Supplementary problems.

(1) Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables each $X_i$ having symmetric distribution, that is
\[ P(\xi \in A) = P(X_i \in -A) \]
for any Borel set $A \subset \mathbb{R}$. Assume that $E(X_i^4) < \infty$ for $i = 1 \ldots n$. Consider the sums $S_n = \sum_{j=1}^n X_j$. Show that
\[
P(\max_{k<n} |S_k| > t) \leq \frac{\mathbb{E}(S_n^4)}{t^4}.
\]

(2) Let $X_1, X_2, \ldots, X_n \ldots$ be independent random variables such that for large $n$
\[
P(X_n = 1) = \frac{1}{n^\alpha}, \quad P\left(X_n = \frac{1}{n^\beta}\right) = \frac{1}{2}, \quad P\left(X_n = -\frac{1}{n^\beta}\right) = \frac{1}{2} - \frac{1}{n^\alpha}.
\]
For which $\alpha$ and $\beta$ does $\sum_{n=1}^\infty X_n$ converge?

(3) Let $X_1, X_2, \ldots, X_n \ldots$ be iid having Gaussian distribution with zero mean and variance $\sigma^2$. Let $S_n = \sum_{j=1}^n X_j$. Compute
\[
I(a) = \lim_{n \to \infty} \frac{\ln P(S_n \geq an)}{n}.
\]

(4) Let $X_1, X_2, \ldots, X_n \ldots$ be iid such that $M_X(t) = \mathbb{E}(e^{tX}) < \infty$ for all $t \in \mathbb{R}$. Set $S_n = \sum_{j=1}^n X_j$,
\[
I(a) = \lim_{n \to \infty} \frac{\ln P(S_n \geq an)}{n}.
\]
Show that $I(a) \to -\infty$ as $a \to +\infty$.

(5) Let $X$ be a bounded random variable. Define
\[
\text{ess sup}(X) = \{\sup x : P(X > x) > 0\}, \quad \text{ess inf}(X) = \{\inf x : P(X < x) > 0\}.
\]
Consider the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$. Show that
\[
\lim_{t \to \infty} \frac{\ln M_X(t)}{t} = \text{ess sup}(X), \quad \lim_{t \to -\infty} \frac{\ln M_X(t)}{t} = \text{ess inf}(X).
\]

(6) Let $(X_1, X_2, X_3, X_4)$ be a Gaussian random vector with zero mean and covariance matrix
\[
\begin{pmatrix}
5 & 1 & 2 & 2 \\
1 & 10 & 1 & 2 \\
2 & 1 & 8 & 2 \\
2 & 2 & 2 & 10
\end{pmatrix}.
\]
Compute $P(X_1, X_2|X_3, X_4)$.

(7) A fair coin is tossed 12 times. Let $M$ be the number of heads during the first 6 tosses and $N$ be the total number of heads. Compute $P(M|N = 8)$. 