1.3. Expanding Endomorphisms of the circle. Let $E_{10}: S^1 \to S^1$ be given by $E_{10}(x) = 10x \mod 1$.

**Exercise 1.** Show that there exists a point $x$ such that $E_{10}$-orbit of $x$ is neither eventually periodic nor dense.

1.5. Quadratic maps. Let $q_{\mu}(x) = \mu x(1-x)$. It has fixed points 0 and $1 - \frac{1}{\mu}$. Observe that $q_{\mu}^{-1}(0) = \{0, 1\}$, $q_{\mu}^{-1}(1 - \frac{1}{\mu}) = \{1 - \frac{1}{\mu}, \frac{1}{\mu}\}$.

**Lemma 1.** Consider the map $f: x \to ax^2 + bx + c$, $a \neq 0$. Then either for all $x$ we have $x_n \to \infty$ as $n \to \infty$ or $f$ is conjugated to some $q_{\mu}, \mu > 0$.

**Proof.** By changing coordinates $x \to -x$ if necessary we can assume that $a < 0$. Then $f(x) < x$ for large $|x|$. Consider two cases

(1) $f(x) < x$ for all $x$. Then $x_n$ is decreasing so it either has a finite limit or goes to $-\infty$. Since $f$ has no fixed points the second alternative holds.

(2) $f(x) = x$ has two (maybe coinciding solutions) $x_1$ and $x_2$, let $x_3$ be the solution of $f(x_3) = f(x_1)$ and $x_4$ be the solution of $f(x_4) = f(x_2)$. We have

$$x_1 + x_2 = -\frac{b-1}{a}, \quad x_1 + x_3 = -\frac{b}{a}, \quad x_2 + x_4 = -\frac{b}{a}.$$ 

Hence $(x_3 - x_1) + (x_4 - x_2) = -\frac{2}{a} > 0$. So either $x_3 > x_1$ or $x_4 > x_2$. In the first case make a change of coordinates $y = \frac{x_3 - x_1}{x_3 - x_2}$. In this coordinates $f$ takes the form $y \to g(y)$ where $g$ is quadratic with negative leading term. Also $g(0) = 0, g(1) = g(0) = 0$. Thus $g = q_{\mu}$ for some $\mu$. In the second case make a change of coordinates $y = \frac{x_4 - x_2}{x_4 - x_2}$. □

**Exercise 2.** Let $\mu = 4$. Then $I = [0, 1]$ is invariant. Show that

(a) If $x \not\in I$ then $q_4^n(x) \to -\infty$.

(b) Show that the changes of variables $y = 2x - 1, y = \cos z$ conjugate $q_4|_I$ to a piecewise linear map.
Lemma 2. If $0 < \mu < 1$ then either

1. $q^n(x) \to -\infty$
2. $q^n(x) \to 0$
3. $q^n(x) = 1 - \frac{1}{\mu}$, $n \geq 2$.

Proof. There are several cases to consider. (1) $x < 1 - \frac{1}{\mu}$, (2) $1 - \frac{1}{\mu} < x < 0$, (3) $0 < x < 1$, (4) $1 < x < \frac{1}{\mu}$, $x > \frac{1}{\mu}$. We consider case (2), others are similar. In this case by induction $x_n < x_{n+1} < 0$. Let $y = \lim_{n \to \infty} x_n$. Then $q(\mu)(y) = \lim_{n \to \infty} x_{n+1} = y$. So $y$ is fixed. Also $y > x_0 > 1 - \frac{1}{\mu}$ since we are in case (2). It follows that $y = 0$. □

Exercise 3. Complete the proof of Lemma 2.

1.6 Gauss map. Let $A$ be $2 \times 2$ matrix. Since $A(0) = 0$ and $A$ moves lines to lines, it acts on the projective line. Let $P_A$ denote this action. Coordinatizing projective space, by making the coordinate of a line its intersection with $\{y = 1\}$ we get

$$P_A(x) = \frac{ax + b}{cx + d}, \quad x \in \mathbb{R} \cup \{\infty\} \quad \text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Note that $P_A P_B = P_{AB}$.

Exercise 4. Describe the dynamics of $P_A$. When is $P_A$ conjugated to a rotation?

