\Sigma^\tau)$. In this case $R_{A,B}(F,t)$ is a smooth function and $\frac{\partial}{\partial t}R_{A,B}(F,t) = -R_{\partial_t A,B}$ ∂_t being the derivative along the orbits. Consider the Taylor expansion

$$R_{A,B}(t) = \sum_{j=0}^{N-1} \frac{R_{A,B}^{(j)}(0)}{j!}t^j + \int\limits_0^t R_{\partial_t A,B}(s)\frac{(s-t)^N}{(N-1)!}\,ds.$$

Laplace transform of the last term decays not slower than

$$C_{23}\|A\|_N\|B\|_1|b|^{\alpha_{10}-N}$$

in the region above (Here $\|1_{\mathcal{C}}(\omega)a(t)\|_N$ means $\left(\frac{1}{\theta}\right)^{l(\mathcal{C})}\|a\|_{C^N}$.) It has the only pole of the $N-$th order at 0.
Applying the inversion formula for Laplace transform and moving the contour of the integration to $\{-a = \min(\varepsilon, |b|^{-\alpha_2})\}$ we get

$$|\sum_{j=0}^{N-1} \frac{R_{A,B}^{(j)}(0)}{j!}t^j + \int\limits_0^t R_{\partial_t A,B}(s)\frac{(s-t)^N}{(N-1)!}\,ds| \leq$$

$$C_{24}\|A\|_N\|B\|_1 \int\limits_1^\infty |b|^{\alpha_{10}-N}\exp(-t|b|^{-\alpha_2})\,db$$

which implies (ii). ∎

11. Polefree regions. In this section we prove that (vii) $\Rightarrow$ (ix). To this end it is enough to show that if for some f (2.ii) holds then the poles of $(1-\mathcal{L}_\xi)^{-1}$ accumulate to the imaginary axis faster than any power of ξ ($\mathcal{L}_\xi$ is a shorthand for $\mathcal{L}_{f+\xi\tau}$). The idea is to establish the analyticity of the leading eigenvalue of $\mathcal{L}_\xi$ and then use Rouche's theorem to show that $1 \in Sp(\mathcal{L}_\xi)$. v
So assume that for some $\bar{b}$ there exists G, $|G| \equiv 1$, $\|G\|_\theta \leq |\bar{b}|^{\alpha_3}$ such that $\mathcal{M}h = \frac{1}{G}\mathcal{L}_{ib_0}^{n(\beta,\bar{b})}(Gh)$ satisfies

$$\|\mathcal{L}_f^{n(\beta,\bar{b})} - \mathcal{M}\|_0 \leq |\bar{b}|^{-\alpha}.$$

Denote $\mathcal{M}_\xi = \frac{1}{G}\mathcal{L}_\xi{}^{n(\beta,\bar{b})}(Gh)$. $\mathcal{L}_\xi$ is invertible if and only if $\mathcal{M}_\xi$ is invertible. We show that $\mathcal{M}_\xi$ has 1 as an eigenvalue for ξ close to $i\bar{b}$. A direct calculation shows that

$$\|\mathcal{M} - \mathcal{L}_f^{n(\beta,\bar{b})}\|_{(N)} \leq \frac{1}{|\bar{b}|^\alpha} + \frac{C_{23}|\bar{b}|^{\tilde{\alpha}}}{N}$$

where $\tilde{\alpha} = 2\alpha_3 + 1$ (since $\|\bar{b}\tau\|_\theta = O(|\bar{b}|)$, $\|G\|_\theta = \|G^{-1}\|_\theta = O(|\bar{b}|^{\alpha_3})$). Take $N = |\bar{b}|^{\alpha+\tilde{\alpha}}$.

