ERRATIC BEHAVIOR FOR 1-DIMENSIONAL RANDOM WALKS IN GENERIC QUASI-PERIODIC ENVIRONMENT

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Abstract. We show that one-dimensional random walks in a quasi-periodic environment with Liouville frequency generically have an erratic statistical behavior. In the recurrent case we show that neither quenched nor annealed limit theorems hold and both drift and variance exhibit wild oscillations, being logarithmic at some times and almost linear at other times. In the transient case we show that the annealed Central Limit Theorem fails generically. These results are in stark contrast with the Diophantine case where the Central Limit Theorem with linear drift and variance was established by Sinai.

1. Introduction

1.1. **Quasiperiodic random walks.** Denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $C^\infty(\mathbb{T}, (0,1))$ be the set of smooth functions from $\mathbb{R}$ to $(0,1)$ that are 1-periodic. Let $\mathfrak{p} \in C^\infty(\mathbb{T}, (0,1))$ and $\mathfrak{q}(x) = 1 - \mathfrak{p}(x)$. Given $\alpha \in \mathbb{T}$, consider the Markov process X_n introduced in [25]:

$$\mathbb{P}(X_{n+1} = x + \alpha | X_n = x) = \mathfrak{p}(x), \quad \mathbb{P}(X_{n+1} = x - \alpha | X_n = x) = \mathfrak{q}(x). \tag{1.1}$$

Associated to X_n one defines a random walk on $\mathbb{Z}$ given by Z_n such that

$$X_n = X_0 + Z_n \alpha.$$

We also consider the walk on the circle given by $\bar{X}_n = X_n \bmod 1$. When $\alpha \notin \mathbb{Q}$, we call Z_n a *one-dimensional random walk in quasi-periodic environment*. Following [25], we call the walk *symmetric* if

$$\int_{\mathbb{T}} \ln \mathfrak{p}(x) dx = \int_{\mathbb{T}} \ln \mathfrak{q}(x) dx \tag{1.2}$$

and call it *asymmetric* otherwise. Let $\mathcal{P} \subset C^\infty(\mathbb{T}, (0,1))$ be the set of functions satisfying the symmetry condition (1.2).

Recall that $\alpha \in \mathbb{R}$ is said to be *Diophantine* if there exists $\gamma > 0$ and $\tau \geq 0$ such that for any $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{\gamma}{q^{2+\tau}}.$$

We then say that $\alpha \in \mathrm{DC}(\gamma, \tau)$. An irrational real number that is not Diophantine is called *Liouville*.

An elementary but noticeable fact of number theory is that Liouville numbers form a dense G^δ set of $\mathbb{R}$, while the Lebesgue measure of this set is zero.

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One-dimensional random walks in a Diophantine quasi-periodic environment are well understood. In fact, Diophantine walks behave similarly to simple random walks. Namely they have linear (possibly null) drift and satisfy the Central Limit Theorem. The precise statements for Diophantine walks will be recalled in §2.2. By contrast nothing was known for Liouville environment beyond the results which hold in any uniquely ergodic environments. In this paper we study the Liouville case. We show that for any given Liouville α the walk driven by that α and a generic $\mathfrak{p} \in C^\infty(\mathbb{T}, (0,1))$ has a very erratic statistical behavior. By *generic* we mean of first category for the C^∞ topology. Since a generic irrational α is Liouville, our results imply the erratic behavior for one-dimensional random walks in a generic quasi-periodic environment.

Our main results can be summarized as follows. For symmetric random walks we show that the following behavior is generic

• The spread of the walk (as measured, for example, by standard deviation) oscillates wildly. Sometimes the walk is localized at a logarithmic scale while at other times the variance grows faster than $n^{1-\varepsilon}$.

• The drift of the walk oscillates wildly: sometimes it is larger than $n^{1-\varepsilon}$, sometimes it is smaller than $-n^{1-\varepsilon}$, sometimes it is of order 1.

• The walk does satisfies neither annealed nor quenched limit theorem: the set of limit distributions includes the normal distribution as well as a distribution with atoms.

We will also show that

• A one-dimensional random walk in a generic asymmetric quasi-periodic environment does not have an annealed limit law.