Let $f(x) = \{1/x\}$. Hence $x_1 = (1/x) - a_1$ for some $a_1 \in \mathbb{N}$. Thus

$$x = \frac{1}{a_1 + x_1} = P \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} x_1.$$ 

Continuing we get

$$x = P \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} x_1 = P \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} P \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} x_2 = \cdots = P_{M_n} x_n$$

where

$$M_{n+1} = M_n \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix}.$$ 

Denoting the elements of $M_n$

$$M_n = \begin{pmatrix} A_n & p_n \\ B_n & q_n \end{pmatrix}$$

we get $A_n = p_{n-1}$, $B_n = q_{n-1}$ and

$$p_{n+1} = p_{n-1} + a_{n+1} p_n, \quad q_{n+1} = q_{n-1} + a_{n+1} q_n.$$

Exercise 5. $q_n \geq f_n$ where $f_n$ is the $n$-th Fibonacci number.
Exercise 6. Find all $x$ such that $q_n = f_n$.

Lemma 3. $p_{n-1}q_n - q_{n-1}p_n = (-1)^n$.

Proof.

$$\det(M_n) = \prod_{j=1}^n \det \begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix}.$$ 

Lemma 4. $\frac{p_n}{q_n} \to x, n \to \infty$. Moreover

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q^2_n}.$$ 

Proof. We have

$$\left| x - \frac{p_n}{q_n} \right| = \frac{|p_n - x q_n|}{q_n (q_n - 1 x_n + q_n)} = \frac{|q_n (q_n - 1 x_n + q_n) - x_n q_n|}{q_n (q_n - 1 x_n + q_n)}$$

(by Lemma 3) (since $0 \leq x_n \leq 1$).

Thus for every number $|x - p/q| \leq 1/q^2$ has infinitely many solutions.

Lemma 5. Suppose that $x$ is an irrational number satisfying the quadratic equation

$$F(x) = ax^2 + bx + c = 0$$

with integer coefficients. When there is a constant $C$ such that $|x - p/q| \geq \frac{C}{q^2}$.

Proof. Consider two cases

1. $|x - p/q| > 1$. Then $|x - p/q| > 1/q^2$ since $q > 1$.
2. $|x - p/q| \leq 1$. Decompose $F(z) = a(z - x)(z - y)$ where $y$ is the second root of $F$. Then

$$F \left( \frac{p}{q} \right) = \frac{a p^2 + b pq + c q^2}{q^2} \geq \frac{1}{q^2}$$

since the denominator is a non-zero integer. On the other hand

$$\left| y - \frac{p}{q} \right| \leq |y| + \left| \frac{p}{q} \right| \leq |y| + |x| + 1$$

we have

$$\left| F \left( \frac{p}{q} \right) \right| \leq |a| \left| x - \frac{p}{q} \right| \left| y - \frac{p}{q} \right| \leq |a| \left| x - \frac{p}{q} \right| (|y| + |x| + 1)$$

(1) and (2) imply the result.
Exercise 7. Let $x$ be an irational root of an equation $F(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots a_1 z + a_0 = 0$. Prove that there exists a constant $C$ such that $|x - p/q| \geq C/q^m$.

Lemma 3 implies
\[
\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^{n+1}}{q_n q_{n+1}} \frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+2}}{q_{n+2}} = \frac{(-1)^{n}}{q_{n+1} q_{n+2}}.
\]

so
\[
\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^{n+1}}{q_{n+1}} \left( \frac{1}{q_n} - \frac{1}{q_{n+2}} \right).
\]

Corollary 6. $p_{2k}/q_{2k}$ is increasing, $p_{2k+1}/q_{2k+1}$ is decreasing. In particular
\[
\frac{p_{2k}}{q_{2k}} < x < \frac{p_{2k+1}}{q_{2k+1}}.
\]

Observe that $f(0)$ is not defined.

Lemma 7. The orbit of $x$ contains 0 if and only if $x$ is rational.

Proof. If $x_n = 0$ then $x = p_n/q_n$. Conversely if $x$ is rational then $f(x)$ is rational with smaller denominator. \qed

Theorem 1. $x$ is eventually periodic if and only if it is a quadratic irrational.