LEMMA 8. *There are constants C_{26}, C_{27} such that uniformly in $\bar{b}, \xi$ such that $\xi \ - \ i\ \bar{b}\ | \ \leq \ |\bar{b}|^{-1}$*

$$\|\frac{\partial}{\partial\xi}\mathcal{M}\|_{(N)} \leq C_{26} \ln|\bar{b}|,$$

$$\|\frac{\partial^2}{\partial\xi^2}\mathcal{M}\|_{(N)} \leq C_{27} \ln^2|\bar{b}|.$$

PROOF: We prove only the first estimate, the proof of the second one is similar. Since multiplications by G, G^{-1} are (uniformly) bounded operators it is enough to bound $\|\frac{\partial}{\partial\xi}\mathcal{L}_\xi^{n(\beta,\bar{b})}\|_{(N)}$.

$$\|\frac{\partial}{\partial\xi}\mathcal{L}_\xi^{n(\beta,\bar{b})}\|_{(N)} = \|\sum_{j<n(\beta,\bar{b})} \mathcal{L}_\xi^j(\frac{\partial}{\partial\xi}\mathcal{L}_\xi)\mathcal{L}_\xi^{n(\beta,\bar{b})-j-1}\|_{(N)}$$

Applying Proposition 2 to $\frac{\mathcal{L}_\xi}{\exp[Pr(f+\Re\xi\tau)]}$ and using the bound $Pr(f+\Re(\xi)\tau) \leq C_{28}|\Re\xi|$ we get

$$\|\mathcal{L}_\xi^j\|_{(N)} \leq C_{29} n(\beta, b)$$

and we are done. ■

Let γ be the circle $\{|z| = \frac{1+\delta}{2}\}$, where δ is the constant from Section 6. By Lemma 2 for $\lambda \in \gamma$

$$\|(\lambda - \mathcal{L}_f^{n(\beta,\bar{b})})^{-1}\|_{(N)} \leq C_{30} \ln|\bar{b}|$$

and so

$$\|(\lambda - \mathcal{M})^{-1}\|_{(N)} \leq C_{31} \ln|\bar{b}|.$$

From Lemma 8 it follows that in $|\xi - i\bar{b}| \leq |\bar{b}|^{-1}$ $\mathcal{M}_\xi$ has only one eigenvalue $\rho(\xi)$, this eigenvalue is simple and the corresponding eigenfunction is

$$g(\xi, \omega) = 1 - \frac{1}{2\pi i}\int_\gamma (z - \mathcal{M}_\xi)^{-1} 1\, dz.$$

Moreover $\rho(\xi)$ is analytic in $\{|\xi - i\bar{b}| < \frac{1}{|\bar{b}|}\}$. By standard perturbation theory

$$|\rho(i\bar{b}) - 1| \le C_{32} |\bar{b}|^{-\alpha} \ln |\bar{b}|,$$

$$|\frac{\partial}{\partial \xi}\rho(\xi)| \le C_{33} \ln^2 |\bar{b}|,$$

$$|\frac{\partial^2}{\partial \xi^2}\rho(\xi)| \le C_{34} \ln^3 |\bar{b}|,$$

$$\|g(i\bar{b}, \omega) - 1\|_0 \le C_{35} |\bar{b}|^{-\alpha} \ln |\bar{b}|$$

and

$$\|\frac{\partial}{\partial \xi} g(\xi, \omega)\|_0 \le C_{36} \ln |\bar{b}|.$$

12. Derivative of the eigenvalue. It remains to estimate $\frac{\partial}{\partial \xi}\rho(\xi)$ from below. Differentiating

$$\frac{1}{G(\omega)} \sum_{\sigma^{n(\beta,\bar{b})}\varpi = \omega} e^{f_{n(\beta,\bar{b})}(\varpi) + \xi \tau_{n(\beta,\bar{b})}(\varpi)} G(\varpi) g(\xi, \varpi) = \rho(\xi) g(\xi, \omega)$$

at $\xi = i\bar{b}$ we obtain

$$\mathcal{M}_{i\bar{b}} \left(\tau_{n(\beta,\bar{b})} \circ \sigma^{n(\beta,\bar{b})} g \right) + \mathcal{M}_{i\bar{b}} \left(\frac{\partial}{\partial \xi} g \right) = \left(\frac{\partial}{\partial \xi} \rho(i\bar{b}) \right) g + \rho(i\bar{b}) \frac{\partial}{\partial \xi} g.$$

Integrate this identity against ν_f.

