MATH642. COMPLEMENTS TO "INTRODUCTION TO DYNAMICAL SYSTEMS" BY M. BRIN AND G. STUCK

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1.3. Expanding Endomorphisms of the circle. Let $E_{10} : S^1 \to S^1$ be given by $E_{10}(x) = 10x \mod 1$.

Exercise 1. *Show that there exists a point x such that E_{10}-orbit of x is neither eventually periodic nor dense.*

1.5. Quadratic maps. Let $q_\mu(x) = \mu x(1-x)$. It has fixed points 0 and $1 - \frac{1}{\mu}$. Observe that $q_\mu^{-1}(0) = \{0, 1\}$, $q_\mu^{-1}(1 - \frac{1}{\mu}) = \{1 - \frac{1}{\mu}, \frac{1}{\mu}\}$.

Lemma 1. *Consider the map $f : x \to ax^2 + bx + c, a \neq 0$. Then either for all x we have $x_n \to \infty$ as $n \to \infty$ or f is conjugated to some $q_\mu, \mu > 0$.*

Proof. By changing coordinates $x \to -x$ if necessary we can assume that $a < 0$. Then $f(x) < x$ for large $|x|$. Consider two cases

(1) $f(x) < x$ for all x. Then x_n is decreasing so it either has a finite limit or goes to $-\infty$. Since f has no fixed points the second alternative holds.

(2) $f(x) = x$ has two (maybe coinciding solutions) x_1 and x_2. let x_3 be the solution of $f(x_3) = f(x_1)$ and x_4 be the solution of $f(x_4) = f(x_2)$. We have

$$x_1 + x_2 = -\frac{b-1}{a}, \quad x_1 + x_3 = -\frac{b}{a}, \quad x_2 + x_4 = -\frac{b}{a}.$$

Hence $(x_3 - x_1) + (x_4 - x_2) = -\frac{2}{a} > 0$. So either $x_3 > x_1$ or $x_4 > x_2$. In the first case make a change of coordinates $y = \frac{x - x_1}{x_3 - x_1}$. In this coordintes f takes the form $y \to g(y)$ where g is quadratic with negative leading term. Also $g(0) = 0, g(1) = g(0) = 0$. Thus $g = q_\mu$ for some μ. In the second case make a change of coordinates $y = \frac{x - x_2}{x_4 - x_2}$. □

Exercise 2. *Let $\mu = 4$. Then $I = [0, 1]$ is invariant. Show that*

(a) If $x \notin I$ then $q_4^n(x) \to -\infty$.

(b) Show that the changes of variables $y = 2x - 1$, $y = \cos z$ conjugate $q_4|_I$ to a piecewise linear map.

Lemma 2. *If* $0 < \mu < 1$ $x \in \mathbb{R}$ *then either*

(1) $q_\mu^n(x) \to -\infty$ *or*
(2) $q_\mu^n(x) \to 0$ *or*
(3) $q_\mu^n(x) = 1 - \frac{1}{\mu}$, $n \geq 2$.

Proof. There are several cases to consider. (1) $x < 1 - \frac{1}{\mu}$, (2) $1 - \frac{1}{\mu} < x < 0$, (3) $0 < x < 1$, (4) $1 < x < \frac{1}{\mu}$, $x > \frac{1}{\mu}$. We consider case (2), others are similar. In this case by induction $x_n < x_{n+1} < 0$. Let $y = \lim_{n\to\infty} x_n$. Then $q_\mu(y) = \lim_{n \to =infty} x_{n+1} = y$. So y is fixed. Also $y > x_0 > 1 - \frac{1}{\mu}$ since we are in case (2). It follows that $y = 0$. □

Exercise 3. *Complete the proof of Lemma 2.*

1.6 Gauss map. Let A be 2×2 matrix. Since $A(0) = 0$ and A moves lines to lines, it acts on the projective line. Let P_A denote this action. Coordinatizing projective space, by making the coordinate of a line its intersection with $\{y = 1\}$ we get

$$P_A(x) = \frac{ax+b}{cx+d}, \quad x \in \mathbb{R} \bigcup \{\infty\} \quad \text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that $P_A P_B = P_{AB}$.