Proof. (1) Observe that if $x$ satisfies a quadratic equation then $ax^2 + bx + c$ then $x_n$ satisfies the equation $A_n x_n^2 + B_n x_n + C_n = 0$ where
\[
A_n = \frac{ap_{n-1}^2 + bp_{n-1} q_{n-1} + c}{q_n^2}, \\
B_n = 2ap_{n-1} p_n + b(p_{n-1} q_n + q_{n-1} p_n) + 2c q_{n-1} q_n, \\
C_n = \frac{ap_n^2 + b p_n q_n + c}{q_n^2}.
\]

We claim that $A_n, B_n$ and $C_n$ are uniformly bounded so they must eventually repeat giving eventual periodicity of $x$. Indeed using the notation of Lemma 5 we get
\[
|C_n| = q_n^2 |F(p/q)| = q_n^2 |a||x-p/q||y-p/q| \\
\leq q_n^2 |a|(|y|+1)|x-p/q| \\
\leq |a|(|y|+1) (|x| < 1) \\
\text{(Lemma 5)}
\]

Likewise $|A_n| \leq |a|(|y|+1).

Exercise 8. Show (e.g. by induction) that $b^2 - 4ac = B_n^2 - 4A_n C_n$.

Hence
\[
|B_n| \leq \sqrt{|b^2 - 4ac| + 4a^2(|y|+1)^2}
\]

Thus $x$ is eventually periodic.
(2) If $x$ is periodic then for some $n$

$$x = \frac{p_{n-1}x + p_n}{q_{n-1}x + q_n}.$$ 

Hence $x$ satisfies the equation

$$q_{n-1}x^2 + (q_n - p_{n-1})x - p_n = 0.$$ 

Next if

$$x = \frac{p_{m-1}y + p_m}{q_{m-1}y + q_m}$$

with $y$ periodic then $y$ satisfies a quadratic equation an a computation similar to the one done part (1) shows that $x$ satisfies a quadratic equation as well. \hfill \Box

5.12 Markov partitions.

Exercise 9. Show that any linear hyperbolic automorphism of $\mathbb{T}^2$ has a Markov partition.

Exercise 10. Show that no Markov partition of $\mathbb{T}^2$ gives a full shift.

Hint. Compare periodic points.

2.4 Expansive transformations.

Exercise 11. Show that no isometry of infinite compact metric space is expansive.

2.8 Applications of topological dynamics.

Exercise 12. Let $T_1, T_2 \ldots T_N$ be commuting homeomorphisms of a compact metric space $X$. Prove that there exist $x \in X$ and a sequence $n_k \to \infty$ such that $d(x, T_j^{n_k}x) \to 0$ for all $j$.

Hint. Let $F(x) = \inf_{n \geq 1} \max_j d(x, T_j x)$. Let $A_\varepsilon = \{ F < \varepsilon \}$. Show that $A_\varepsilon$ is open and dense.

3.3 The Perron-Frobenius Theorem. A subset $K \subset \mathbb{R}^d$ is called a cone if for any $v \in K$, $\lambda > 0 \lambda v \in K$. Let $K$ be a convex closed cone satisfying

(K1) $K \cap (-K) = \{0\}$ and

(K2) Any vector $u$ in $\mathbb{R}^d$ can be represented as $u = v_1 - v_2$ with $v_j \in K$.

Lemma 8. Any line $l$ containing points inside $K$ intersects the boundary of $K$. 
Proof. Take two points \( v, u \in K \cap K \). Then \( l = \{ z_t = tu + (1-t)v \} \). Rewrite \( z_t = v + t(u - v) \). Let \( z_t \in K \) for all positive \( t \). Since \( K \) is a cone, \( (v/t) + u - v \in K \) and since \( K \) is closed \( u - v \in K \). Likewise if \( z_t \in K \) for all negative \( t \) then \( v - u \in K \). By (K2) both inclusions cannot be true. \( \square \)

Let \( \tilde{K} \) be a subset of the \( \mathbb{RP}^{d-1} \) consisting of directions having representatives in \( K \). Define a distance on \( \tilde{K} \) as follows. If \( \tilde{u}_1, \tilde{u}_2 \) are rays in \( K \) choose \( b \in \tilde{u}_1, c \in u_2 \) and let \( l \) be the line through \( b \) and \( c \). Let \( a \) and \( d \) be the points where \( l \) crosses the boundary of \( K \) such that \( a, b, c, d \) is the correct order on this line and let \( t \) be an affine parameter on \( l \). Define

\[
d_K(u_1, u_2) = \ln \left( \frac{(t_c - t_a)(t_d - t_b)}{(t_b - t_a)(t_d - t_c)} \right).
\]