$$\nu_f \left(\mathcal{M}_{i\bar{b}} \frac{\partial}{\partial \xi} g \right) = \nu_f \left(\mathcal{L}_f^{n(\beta,\bar{b})} \frac{\partial}{\partial \xi} g \right) + O\left(\frac{\ln |\bar{b}|}{|\bar{b}|^\alpha} \right) = \nu_f \left(\frac{\partial}{\partial \xi} g \right) + O\left(\frac{\ln |\bar{b}|}{|\bar{b}|^\alpha} \right),$$

$$\rho(i\bar{b}) \nu_f \left(\frac{\partial}{\partial \xi} g \right) = \nu_f \left(\frac{\partial}{\partial \xi} g \right) + O\left(\frac{\ln^2 |\bar{b}|}{|\bar{b}|^\alpha} \right),$$

$$\frac{\partial}{\partial \xi} \rho(i\bar{b}) \nu_f(g) = 1 + O\left(\frac{\ln^2 |\bar{b}|}{|\bar{b}|^\alpha} \right),$$

$$\nu_f \left(\mathcal{M}_{i\bar{b}} (\tau_{n(\beta,\bar{b})} \circ \sigma^{n(\beta,\bar{b})} g) \right) = \nu_f \left(\mathcal{L}_{i\bar{b}}^{n(\beta,\bar{b})} (\tau_{n(\beta,\bar{b})} \circ \sigma^{n}(\beta,\bar{b}) g) \right) + O\left(\frac{\ln^2 |\bar{b}|}{|\bar{b}|^\alpha} \right) =$$

$$\nu_f \left(\tau_{n(\beta,\bar{b})} \circ \sigma^{n(\beta,\bar{b})} \right) + O\left(\frac{\ln^2 |\bar{b}|}{|\bar{b}|^\alpha} \right) = n(\beta,\bar{b}) \nu_f(\tau) + O\left(\frac{\ln^2 |\bar{b}|}{|\bar{b}|^\alpha} \right).$$

Hence

$$\frac{\partial}{\partial\xi}\rho(i\bar{b}) = n(\beta,\bar{b})\nu_f(\tau) + O\left(\frac{\ln^2|\bar{b}|}{|\bar{b}|^\alpha}\right).$$

REMARK. Actually we have shown that

$$\frac{\partial}{\partial\xi}|_{\xi=i\bar{b}}\,\rho(\mathcal{M}_\xi) = \frac{\partial}{\partial\xi}|_{\xi=0}\left(\rho^{n(\beta,b)}(\mathcal{L}_\xi)\right) + O\left(\frac{\ln^2|\bar{b}|}{|\bar{b}|^\alpha}\right).$$

Taking into account the bound on the second derivative of $\rho(\xi)$ we conclude by Rouche's theorem that in $\{|\xi-\bar{b}|\leq|\bar{b}|^{-1}\}$ the equations $\rho(\xi)=1$ and $\rho(i\bar{b})+(\xi-i\bar{b})\nu(\tau)=1$ have the same number of solutions. That is both have one solution. ■

13. Proof of Theorem 3. In this section we show that conditions of Theorem 3 imply (xii). Indeed let H be $(1,|b|^{\alpha_5},|b|^{-\alpha})-$ approximate eigenfunction of $\mathcal{V}_b^{n(\beta,b)}$, that is

$$\left|\frac{e^{ib\tau_{n(\beta,b)}(\omega)}H(\sigma^{n(\beta,b)}\omega)}{H(\omega)}\right| \leq |b|^{-\alpha}.$$

Substituting $\omega=\omega_1$ we get

$$|e^{ibn(\beta,b)l_1}-1|\leq|b|^{-\alpha}.$$

So $bn(\beta,b)l_1 = 2\pi m_1 + O\left(|b|^{-\alpha}\right)$ for some $m_1\in\mathbf{Z}$. Similarly $bn(\beta,b)l_2 = 2\pi m_2 + O\left(|b|^{-\alpha}\right)$. Hence

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} + O\left(|b|^{-\alpha}\right) = \frac{m_1}{m_2} + O\left(|m_2|^{-\alpha}\right).$$

If $|b|$ is large enough this implies $\alpha\leq\alpha_6$ where α_6 is Diophantine exponent of $\frac{l_1}{l_2}$.■

14. Concluding remarks. As we have seen above topological power mixing plays quite important role for suspension flows over subshift of a finite type. Now several examples of dynamical systems satisfying this condition are known. We mention just few of them: any topologically mixing Anosov flow, some maps on the boundary of Anosov diffeomorphisms, certain Henon maps etc. We expect that some convenient set of axioms could be formulated which would cover all this cases. Of course, in the general case one cannot get anything better then powerlike correlation decay.