Exercise 4. *Describe the dynamics of* P_A. *When is* P_A *conjugated to a rotation?*

Let $f(x) = \{1/x\}$. Hence $x_1 = (1/x) - a_1$ for some $a_1 \in \mathbb{N}$. Thus

$$x = \frac{1}{a_1 + x_1} = P_{\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}} x_1.$$

Continuing we get

$$x = P_{\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}} x_1 = P_{\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}} P_{\begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix}} x_2 = \cdots = P_{M_n} x_n$$

where

$$M_{n+1} = M_n \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix}.$$

Denoting the elements of M_n

$$M_n = \begin{pmatrix} A_n & p_n \\ B_n & q_n \end{pmatrix}$$

we get $A_n = p_{n-1}$, $B_n = q_{n-1}$ and

$$p_{n+1} = p_{n-1} + a_{n+1} p_n, \quad q_{n+1} = q_{n-1} + a_{n+1} q_n.$$

Exercise 5. $q_n \geq f_n$ *where* f_n *is the* n*-th Fibonacci number.*

Exercise 6. *Find all x such that $q_n = f_n$.*

Lemma 3. $p_{n-1}q_n - q_{n-1}p_n = (-1)^n$.

Proof.

$$\det(M_n) = \prod_{j=1}^{n} \det \begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix}.$$

□

Lemma 4. $\frac{p_n}{q_n} \to x$, $n \to \infty$. *Moreover*

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}.$$

Proof. We have

$$\begin{aligned} \left| x - \tfrac{p_n}{q_n} \right| &= \left| \tfrac{p_{n-1}x_n + p_n}{q_{n-1}x_n + q_n} - \tfrac{p_n}{q_n} \right| && = \tfrac{x\left|p_{n-1}q_n - q_{n-1}p_n\right|}{q_n(q_{n-1}x_n + q_n)} \\ &= \tfrac{x}{q_n(q_{n-1}x_n + q_n)} && \text{(by Lemma 3)} \\ &\leq \tfrac{1}{q_n^2} && \text{(since } \quad 0 \leq x_n \leq 1). \end{aligned}$$

□

Thus for every number $|x - p/q| \leq 1/q^2$ has infinitely many solutions.

Lemma 5. *Suppose that x is an irrational number satisfying the quadratic equation*

$$F(x) = ax^2 + bx + c = 0$$

with integer coefficients. When there is a constant C such that $|x - p/q| \geq \frac{C}{q^2}$.

Proof. Consider two cases

(1) $|x - p/q| > 1$. Then $|x - p/q| > 1/q^2$ since $q > 1$.

(2) $|x - p/q| \leq 1$. Decompose $F(z) = a(z - x)(z - y)$ where y is the second root of F. Then

$$F\left(\frac{p}{q}\right) = \frac{ap^2 + bpq + cq^2}{q^2} \geq \frac{1}{q^2} \tag{1}$$

since the denominator is a non-zero integer. On the other hand

$$\left| y - \frac{p}{q} \right| \leq |y| + \left| \frac{p}{q} \right| \leq |y| + |x| + 1$$

we have

$$\left| F\left(\frac{p}{q}\right) \right| \leq |a| \left| x - \frac{p}{q} \right| \left| y - \frac{p}{q} \right| \leq |a| \left| x - \frac{p}{q} \right| (|y| + |x| + 1) \tag{2}$$

(1) and (2) imply the result. □

Exercise 7. *Let x be an irational root of an equation $F(z) = a_m z^m + a_{m-1}z^{m-1} + \dots a_1 z + a_0 = 0$. Prove that there exists a constant C such that $|x - p/q| \geq C/q^m$.*

Lemma 3 implies

$$\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^{n+1}}{q_n q_{n+1}} \quad \frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+2}}{q_{n+2}} = \frac{(-1)^n}{q_{n+1}q_{n+1}}$$

so

$$\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^{n+1}}{q_{n+1}} \left(\frac{1}{q_n} - \frac{1}{q_{n+2}} \right).$$

Corollary 6. *p_{2k}/q_{2k} is increasing, p_{2k+1}/q_{2k+1} is decreasing. In particular*

$$\frac{p_{2k}}{q_{2k}} < x < \frac{p_{2k+1}}{q_{2k+1}}.$$

Observe that $f(0)$ is not defined.