To see that this distance correctly defined it is enough to consider the case of the plane since \( d_K \) only depends on the section of \( K \) be the plane containing \( u_1 \) and \( u_2 \). Now let \( v_1 \) and \( v_2 \) are two vectors on the boundary of the cone and \( u \) is a vector on the line joining \( v_1 \) and \( v_2 \). Thus \( u = tv_1 + (1-t)v_2 \). Now if we consider another line say through \( v'_1 \) and \( v'_2 \) then this line crosses the ray through \( u \) at a point \( \tilde{u} = sv'_1 + (1-s)v'_2 \). Denoting by \( \times \) the vector product we get

\[
(v_2 + t(v_1 - v_2)) \times (v'_2 + s(v'_1 - v'_2)) = 0
\]

Thus

\[
s = \frac{v'_2 \times (v_2 + t(v_1 - v_2))}{(v_2 + t(v_1 - v_2)) \times (v'_1 - v'_2)}.
\]

That is, the map between the affine parameters corresponding to different lines is fractional linear. Since fractional linear maps preserve the cross ratio \( d_K \) is correctly defined.

**Exercise 13.** Let \( K \) be the cone of vectors with non-negative components and

\[
K_L = \{ u \in K : \max_i u_i \leq L \min_i u_i \}.
\]

Show that \( K_L \) has finite \( d_K \)-diameter.

**Theorem 2.** Let \( A \) be a matrix with positive entries. Then

(a) \( A \) has a positive eigenvalue \( \lambda \);

(b) The corresponding eigenvector \( v \) is positive;

(c) All other eigenvalues have absolute value less than \( \lambda \);

(d) There are no other positive eigenvectors.
Lemma 9. Let $P_A$ be the projective transformation defined by $A$. Then there exists $\tilde{v} \in \tilde{K}$ and $\theta < 1$ such that $P_A(\tilde{v}) = \tilde{v}$ and for all $\tilde{u} \in \tilde{K}$

$\text{(3)} \quad \text{dist}(P_A^n \tilde{u}, \tilde{v}) \leq \text{Const} \theta^n.$

Proof of Theorem 2. Let $\tilde{v}$ be as in Lemma 9 and $v$ be a positive vector projecting to $\tilde{v}$. Then $P_A(\tilde{v}) = (\tilde{v})$ means that $A(v) = \lambda(v)$, so $A$ has positive eigenvector. Take $i$ such that $v_i \neq 0$. Then $\lambda = (Av)_i/v_i$ is positive. Also it $v'$ is another positive eigenvector then $P_A(\tilde{v}') = (\tilde{v}')$ contradicting Lemma 9, so there are no other positive eigenvectors.

We can assume without the loss of generality that $||v|| = 1$. Next we claim that for all $u \in \mathbb{R}^d$ there exist the limit

$\text{(4)} \quad l(u) = \lim_{n \to \infty} \frac{||A^nu||}{\lambda^n}$

and moreover

$\text{(5)} \quad ||A^nu - l(u)\lambda^n v|| \leq \text{Const} \lambda^n \theta^n.$

Indeed assume first that $u \in K$. denote $S(\tilde{u}) = \frac{||Au||}{||u||}$ (this definition is clearly independent of the choice of the vector projecting to $\tilde{u}$). We have

$$||A^nu|| = ||u|| \prod_{j=0}^{n-1} \frac{||A^{j+1}u||}{||A^ju||} = ||u|| \prod_{j=0}^{n-1} S(P_A^j(u)) = (||u||\lambda^n) \exp \left[ \sum_{j=0}^{n-1} \left( \ln S(P_A^j(u)) - \ln S(v) \right) \right].$$