Another problem is whether generic Axiom A flow is actually exponentially mixing. The same question may also be asked for symbolic flows. Even if the answers to these two cases are the same the proofs would probably be different. We expect that the question about the exponential bound is of topological rather than arithmetic nature.
It is also interesting to know to what extent Theorem 1 can be generalized to deal with exponential mixing. More specifically, we would like to ask two questions. Is it true that topological exponential mixing imply exponential correlation decay for any Gibbs measure? In particular is it true that if correlations decay exponentially for some Gibbs measure then the same holds for any potential? Of course, the conditions of this theorem if true would be even more difficult to verify than for power mixing.

Appendix.

Dimension and Rapid Mixing.

1. Two-sided subshifts. Above we gave several necessary and sufficient conditions for a symbolic flow to mix faster than any power of time. We also described an arithmetic property (incommensurability of periods) which imply these conditions. Here we present a topological mechanism for a rapid mixing.
Here we let (Σ, σ) be a two-sided subshift of a finite type since to formulate our condition for one-sided subshift we need to pass to its natural extention. If $\omega\Sigma$ ω^- and ω^+ stand for $\{\omega_j\}_{j\leq 0}$ and $\{\omega_j\}_{j\geq 0}$ respectively. $C_\theta^-(\Sigma)$ and $C_\theta^+(\Sigma)$ are subspaces of $C_\theta(\Sigma)$ depending only on ω^- (ω^+). As before (Σ^τ, g^t) means the suspension flow with the roof function τ.
Recall that any symbolic flow has a certain geometric structure, namely the structure of a Smale space ([R1]). For the suspension (Σ^τ, g^t) of the two-sided subshift of a finite type (Σ, σ) with the roof function $\tau \in C_\theta(\Sigma)$ the Smale structure consists of the following objects.
–the unstable manifold $W^u(\omega, t) = \{(\bar{\omega}, \bar{t}) : \bar{\omega}^+ = \omega^+\}$;
–the stable manifold $W^s(\omega, t) = \{(\bar{\omega}, \bar{t}) : \bar{\omega}^- = \omega^-\}$;
–the strong unstable manifold $W^{su} = \{(\bar{\omega}, \bar{t}) \in W^u(\omega, t) : t - \bar{t} = \Delta^-(\bar{\omega}, \omega)\}$ where $\Delta^-(\bar{\omega}, \omega) = \sum_{j=1}^{\infty} [\tau(\sigma^{-j}\bar{\omega}) - \tau(\sigma^{-j}\omega)]$;

–the strong stable manifold $W^{ss} = \{(\bar\omega, \bar t) \in W^s(\omega,t) : t-\bar t = \Delta^+(\bar\omega,\omega)\}$ where $\Delta^-(\bar\omega,\omega) = \sum_{j=0}^{\infty}[\tau(\sigma^j\bar\omega) - \tau(\sigma^j\omega)]$;
–the local product structure: if $\omega_0 = \bar\omega_0$ then $[\omega,\bar\omega]$ is defined as following.

$$([\omega,\bar\omega])_j = \begin{cases} \omega_j, & \text{if } j \leq 0 \\ \bar\omega_j, & \text{if } j \geq 0 \end{cases};$$

–the temporal distance function (see [Ch])

$$\varphi(\omega,\bar\omega) = \Delta^+([\bar\omega,\omega],\omega) + \Delta^-([\bar\omega,\omega],\bar\omega) + \Delta^-([\omega,\bar\omega],\omega) + \Delta^+([\omega,\bar\omega],\bar\omega).$$