Lemma 7. *The orbit of x contains 0 if and only if x is rational.*

Proof. If $x_n = 0$ then $x = p_n/q_n$. Conversely if x is rational then $f(x)$ is rational with smaller denominator. □

Theorem 1. *x is eventually periodic if and only if it is a quadratic irrational.*

Proof. (1) Observe that if x satisfies a quadratic equation then $ax^2 + bx + c$ then x_n satisfies the equation $A_n x_n^2 + B_n x_n + C_n = 0$ where

$$\begin{aligned} A_n &= ap_{n-1}^2 + bp_{n-1}q_{n-1} + c \\ B_n &= 2ap_{n-1}p_n + b(p_{n-1}q_n + q_{n-1}p_n) + 2cq_{n-1}q_n \\ C_n &= ap_n^2 + bp_n q_n + c \end{aligned}$$

We claim that A_n, B_n and C_n are uniformly bounded so they must eventually repeat giving eventual periodicity of x. Indeed using the notation of Lemma 5 we get

$$\begin{aligned} |C_n| &= q_n^2 |F(p/q)| = q_n^2 |a||x - p/q||y - p/q| \\ &\leq q_n^2 |a|(|y| + 1)|x - p/q| && (|x| < 1) \\ &\leq |a|(|y| + 1) && \text{(Lemma 5)} \end{aligned}$$

Likewise $|A_n| \leq |a|(|y| + 1)$.

Exercise 8. *Show (e.g. by induction) that $b^2 - 4ac = B_n^2 - 4A_n C_n$.*

Hence

$$|B_n| \leq \sqrt{|b^2 - 4ac| + 4a^2(|y| + 1)^2}$$

Thus x is eventually periodic.

(2) If x is periodic then for some n

$$x = \frac{p_{n-1}x + p_n}{q_{n-1}x + q_n}.$$

Hence x satisfies the equation

$$q_{n-1}x^2 + (q_n - p_{n-1})x - p_n = 0.$$

Next if

$$x = \frac{p_{m-1}y + p_m}{q_{m-1}y + q_m}$$

with y periodic then y satisfies a quadratic equation an a computation similar to the one done part (1) shows that x satisfies a quadratic equation as well. □

5.12 Markov partitions.

Exercise 9. *Show that any linear hyperbolic automorhism of* $\mathbb{T}^2$ *has a Markov partition.*

Exercise 10. *Show that no Markov partition of* $\mathbb{T}^2$ *gives a full shift.*

Hint. Compare periodic points.

2.4 Expansive transformations.

Exercise 11. *Show that no isometry of infinite compact metric space is expansive.*

2.8 Applications of topological dynamics.

Exercise 12. *Let* $T_1, T_2 \dots T_N$ *be commuting homeomorphisms of a compact metrix space* X*. Prove that there exist* $x \in X$ *and a sequence* $n_k \to \infty$ *such that* $d(x, T_j^{n_k} x) \to 0$ *for all* j*.*

Hint. Let $F(x) = \inf_{n \geq 1} max_j d(x, T_j x)$. Let $A_\varepsilon = \{F < \varepsilon\}$. Show that A_ε is open and dense.

3.3 The Perron-Frobenius Theorem. A subset $K \subset reals^d$ is called a cone if for any $v \in K$, $\lambda > 0$ $\lambda v \in K$. Let K be a convex closed cone satisfying

(K1) $K \bigcap (-K) = \{0\}$ and

(K2) Any vector u in $\mathbb{R}^d$ can be represented as $u = v_1 - v_2$ with $v_j \in K$.