The precise statements of the results outlined above are contained in §2.4.

Plan of the paper and outline of the proofs. §2.4 contains the precise statements about Liouvillian walks. It turns out that their behavior is quite different from the Diophantine walks, but is, in fact, quite similar to the walks in generic environments. By a generic environment we mean a dense G_δ set in the space of all elliptic nearest neighbor Markov chains on $\mathbb{Z}$ endowed with a product topology. The corresponding statements about generic walks are presented in §2.3. Since we could not find these results in the literature, we provide the proofs in Section 4. Before stating our results, we recall for comparison some known results about two most studied classes of environments. Independent environments are discussed in §2.1, and quasi-periodic Diophantine environments are discussed in §2.2.

Section 3 contains the necessary preliminaries. We introduce the crucial martingale (3.6) that is instrumental in the study of one dimensional random walks in a fixed environment given by the probabilities $\mathrm{p}_j, j \in \mathbb{Z}$ and $\mathrm{q}_j = 1 - \mathrm{p}_j$. The important quantities that are involved in this martingale, and that determine the behavior of the walk, are the sums

$$\Sigma(n) = \begin{cases} \sum_{j=1}^{n} \ln \mathrm{q}_j - \ln \mathrm{p}_j & n \geq 1, \\ 0, & n = 0, \\ \sum_{j=n+1}^{0} \ln \mathrm{p}_j - \ln \mathrm{q}_j & n \leq -1. \end{cases} \tag{1.3}$$

The function $n \mapsto \Sigma(n)$ is known as the *potential*. A direct inspection shows that if $\mathrm{p}_j > \mathrm{q}_j$ for all j in some interval $I \subset \mathbb{Z}$ then Σ is decreasing on I, while if $\mathrm{q}_j > \mathrm{p}_j$ for all $j \in I$ then Σ is increasing on I. Thus, the guiding intuition is that the walker tends to go downwards on the graph of Σ and spends a lot of time near local minima of the potential. The study of the potential plays a crucial role in the study of random walks on $\mathbb{Z}$ starting with the pioneering work of Sinai [24] and it is also important in the present paper.

In the generic environment we are allowed to perturb the environment outside of a finite set so we can directly prescribe the values of the potential to enforce a desirable behavior of the walk.

In the case of quasi-periodic walks defined by some $\mathfrak{p} \in C^\infty(\mathbb{T}, (0,1))$ as in (1.1), both the transition probabilities $\mathrm{p}_{j,x} = \mathfrak{p}(x + j\alpha)$ and the sums $\Sigma_x(n)$ defined by (1.3) are dependent on $x \in \mathbb{T}$.

If the walk is symmetric (see (1.2)) and α is Diophantine, the fact that $\ln(1-\mathfrak{p}) - \ln \mathfrak{p}$ is a coboundary above R_α implies that the sums $\Sigma_x(n)$ are bounded, which renders the Diophantine walk very similar to the simple symmetric random walk (we will come back to this in §2.2 and we refer the reader to [6, 8] for a detailed discussion of random walks in bounded potentials).

A contrario, obtaining various specific behaviors for the sums $\Sigma_x(n)$ of a generic function $\mathfrak{p} \in C^\infty(\mathbb{T}, (0,1))$ when α is Liouville, underlies all our findings. Displaying very different behaviors of $\Sigma_x(n)$ at different time scales n and different initial conditions x is the key behind the erratic behavior of the Liouville walks.

To keep the exposition as clear as possible, we split the analysis of Liouville walks into two separate parts. In the first part (Sections 5–7) we deal with a fixed environment and describe several criteria based on the behavior of the potential, that imply various types of behaviors for the random walk. These are inspired by the proofs of the erratic behavior of generic walks, but are necessarily more sophisticated and tailored in a way that makes it convenient to verify their validity for a generic Liouville walk.

In Section 5, we formulate criteria for localization, one-sided drift and two-sided drift for random walks in a fixed environment. The proofs are given in Section 7 and they rely on auxillary estimates on exit times for random walks in a fixed environment presented in Section 6.