The geometric meaning of the temporal distance is the following. Let $p_*^{(\bar\omega,\bar t)}$ denote the projection to $W^*(\bar\omega,\bar t)$ (along the leaves of the complimentary foliation). Then

$$p_s^{(\omega,0)} \circ p_u^{(\omega,0)} \circ p_s^{(\bar\omega,0)} \circ p_u^{(\bar\omega,0)}(\omega,t) = (\omega, t+\varphi(\omega,\bar\omega)).$$

Since φ is defined geometrically it is clear that it is invariant in the sense that depends only on the cohomology class of τ. This can also be seen directly since the replacement of $\tau(\omega)$ by $\tau(\omega)+T(\omega)-T(\sigma\omega)$. alter $\Delta^+(\bar\omega,\omega)$ by $T(\bar\omega)-T(\omega)$ and $\Delta^-(\bar\omega,\omega)$ by $T(\omega)-T(\bar\omega)$. If $\tau \in C^+(\omega)$ then Δ^+ vanishes and

$$\varphi(\omega,\bar\omega) = \Delta^-([\bar\omega,\omega],\bar\omega) + \Delta^-([\omega,\bar\omega],\omega).$$

2. Local integrability.

Definition. $\{g^t\}$ is called locally integrable if $\varphi \equiv 0$.

The following statement essentially goes back to Anosov.

Proposition 4. (Anosov alternative for symbolic flows.) *If $\{g^t\}$ is locally integrable $\tau \sim \bar\tau$ where $\bar\tau$ depends only on ω_0.*

Before giving the proof recall some standard results about the cohomology of subshifts of finite type. First, note that $\Delta^-([\bar\omega,\omega],\bar\omega)$ does not depend on ω^- so we can write $\Delta^-([\bar\omega,\omega],\bar\omega) = \delta^-(\omega^+,\bar\omega^+,\bar\omega^-)$. Similarly $\Delta^+([\bar\omega,\omega],\omega) = \delta^-(\bar\omega^-,\omega^-,\bar\omega^+)$.

Proposition 5. ([S]) *There is a constant C_{37} such that*

$$\left|\delta^-(\omega_1^+,\omega_2^+,\omega_3^-) - \delta^-(\bar\omega_1^+,\bar\omega_2^+,\bar\omega_3^-)\right| \leq C_{37}\sqrt{d_\theta(\omega_1^+,\bar\omega_1^+) + d_\theta(\omega_2^+,\bar\omega_2^+) + d_\theta(\omega_3^-,\bar\omega_3^-)}$$

and

$$\left|\delta^-(\omega_1^-,\omega_2^-,\omega_3^+) - \delta^-(\bar\omega_1^-,\bar\omega_2^-,\bar\omega_3^+)\right| \leq C_{37}\sqrt{d_\theta(\omega_1^-,\bar\omega_1^-) + d_\theta(\omega_2^-,\bar\omega_2^-) + d_\theta(\omega_3^+,\bar\omega_3^+)}.$$

Corollary 5. ([S]) *Choose for any* i *a sequence* $\omega(i)$ *such than* $(\omega(i))_0 = i$. *Set* $T_+(\omega) = \delta^+(\omega(i)^-, \omega^-, \omega(i)^+)$, $\tau^+(\omega) = \tau(\omega) + T_+(\omega) - T_+(\sigma\omega)$, $T_+(\omega) = \delta^+(\omega^+, \omega(i)^+, \omega(i)^-)$, $\tau^-(\omega) = \tau(\omega) + T_-(\omega) - T_-(\sigma\omega)$. *Then* $\tau^+ \in C^+_{\sqrt{\theta}}(\Sigma)$, $\tau^- \in C^-_{\sqrt{\theta}}(\Sigma)$.

Proof of Proposition 4: By the discussion above we may assume that $\tau \in C^+_\theta(\Sigma)$. Consider $\tau^-(\omega) = \tau(\omega) - T_-(\omega) + T_-(\sigma\omega)$. τ^- does not depend on $\omega_j,\ j > 0$ by Corollary 5. On the other hand

$$\varphi(\omega, \bar{\omega}) = \delta^-(\omega^+, \bar{\omega}^+, \bar{\omega}^-) - \delta^-(\omega^+, \bar{\omega}^+, \omega^-). \tag{6}$$