Lemma 8. *Any line* l *containing points inside* K *intersects the boundary of* K*.*

Proof. Take two points $v, u \in K \bigcap K$. Then $l = \{z_t = tu + (1-t)v\}$. Rewrite $z_t = v + t(u-v)$. Let $z_t \in K$ for all positive t. Since K is a cone, $(v/t) + u - v \in K$ and since K is closed $u - v \in K$. Likewise if $z_t \in K$ for all negative t then $v - u \in K$. By (K2) both inclusions can not be true. □

Let $\tilde{K}$ be a subset of the $\mathbb{RP}^{d-1}$ consisting of directions having representatives in K. Define a distance on $\tilde{K}$ as follows. If $\tilde{u}_1, \tilde{u}_2$ are rays in K choose $b \in \tilde{u}_1$, $c \in u_2$ and let l be the line through b and c. Let a and d be the points where l crosses the boundary of K such that a, b, c, d is the correct order on this line and let t be an affine parameter on l. Define

$$d_K(u_1, u_2) = \ln\left(\frac{(t_c - t_a)(t_d - t_b)}{(t_b - t_a)(t_d - t_c)}\right).$$

To see that this distance correctly defined it is enough to consider the case of the plane since d_K only depends on the section of K be the plane containing u_1 and u_2. Now let v_1 and v_2 are two vectors on the boundary of the cone and u is a vector on the line joining v_1 and v_2. Thus $u = tv_1 + (1-t)v_2$. Now if we consider another line say through v_1' and v_2' then this line crosses the ray through u at a point $\bar{u} = sv_1' + (1-s)v_2'$. Denoting by $\times$ the vector product we get

$$(v_2 + t(v_1 - v_2)) \times (v_2' + s(v_1' - v_2')) = 0$$

Thus

$$s = \frac{v_2' \times (v_2 + t(v_1 - v_2))}{(v_2 + t(v_1 - v_2)) \times (v_1' - v_2')}.$$

That is, the map between the affine parameters corresponding to different lines is fractional linear. Since fractional linear maps preserve the cross ratio d_K is correctly defined.

Exercise 13. *Let K be the cone of vecors with non-negative components and*

$$K_L = \{u \in K : \max_i u_i \le L \min_i u_i.\}$$

Show that K_L has finite d_K-diameter.

Theorem 2. *Let A be a matrix with positive entries. Then*

(a) *A has a positive eigenvalue λ;*
(b) *The corresponding eigenvector v is positive;*
(c) *All other eigenvalues have absolute value less than λ;*
(d) *There are no other positive eigenvectors.*

Lemma 9. *Let P_A be the projecive transformation defined by A. Then there exists $\tilde{v} \in \tilde{K}$ and $\theta < 1$ such that $P_A(\tilde{v}) = \tilde{v}$ and for all $\tilde{u} \in \tilde{K}$*

$$\text{dist}(P_A^n \tilde{u}, \tilde{v}) \le \text{Const}\theta^n. \tag{3}$$

Proof of Theorem 2. Let $\tilde{v}$ be as in Lemma 9 and v be a positive vector projecting to $\tilde{v}$. Then $P_A(\tilde{v}) = (\tilde{v})$ means that $A(v) = \lambda(v)$, so A has positive eigenvector. Take i such that $v_i \neq 0$. Then $\lambda = (Av)_i/v_i$ is positive. Also it v' is another positive eigenvector then $P_A(\tilde{v}') = (\tilde{v}')$ contradicting Lemma 9, so there are no other positive eigenvectors.

We can assume without the loss of generality that $||v|| = 1$. Next we claim that for for all $u \in \mathbb{R}^d$ there exist the limit

$$l(u) = \lim_{n\to\infty} \frac{||A^n(u)||}{\lambda^n} \tag{4}$$

and moreover

$$||A^n u - l(u)\lambda^n v|| \le \text{Const}\lambda^n\theta^n. \tag{5}$$

Indeed assume first that $u \in K$. denote $S(\tilde{u}) = \frac{||Au||}{||u||}$ (this definition is clearly independent of the choice of the vector projecting to $\tilde{u}$. We have

$$||A^n u|| = ||u|| \prod_{j=0}^{n-1} \frac{||A^{j+1}u||}{||A^j u||} = ||u|| \prod_{j=0}^{n-1} S(P_A^j(u))$$

$$= (||u||\lambda^n) \prod_{j=0}^{n-1} \frac{S(P_A^j(u))}{S(v)} = (||u||\lambda^n) \exp\left[\sum_{j=0}^{n-1} \left(\ln S(P_A^j(u)) - \ln S(v)\right)\right].$$