Since $S$ is Lipshitz we have

$\text{(6)} \quad |\ln S(P_A^j(u)) - \ln S(v)| \leq \text{Const} \theta^j$

which proves (4) with $l(u) = ||u|| \prod_{j=0}^{\infty}$. Moreover the exponential convergence of (6) implies $||A^nu|| = l(u)\lambda^n (1 + O(\theta^n))$. Now by Lemma 9 we have

$$\frac{A^nu}{||A^nu||} = v + O(\theta^n)$$

This proofs (5) for positive vectors. Now (5) in general case follows by (K2). From the properties of limit it follows that $l$ is a linear functional. Let $L = \text{Ker}(l)$. Then $L$ is $d - 1$ dimensional hypersurface and since $l(A(u)) = \lambda l(u)$, $L$ is $A$-invariant. Thus all other eigenvectors lie in $L$. It follows from (5) that

$$||A^n|_L|| \leq \text{Const} \lambda^n \theta^n$$

so all eigenvalues are less than $\lambda \theta$ in absolute value.
Proof of Lemma 9. For any $u$ in $K$ we have
\[ \min_{ij} A_{ij} \max_j u_j \leq (Au)_i \leq d \max_{ij} A_{ij} \max_j u_j \]
so $A(u) \in K_L$ with $L = d \max_{ij} A_{ij}$. Now $K_L$ is compact in both dist and $d_K$ metrics so it is enough to establish (3) for $d_K$.

Lemma 10. Given $D$ there exists $\theta < 1$ such that for and $d$ for any linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, such that $A(K) \subset K$ and $P_A(\tilde{K})$ has diameter less than $D$ in $\tilde{K}$ $d(P_A\tilde{v}_1, P_A\tilde{v}_2) \leq \theta d(\tilde{v}_1, \tilde{v}_2)$.

Proof of Lemma 10. Since the definition of $d_K(\tilde{v}_1, \tilde{v}_2)$ deepends only on the section of $K$ by the plane through $v_1$ and $v_2$ it is enough to establish the result for linear map from plane to plane. Now on the plane we can use $y/x$ as a coordinate for the point $(x, y)$. In this case $\tilde{K} = [0, \infty]$ and $P_A$ is a fractional linear transformation. Also for $z_1 < z_2$

\[ d_K(z_1, z_2) = \ln \left( \frac{z_2 - 0}{z_1 - 0} \lim_{z \to \infty} \frac{z_1 - z}{z_2 - z} \right) = \ln \frac{z_2}{z_1} = \int_{z_1}^{z_2} \frac{dz}{z}. \]

So we have to prove that any fractional linear transformation $s$ from $[0, \infty]$ to itself such that $s(\infty)/s(0) < e^D$ contracts distance (7) by a factor which depends only on $D$. Since dilations preserve (7) we can assume that $s(\infty) = 1$ thus we have

\[ w := s(z) = \frac{z + a}{z + b} \]

with $a/b > e^{-K}$. We have

\[ \frac{dw}{dz} = \frac{b - a}{(z + b)^2} \]

so that

\[ \frac{dw}{w} = \frac{z(b - a)}{(z + a)(z + b)} \frac{dz}{z} \]

But

\[ \frac{z(b - a)}{(z + a)(z + b)} \leq \frac{b - a}{b} < 1 - e^{-D}. \]

It follows that

\[ d_K(s(z_1), s(z_2)) = \int_{w_1}^{w_2} \frac{dw}{w} = \int_{z_1}^{z_2} \frac{z dw}{w dz} \]

\[ < (1 - e^{-D}) \int_{z_1}^{z_2} \frac{dz}{z} < (1 - e^{-D})d(z_1, z_2). \]

(3) follows from Lemma 10 and contraction mapping principle.
Theorem 3. Let $A$ be the matrix with non-negative entries such that for all $i, j$ there exists $n$ such that $A^n_{ij} > 0$. Then there exist lambda > 0 and $c$ such that
(a) $\lambda e^{2\pi ir/c}$ are eigenvalues of $A$.
(b) $A^c$ has $c$ linearly independent positive eigenvectors with eigenvalues $\lambda^c$.
(c) All other eigenvalues have absolute value less than $\lambda$.
(d) There are no other positive eigenvectors.

Proof. Let $Z_{ij} = \{n : A^n_{ij} > 0\}$. Observe that $Z_{ij} + Z_{jk} \subset Z_{ik}$ in particular $Z_{ii}$ are semigroups.

Lemma 11. If $a, b \in Z_{ii}$ let $q = \gcd(a, b)$. Then for large $N$ $Nq \in Z_{ii}$.