Thus $\varphi \equiv 0$ just means that $\delta^-(\omega_1^+, \omega_2^+, \omega_3^-)$ does not depend on the third variable. Hence τ^- does not depend on $\omega_j,\ j < 0$ as well and we are done. ■

3. The result. The aim of this appendix is to prove the following result.

Theorem 4. *If* g^t *is not rapidly mixing then the lower box counting dimension of the range of* φ *vanishes:*

$$\underline{\mathrm{BD}}(\mathrm{Range}(\varphi)) = 0.$$

Proof: By the discussion in Section 1 we may assume that $\tau \in C^+_\theta(\Sigma)$. By Theorem 2 we can find sequances $\{b_k\}$ and $\{h_k\}$ satisfing conditions (2.v). Let N be some large natural number which we assume to be a a multiple of $n(\beta, b_k)$. Take some $\omega = \omega^-\omega^+$, $\bar{\omega} = \bar{\omega}^-\bar{\omega}^+$ and moreover let $\omega^- = \omega^\Omega \xi_N$, $\bar{\omega}^- = \bar{\omega}^\Omega \bar{\xi}_N$ where ξ_N and $\bar{\xi}_N$ are words of length N. Inequalities

$$|e^{-ib_k\tau_N(\xi_N\omega^+)}h_k(\xi_N\omega^+) - h_k(\omega^+)| \leq N|b_k|^{-\alpha},$$

$$|e^{-ib_k\tau_N(\bar{\xi}_N\omega^+)}h_k(\bar{\xi}_N\omega^+) - h_k(\omega^+)| \leq N|b_k|^{-\alpha}$$

imply that

$$|e^{-ib_k\tau_N(\xi_N\omega^+)}h_k(\xi_N\omega^+) - e^{-ib_k\tau_N(\bar{\xi}_N\omega^+)}h_k(\bar{\xi}_N\omega^+)| \leq 2N|b_k|^{-\alpha}.$$

Using (2.v) once more we get

$$\left|\exp\left(ib_k[\tau_N(\xi_N\omega^+) - \tau(\bar{\xi}_N\omega^+)]\right) - \frac{h_k(\bar{\xi}_N)}{h_k(\xi_N)}\right| \leq C_{38}\left(N|b_k|^{-\alpha} + |b_k|^{\alpha_5}\theta^N\right).$$

Similarly

$$\left|\exp\left(ib_k[\tau_N(\xi_N\bar{\omega}^+) - \tau(\bar{\xi}_N\bar{\omega}^+)]\right) - \frac{h_k(\bar{\xi}_N)}{h_k(\xi_N)}\right| \leq C_{38}\left(N|b_k|^{-\alpha} + |b_k|^{\alpha_5}\theta^N\right).$$

Thus

$$\left|\exp\left(ib_k\left[\tau_N(\xi_N\omega^+) - \tau_N(\bar{\xi}_N\omega^+) - \tau_N(\xi_N\bar{\omega}^+) + \tau_N(\bar{\xi}_N\bar{\omega}^+)\right]\right)\right|$$

$$\leq C_{39}\left(N|b_k|^{-\alpha} + |b_k|^{\alpha_5}\theta^N\right).$$

In view of (6) this gives

$$|\exp(ib_k\varphi(\omega,\bar{\omega})) - 1| \leq C_{39}\left(N|b_k|^{-\alpha} + |b_k|^{\alpha_5}\theta^N\right).$$

Choose $N \sim \frac{\alpha \ln |b_k|}{ln(\theta^{-1})}$, then

$$|\exp(ib_k\varphi(\omega,\bar{\omega})) - 1| \leq C_{40}|b_k|^{\alpha_5-\alpha}.$$

If also $|\varphi| \leq M$ then

$$\text{Range}(\varphi) \subset \bigcup_{|m|\leq \frac{M|b_k|}{2\pi}} \left[\frac{2\pi m}{|b_k|} - C_{41}|b_k|^{\alpha_5-\alpha}, \frac{2\pi m}{|b_k|} + C_{41}|b_k|^{\alpha_5-\alpha}\right],$$

which implies $\underline{\text{BD}}(\text{Range}(\varphi)) \leq (\alpha-\alpha_5)^{-1}$. Since α is arbitrary, Theorem 4 is proven. ∎

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