Since S is Lipshitz we have

$$\left|\ln S(P_A^j(u)) - \ln S(v)\right| \le \text{Const}\theta^j \tag{6}$$

which proves (4) with $l(u) = ||u|| \prod_{j=0}^{\infty}$. Moreover the exponential convergence of (6) implies $||A^n u|| = l(u)\lambda^n(1+O(\theta^n))$. Now by Lemma 9 we have

$$\frac{A^n u}{||A^n u||} = v + O(\theta^n)$$

This proofs (5) for positive vectors. Now (5) in general case follows by (K2). From the properties of limit it follows that l is a linear functional. Let $L = \text{Ker}(l)$. Then L is $d-1$ dimensional hypersurface and since $l(A(u)) = \lambda l(u)$, L is A-invariant. Thus all other eigenvectors lie in L. It follows from (5) that

$$||A^n|_L|| \le \text{Const}\lambda^n\theta^n$$

so all eigenvalues are less than $\lambda\theta$ in asbsolute value. □

Proof of Lemma 9. For any u in K we have

$$\min_{ij} A_{ij} \max_j u_j \leq (Au)_i \leq d \max_{ij} A_{ij} \max_j u_j$$

so $A(u) \in K_L$ with $L = d\frac{\max_{ij} A_{ij}}{\min_{ij} A_{ij}}$. Now K_L is compact in both dist and d_K metrics so it is enough to establish (3) for d_K.

Lemma 10. *Given D there exists $\theta < 1$ such that for and d for any linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, such that $A(K) \subset K$ and $P_A(\tilde{K})$ has diameter less than D in $\tilde{K}$ $d(P_A\tilde{v}_1, P_A\tilde{v}_2) \leq \theta d(\tilde{v}_1, \tilde{v}_2)$.*

Proof of Lemma 10. Since the definition of $d_K(\tilde{v}_1, \tilde{v}_2)$ deepends only on the section of K by the plane through v_1 and v_2 it is enough to establish the result for linear map from plane to plane. Now on the plane we can use y/x as a coordinate for the point (x, y). In this case $\tilde{K} = [0, \infty]$ and P_A is a fractional linear transformation. Also for $z_1 < z_2$

$$d_K(z_1, z_2) = \ln\left(\frac{z_2 - 0}{z_1 - 0} \lim_{z\to\infty} \frac{z_1 - z}{z_2 - z}\right) = \ln\frac{z_2}{z_1} = \int_{z_1}^{z_2} \frac{dz}{z}. \tag{7}$$

So we have to prove that any fractional linear transformation s from $[0, \infty]$ to itself such that $s(\infty)/s(0) < e^D$ contracts distance (7) by a factor which depoends only on D. Since dilations preserve (7) we can assume that $s(\infty) = 1$ thus we have

$$w := s(z) = \frac{z + a}{z + b}$$

with $a/b > e^{-K}$. We have

$$\frac{dw}{dz} = \frac{b - a}{(z + b)^2}$$

so that

$$\frac{dw}{w} = \frac{z(b - a)}{(z + a)(z + b)} \frac{dz}{z}$$

But

$$\frac{z(b - a)}{(z + a)(z + b)} \leq \frac{b - a}{b} < 1 - e^{-D}.$$

It follows that

$$d_K(s(z_1), s(z_2)) = \int_{w_1}^{w^2} \frac{dw}{w} = \int_{z_1}^{z_2} \frac{z}{w}\frac{dw}{dz}\frac{dz}{z}$$
$$< (1 - e^{-D}) \int_{z_1}^{z_2} \frac{dz}{z} < (1 - e^{-D}) d(z_1, z_2).$$

□

(3) follows from Lemma 10 and contraction mapping principle. □

Theorem 3. *Let A be the matrix with non-negative entries such that for all i, j there exists n such that $A^n_{ij} > 0$. Then there exist lambda > 0 and c such that*

(a) $\lambda e^{2\pi i r/c}$ are eigenvalues of A.

(b) A^c has c linearly independent positive eigenvectors with eigenvalues λ^c.

(c) All other eigenvalues have absolute value less than λ

(d) There are no other positive eigenvectors.

Proof. Let $Z_{ij} = \{n : A^n_{ij} > 0$. Observe that $Z_{ij} + Z_{jk} \subset Z_{ik}$ in particular Z_{ii} are semigroups.