Proof. We have $q = ma - nb$ for some $m, n \in \mathbb{N}$. Write $N = Lb + k$ then $Nq = (Lq - n)b + mk$. □

Corollary 12. Let $c_i$ be the greatest common divisor of all numbers in $Z_{ii}$. Then $Z_{ii} = c_i\mathbb{N}$ — finitely many numbers.

Proof. Let $c_i^{(1)}$ be any number in $Z_{ii}$. Then $c_i^{(1)}\mathbb{N}$ is in $Z_{ii}$ and if there are no other numbers in $Z_{ii}$ then we are done. Otherwise if $b_i^{(1)} \in Z_{ii} - c_i^{(1)}\mathbb{N}$ then let $c_i^{(2)} = \gcd(c_i^{(1)}, b_i^{(1)})$. Then $c_i^{(2)}\mathbb{N}$ - a finite set is in $Z_{ii}$ and if there are no other numbers in $Z_{ii}$ then we are done. Otherwise if $b_i^{(2)} \in Z_{ii} - c_i^{(2)}\mathbb{N}$ then let $c_i^{(3)} = \gcd(c_i^{(2)}, b_i^{(2)})$ etc. Since $c_i^{(k)}$ is decreasing it must stabilize. □

Lemma 13. (a) Any two numbers in $Z_{ij}$ are comparable mod $c_i$.
(b) Any two numbers in $Z_{ji}$ are comparable mod $c_i$.
(c) $c_i = c$ do not depend on $i$.

Proof. (a) Let $n', n'' \in Z_{ij}, n \in Z_{ji}, m \in Z_{jj}$ then $n + n' = n + n'' = n + n' + m$ mod $c_i$. This prove (a) and (c). (b) is similar to (a). □

Let $V_m = \{i : Z_{ii} = m \mod c\}, L_m = \text{span}(e_i, i \in V_m)$. Then There exists $N$ such that $A_{ij}^N > 0$ if and only if $i$ and $j$ belong to the same $V_m$. Thus by Theorem 2 $A^N$ has unique eigenvector $v_1$ on $L_1$ with positive eigenvalue $\nu$. Let $v_i = A^{i-1}v_1$. Since $A$ commutes with $A^N$ $v_i$ are eigenvectors of $A^N$ and since $A$ has non-negative entries $v_i$ are non-negative. Also $v_i \in L_i \mod c$. In particular by uniqueness of the positive eigenvalue $A^N v_1 = \nu v_1$. Let $\lambda = \nu^{1/c}$. Then $A^c(v_i) = \lambda^c(v_i)$.

Let $w_r = \sum_{j=1}^{c} e^{2\pi ijr/c} v_j$. Then $Aw_r = e^{2\pi ijr/c} w_r$. Finally by Theorem 2 all other eigenvalues of $A^N$ are less than $\nu = \lambda^N$ in absolute value. Theorem 3 is proven. □
4.10 Weak Mixing.

Let $U$ be a unitary operator and $\phi$ be a unit vector. Let
\[ R_n = \langle U^n \phi, \phi \rangle. \]

Lemma 14. There is a probability measure $\sigma_\phi$ on $[0,1]$ such that
\[ R_n = \int e^{2\pi i u} d\sigma(u). \]

Proof. For $0 < \rho < 1$ let
\[ f(u, \rho) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} R_{n-m} e^{2\pi i (m-n)} \rho^{n+m}. \]
Since $R_{n-m} = \langle U^n \phi, U^m \phi \rangle$ it follows that
\[ f(u, \rho) = || \sum_n \rho^n e^{-2\pi i u} U^n \phi ||^2 \]
is a real positive number. Estimating terms in () by their absolute values we get $|f| \leq (1 - \rho)^{-2}$. Thus $f \in L^\infty(du) \subset L^2(du)$. Let us examine its Fourier series. We have
\[ f = \sum_k R_k e^{-2\pi i u} \sum_m \rho^{k+2m} = \sum_k e^{-2\pi k} R_k \rho^k (1 - \rho^2)^{-1}. \]
Consider measures
\[ d\sigma_\rho = (1 - \rho^2) f(u, \rho) du. \]
We have
\[ \int e^{2\pi i u} d\sigma_\rho(u) = R_k \rho^k. \]
Hence as $\rho \to 1$
\[ \int e^{2\pi i u} d\sigma_\rho(u) \to R_k. \]
Since linear combinatorials of $e^{2\pi i u}$ are dense in $C(S^1)$ it follows that for any continuous function $A$
\[ \int A(u) d\sigma_\rho(u) \to \sigma(A). \]