The second part of the paprer (Sections 8–9) deals with quasi-periodic walks. In Section 8 we prove Theorem 8.1, stating that when α is Liouville, then for a residual set of symmetric environments, the criteria for localization, one-sided drift and two-sided drift are satisfied for almost every $x \in \mathbb{T}$. In fact, as we mentioned above, the criteria ask for particular behaviors of the sums $\Sigma_x(n)$ at different time scales n and different initial conditions x. By definition of the criteria, it will be easy to show that the set of $\mathfrak{p} \in \mathcal{P}$ for which these criteria are satisfied contains a countable intersection of countable unions of open sets. These open sets, are subsets of $\mathfrak{p} \in \mathcal{P}$ for which a criterion on $\Sigma_x(n)$ holds for some n and some (not too small) intervals of initial conditions x.

To prove the theorem, we just need to show that the union of these open sets is dense. For this we start by perturbing any given $\mathfrak{p} \in \mathcal{P}$ into a coboundary $\bar{\mathfrak{p}}$ above

R_α. Then, the main construction is to show that any coboundary $\bar{\mathfrak{p}}$ can be perturbed to $\bar{\mathfrak{p}} + e_n(\cdot) \in \mathcal{P}$ that satisfies each of the above mentioned criteria at different scales. This is stated in the main Proposition 8.3. §8.1 contains the reduction of Theorem 8.1 to Proposition 8.3 while the rest of Section 8 is devoted to the proof of Proposition 8.3. The proof proceeds by a Liouville construction in which we obtain e_n and prove the required properties of the ergodic sums as in (1.3) for the function $\bar{\mathfrak{p}} + e_n$.

In Section 9, we obtain the proofs of all the statements about Liouville walks, including the asymmetric walks that are treated in §9.4. We note that the proofs in §9.4 do not use the results of Sections 5–7 so the reader who is only interested in the asymmetric walks could skip those sections.

2. Results

In this section we present our main results about Liouville walks and compare them with other classes of random walks.

2.1. **Random walks in independent environments.** Quasi-periodic random walks are examples of random walks in random environment where the walker moves to the right with probability p_n and to the right with probability $\mathrm{q}_n = 1 - \mathrm{p}_n$ and $\mathrm{p}_n = \mathfrak{p}(\mathcal{T}^n\omega)$ where $\mathcal{T}$ is a map of a space Ω and $\mathfrak{p} : \Omega \to \mathbb{R}$ is a measurable function. It is usually assumed that $\mathcal{T}$ preserves a probability measure μ. The case where ω is distributed according to μ is called *annealed*, and the case where ω is fixed and we wish to obtain the results for μ almost every ω is called *quenched*. The most studied system in this class are iid environments where $\{\mathrm{p}_n\}$ are independent for different n. Since there is a vast literature on this subject, we will just briefly discuss the iid walks here, referring the readers to [27, Part I] for more information. Let $\Delta = \mathbb{E}(\ln \mathrm{p}_n - \ln \mathrm{q}_n)$. We call the walk *symmetric* if $\Delta = 0$ and *asymmetric* otherwise. Denote by Z_n the position of the walker at time n. We assume that the walk starts at the origin. According to [26], the walk is recurrent iff it is symmetric. Moreover, if $\Delta > 0$, then $Z_n \to +\infty$ with probability 1, and if $\Delta < 0$ then $Z_n \to -\infty$ with probability 1. Surprisingly, in the transient case the walk can escape to infinity with zero speed. To fix our notation, let us suppose that $\Delta > 0$. Let s be the unique positive solution of $\mathbb{E}((\mathrm{q}/\mathrm{p})^s) = 1$ if the solution exists and $s = \infty$ otherwise ($s = \infty$ iff $\mathrm{p}_n \geq \mathrm{q}_n$ with probability 1). It is shown in [26] that $\mathbf{v} := \lim\limits_{n\to\infty} \dfrac{Z_n}{n}$ always exists, and $\mathbf{v} > 0$ iff $s > 1$.