Lemma 11. *If $a, b \in Z_{ii}$ let $q = \gcd(a, b)$. Then for large N $Nq \in Z_{ii}$.*

Proof. We have $q = ma - nb$ for some $m, n \in \mathbb{N}$. Write $N = Lb + k$ then $Nq = (Lq - n)b + mka$. □

Corollary 12. *Let c_i be the greatest common divisor of all numbers in Z_{ii}. Then $Z_{ii} = c_i\mathbb{N}-$finitely many numbers.*

Proof. Let $c_i^{(1)}$ be any number in Z_{ii}. Then $c_i^{(1)}\mathbb{N}$ is in Z_{ii} and if there are no other numbers in Z_{ii} then we are done. Otherwise if $b_i^{(1)} \in Z_{ii} - c_i^{(1)}\mathbb{N}$ then let $c_i^{(2)} = gcd(c_i^{(1)}, b_i^{(1)})$. Then $c_i^{(2)}\mathbb{N}-$ a finite set is in Z_{ii} and if there are no other numbers in Z_{ii} then we are done. Otherwise if $b_2^{(1)} \in Z_{ii} - c_i^{(2)}\mathbb{N}$ then let $c_i^{(3)} = gcd(c_i^{(2)}, b_i^{(2)})$ etc. Since $c_i^{(k)}$ is decreasing it must stabilize. □

Lemma 13. *(a) Any two numbers in Z_{ij} are comparable mod c_i.*

(b) Any two numbers in Z_{ji} are comparable mod c_i.

(c) $c_i = c$ do not depend on i.

Proof. (a) Let $n', n'' \in Z_{ij}, n \in Z_{ji}, m \in Z_{jj}$ then $n + n' = n + n'' = n + n' + m \mod c_i$. This prove (a) and (c). (b) is similar to (a). □

Let $V_m = \{i : Z_{1i} = m \mod c\}$, $L_m = \mathrm{span}(e_i, i \in V_m\}$. Then There exists N such that $A^N_{ij} > 0$ if and only if i and j belong to the same V_m. Thus by Theorem 2 A^N has uniques eigenvector v_1 on L_1 with positive eigenvalue ν. Let $v_i = A^{i-1}v_1$. Since A commutes with A^N v_i are eigenvectors of A^N and since A has non-negative entries v_i are non-negative. Also $v_i \in L_{i \mod c}$. In particular by uniqueness of the positve eigenvector $A^c v_1 = \bar{\nu} v_1$. Let $\lambda = \bar{\nu}^{1/c}$. Then $A^c(v_i) = \lambda^c(v_i)$. Let $w_r = \sum_{j=1}^{c} \frac{e^{-2\pi i j r/c}}{\lambda^j} v_j$. Then $Aw_r = e^{2\pi j r/c} w_r$. Finally by Theorem 2 all other eigenvalues of A^N are less than $\nu = \lambda^N$ in absolute value. Theorem 3 is proven. □

4.10 Weak Mixing.

Let U be a unitary operator and ϕ be a unit vector. Let

$$R_n = < U^n\phi, \phi > .$$

Lemma 14. *There is a probability measure σ_ϕ on $[0,1]$ such that*

$$R_n = \int e^{2\pi iu} d\sigma(u).$$

Proof. For $0 < \rho < 1$ let

$$f(u,\rho) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} R_{n-m} e^{2\pi i(m-n)} \rho^{n+m}.$$

Since $R_{n-m} = < U^n\phi, U^m\phi >$ it follows that

$$f(u,\rho) = ||\sum_n \rho^n e^{-2\pi iu} U^n\phi||^2$$

is a real positive number. Estimateing terms in () by their absolute values we get $|f| \leq (1-\rho)^{-2}$. Thus $f \in L^\infty(du) \subset L^2(du)$. Let us examine its Fourier series. We have

$$f = \sum_k R_k e^{-2\pi iu} \sum_m \rho^{k+2m} = \sum_k e^{-2\pi k} R_k \rho^k (1-\rho^2)^{-1}.$$