Exercise 14. Consider a full two shift with Bernoulli measure (that is the measure of each cylinder of size $n$ is $(1/2)^n$) and let
\[ \phi(x) = \sqrt{2} (I_{x_0=1, x_1=1} - I_{x_0=0, x_1=0}). \]
Find the spectral measure of $\phi$ with respect to $U(\phi) = \phi(\sigma x)$. \[ \square \]
**Exercise 15.** Let $\Phi_n(u) = \frac{1}{n^2} \sum_{j,k=1}^n e^{2\pi i(j-k)u}$. Show that $\Phi_n(u) \to 0$, if $u \neq 0$ (and $\Phi_n(0) = 1$).

### 5.1 Expanding endomorphisms.

Let $f : S^1 \to S^1$ be a map such that $|f'| \geq \theta^{-1}$ for some $\theta < 1$. We call such map expanding. Given any diffeomorphism of $S^1$ let $\bar{g} : \mathbb{R} \to \mathbb{R}$ be its lift, that is $\pi \circ \bar{g} = g \circ \pi$, where $\pi : \mathbb{R} \to S^1$ is the natural projection. Define $\deg(g) = \bar{g}(x+1) - \bar{g}(x)$ (this number is easily seen to be independent of $x$ and the lift $\bar{g}$). Let $L$ be the space of maps $\bar{\tau} : \mathbb{R}^1 \to \mathbb{R}^1$ which are lifts of degree 1 maps. That is $\bar{\tau} - x$ is periodic. We endow $L$ with the distance $d(\bar{\tau}_1, \bar{\tau}_2) = \sup_{x \in \mathbb{R}} |\bar{\tau}_1(x) - \bar{\tau}_2(x)| = \max_{x \in [0,1]} |\bar{\tau}_1(x) - \bar{\tau}_2(x)|$.

**Lemma 15.** Let $f$ be an expanding map and $g$ be a map of the same degree. Then given any two lifts $\bar{f}$ and $\bar{g}$ there is unique $\bar{\tau} \in L$ such that $\bar{f} \circ \bar{\tau} = \bar{\tau} \circ \bar{g}$.

**Proof.** $\bar{\tau}$ must satisfy $\bar{\tau}(x) = \bar{f}^{-1}(\bar{\tau}(\bar{g}(x)))$. Define $K : L \to L$ by $K(\bar{\tau})(x) = \bar{f}^{-1}(\bar{\tau}(\bar{g}(x)))$. Since $f$ is expanding it follows from the Intermediate Value Theorem that $|\bar{f}^{-1}(x_1) - \bar{f}^{-1}(x_2)| \leq \theta |x_1 - x_2|$. Hence $d(K(\bar{\tau}_1), K(\bar{\tau}_1)) \leq \theta d(\bar{\tau}_1, \bar{\tau}_2)$. Now the result follows from the contraction mapping principle. □

**Theorem 4.** Any two expanding maps of the same degree are topologically conjugated.

**Proof.** Let $\bar{f}_1$ and $\bar{f}_2$ be lifts of two expanding maps. By Lemma 15 there are maps $\bar{\tau}_1, \bar{\tau}_2$ such that $\bar{\tau}_1 \circ \bar{f}_1 = \bar{f}_2 \circ \bar{\tau}_1$ and $\bar{\tau}_2 \circ \bar{f}_2 = \bar{f}_1 \circ \bar{\tau}_2$. Let $\bar{\tau} = \bar{\tau}_2 \circ \bar{\tau}_1$. Then $\bar{\tau} \circ \bar{f}_1 = \bar{f}_1 \circ \bar{\tau}$. By uniqueness part of Lemma 15 $\bar{\tau}_2 \circ \bar{\tau}_1 = \text{id}$. Likewise $\bar{\tau}_1 \circ \bar{\tau}_2 = \text{id}$. □

**Exercise 16.** Show that this conjugacy is typically NOT $C^1$. 