Thus, depending on the parameter s, the behavior of the random walk in the independent asymmetric environment can be very different from that of a simple random walk. The difference is even more startling in the symmetric case where it was shown by Sinai ([24]) that $\frac{Z_n}{\ln^2 n}$ converges to a non-trivial limit (the density of the limit distribution is obtained in [17]). The quenched distribution has even stronger localization properties, namely, most of the time the walker is localized on the scale $\mathcal{O}(1)$ [12]. More precisely, given $\varepsilon > 0$, we can find an integer $N(\varepsilon)$ such that for each $n \in \mathbb{N}$, for a set of environments ω of measure more than $1 - \varepsilon$, there is a subset $\mathfrak{T}_n(\omega) \subset \mathbb{Z}$ of cardinality N such that $\mathbb{P}_\omega(Z_n \in \mathfrak{T}_n(\omega)) > 1 - \varepsilon$. This strong localization could be used to show that the symmetric walk does not satisfies a quenched limit theorem. The fact that

the fluctuations of the walk in both annealed and quenched case are subpolynomial is referred to as *Sinai-Golosov localization*.

In the asymmetric case it was shown in [18] that $\frac{Z_n - n\mathbf{v}}{n^\sigma}$ has a non-trivial annealed limit where

$$\sigma = \begin{cases} s, & \text{if } s < 1, \\ \frac{1}{s} & \text{if } s \in (1,2), \\ \frac{1}{2} & \text{if } s > 2. \end{cases} \tag{2.1}$$

(the statements for $s = 1$ or 2 are similar, but additional logarithmic corrections are needed). The limit distribution is Gaussian if $s \geq 2$, stable if $1 < s < 2$ and Mittag Leffler if $s \leq 1$. The above results show that in the asymmetric case we have: $Z_n = n^{\eta + o(1)}$ where

$$\eta = \min(1, s), \tag{2.2}$$

and the fluctuations are of order $n^{\sigma + o(1)}$, where σ is given by (2.1). The quenched limit theorem holds if $s > 2$ ([1, 11, 19]), while there is no quenched limit if $s \leq 2$ ([19, 22], see also [5, 10, 21] for more information on the quenched case).

To understand different behaviors in different regimes in the asymmetric case, one needs a notion of a *trap*. Informally, a trap is a short segment I where the drift is pointing to the left (the opposite of where the walker is going) for most of the sites. We can split the time it takes the walker to reach a given level L into two parts: time spent inside the traps and time spent outside the traps. The time spent spent outside the traps has normal behavior: its mean is linear in L and the fluctuations are of order $\sqrt{L}$. On the other hand, the time spent in the traps scales as $L^{1/s}$. Depending on the value of s, the main contribution to either the drift or the fluctuations around the leading behavior may come from either inside or outside the traps, which explains the transitions described in (2.1), (2.2). In order to make the above heuristic arguments precise, one needs a precise mathematical definition of traps. The most convenient way to do this ([24]) is in terms of the *potential*, defined in (1.3). Namely, a segment I is a trap if the minimal value of the potential inside I is much smaller than both boundary values. The creation of traps is our main tool for proving the localization of the walker in the Liouvillian case.

2.2. **The Diophantine case.** As we saw in §2.1, random walks in an independent random environment may behave quite differently from simple random walks. In this section we review the known results about quasiperiodic Diophantine walks. These results show that Diophantine walks are very similar to the simple random walk.

We shall use $\mathbb{P}_x$ for the distribution of the paths of the above processes when $X_0 = x$ and $\mathbb{P}_{\text{Leb}}$ for the case when X_0 is uniformly distributed on $\mathbb{T}$.

The following results about the walks $\bar{X}_n$ and Z_n are known.

Theorem 2.1. ([25, Theorem 1]) **(Invariant measure)** *If any of the following two conditions holds:*

(1) the walk is asymmetric,

(2) α is Diophantine,

then there exists a unique stationary measure ν for the process $\bar{X}_n$ and this measure is absolutely continuous with respect to Lebesgue measure. Moreover, for each $\phi \in C^0(\mathbb{T}, \mathbb{R})$ and for any $x \in \mathbb{T}$

$$\mathbb{E}_x(\phi(\bar{X}_n)) \to \nu(\phi).$$

Under the same conditions of Theorem 2.1, the walk satisfies the Central Limit Theorem, as shown by the following two statements. Denote

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

Theorem 2.2. ([1], [25, Equations (15) and (20)]) **(Annealed CLT)** *If α is Diophantine then there exist $\mathbf{v} \in \mathbb{R}$ and $\sigma > 0$ such that for all x*

$$\lim_{n\to\infty} \mathbb{P}_x(Z_n - n\mathbf{v} < \sigma\sqrt{n}z) = \Phi(z). \tag{2.3}$$

Therefore

$$\lim_{n\to\infty} \mathbb{P}_{\mathrm{Leb}}(Z_n - n\mathbf{v} < \sigma\sqrt{n}z) = \Phi(z). \tag{2.4}$$

Moreover $\mathbf{v} = 0$ iff the walk is symmetric.