Consider measures

$$d\sigma_\rho = (1-\rho^2) f(u,\rho) du.$$

We have

$$\int e^{2\pi iu} d\sigma_\rho(u) = R_k \rho^k.$$

Hence as $\rho \to 1$

$$\int e^{2\pi iu} d\sigma_\rho(u) \to R_k.$$

Since linear combinattions of $e^{2\pi iu}$ are dense in $C(S^1)$ it follows that for any continuous function A

$$\int A(u) d\sigma_\rho(u) \to \sigma(A).$$

□

Exercise 14. *Consider a full two shift with Bernoulli measure (that is the measure of each cylinder of size n is $(1/2)^n$) and let*

$$\phi(x) = \sqrt{2}\left(I_{x_0=1,x_1=1} - I_{x_0=0,x_1=0}\right).$$

Find the spectral measure of ϕ with respect to $U(\phi) = \phi(\sigma x)$.

Exercise 15. *Let* $\Phi_n(u) = \frac{1}{n^2}\sum_{j,k=1}^{n} e^{2\pi i(j-k)u}$. *Show that* $\Phi_n(u) \to 0$, *if* $u \neq 0$ *(and* $\Phi_n(0) = 1$*).*

5.1 Expanding endomorphisms.

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a map such that $|f'| \geq \theta^{-1}$ for some $\theta < 1$. We call such map *expanding*. Given any diffeomorphism of $\mathbb{S}^1$ let $\bar{g} : \mathbb{R} \to \mathbb{R}$ be its lift, that is $\pi \circ \bar{g} = g \circ \pi$, where $\pi : \mathbb{R} \to \mathbb{S}^1$ is the natural projection. Define $\deg(g) = \bar{g}(x+1) - \bar{g}(x)$ (this number is easily seen to be independent of x and the lift $\bar{g}$). Let L be the space of maps $\bar{\tau} : \mathbb{R}^1 \to \mathbb{R}^1$ which are lifts of degree 1 maps. That is $\bar{\tau} - x$ is periodic. We endow L with the distance

$$d(\bar{\tau}_1, \bar{\tau}_2) = \sup_{x\in\mathbb{R}} |\bar{\tau}_1(x) - \bar{\tau}_2(x)| = \max_{x\in[0,1]} |\bar{\tau}_1(x) - \bar{\tau}_2(x)|.$$

Lemma 15. *Let f be an expanding map and g be a map of the same degree. Then given any two lifts $\bar{f}$ and $\bar{g}$ there is unique $\bar{\tau} \in L$ such that $\bar{f} \circ \bar{\tau} = \bar{\tau} \circ \bar{g}$.*

Proof. $\bar{\tau}$ must satisfy $\bar{\tau}(x) = \bar{f}^{-1}(\bar{\tau}(\bar{g}(x)))$. Define $\mathcal{K} : L \to L$ by $\mathcal{K}(\bar{\tau})(x) = \bar{f}^{-1}(\bar{\tau}(\bar{g}(x)))$. Since f is expanding it follows from the Intermidiate Value Theorem that $|\bar{f}^{-1}(x_1) - \bar{f}^{-1}(x_2)| \leq \theta|x_1 - x_2|$. Hence $d(\mathcal{K}(\bar{\tau}_1), \mathcal{K}(\bar{\tau}_1)) \leq \theta d(\bar{\tau}_1, \bar{\tau}_2)$. Now the result follows from the contraction mapping principle. □

Theorem 4. *Any two expanding maps of the same degree are topologically conjugated.*

Proof. Let $\bar{f}_1$ and$\bar{f}_2$ be lifts of two expanding maps. By Lemma 15 there are maps $\bar{\tau}_1$, $\bar{\tau}_2$ such that $\bar{\tau}_1 \circ \bar{f}_1 = \bar{f}_2 \circ \bar{\tau}_1$ and $\bar{\tau}_2 \circ \bar{f}_2 = \bar{f}_1 \circ \bar{\tau}_2$. Let $\bar{\tau} = \bar{\tau}_2 \circ \bar{\tau}_1$. Then $\bar{\tau} \circ \bar{f}_1 = \bar{f}_1 \circ \bar{\tau}$. By uniquesness part of Lemma 15 $\bar{\tau}_2 \circ \bar{\tau}_1 = \mathrm{id}$. Likewise $\bar{\tau}_1 \circ \bar{\tau}_2 = \mathrm{id}$. □

Exercise 16. *Show that this conjugacy is typically NOT C^1.*