Note that Theorem 2.2 shows that the Diophantine walks behave similarly to simple random walks independently of the starting point on the circle.

An asymmetric walk also behaves like a simple random walk, but with a drift that depends on the starting point:

Theorem 2.3. ([6, Theorem B.2]) **(Quenched CLT)** *If the walk is asymmetric then there are functions $b_n(x)$ and a number $\sigma > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}_x(Z_n - b_n(x) < \sigma\sqrt{n}z) = \Phi(z). \tag{2.5}$$

Note that Theorem 2.2 shows that in the Diophantine case one can take $b_n(x) = n\mathbf{v}$. The general formula for b_n will be given in §9.4 (see equation (9.1)).

The results of Theorems 2.1–2.3 have been extended to random walks driven by rotations of $\mathbb{T}^d$, for arbitrary $d \in \mathbb{N}$, to random walks with bounded jumps where the walker can move from x to $x + j\alpha$ with $|j| \le L$ for some $L > 1$ and to quasi-periodic walks on the strip, see [3, 6, 7, 8].

2.3. **Generic deterministic environments.** The results of §2.2 raise a natural question if the same statements hold in the Liouvillian case. It turns out that behavior of Liouvillian walks is quite different. In particular, Liouvillian walks share several features of the walks in independent environments: the localization in the symmetric case (but along a subsequence) and the lack of quenched limit theorems. However, the Liouvillian walks are more erratic than the walks in independent environments, in fact, they are closer to generic environments described below.

Namely, fix a small $\varepsilon_0 \le 0.1$ and let

$$\mathcal{E} = \{\mathrm{p} : \mathbb{Z} \to [\varepsilon_0, 1 - \varepsilon_0]\}.$$

We endow $\mathcal{E}$ with the product topology generated by the sets of the form

$$U_{\varepsilon,K}(\bar{\mathrm{p}}) = \{\mathrm{p} : |\mathrm{p}(n) - \bar{\mathrm{p}}(n)| < \varepsilon \text{ for } |n| \le K\}.$$

Recall that a subset of $\mathcal{E}$ is called *generic* if it contains a countable intersection of open dense sets.

Theorem 2.4. *The following properties are generic*

(a) **(Recurrence)** *The walk is recurrent.*

Moreover, there exist strictly increasing sequences r_k, s_k, t_k such that

(b) **(Localization)** *For $T = r_k$*

$$\mathbb{P}\left(|Z_T| > (\ln T)^2\right) < T^{-1/2} \text{ and } \operatorname{Var}(Z_T) < 2(\ln T)^4. \tag{2.6}$$

(c) **(One-sided drift)** *For $T = s_k$ and some μ_k, σ_k such that*

$$\lim_{k\to\infty} \frac{\mu_k}{s_k} > 0, \quad \lim_{k\to\infty} \frac{\sigma_k}{\sqrt{s_k}} > 0$$

we have that for all z

$$\mathbb{P}\left(\frac{Z_T - \mu_k}{\sigma_k} < z\right) = \Phi(z). \tag{2.7}$$

(d) **(Two-sided drift)** *For $T = t_k$ there exists b_k such that $\lim_{k\to\infty} \frac{b_k}{t_k} > 0$, $\lim_{k\to\infty} \varepsilon_k = 0$ and*

$$\mathbb{P}\left(|Z_T - b_k| < \varepsilon_k b_k\right) > 0.1, \quad \mathbb{P}\left(|Z_T + b_k| < \varepsilon_k b_k\right) > 0.1. \tag{2.8}$$

The proof of this theorem is not difficult. However we will give a sketch of the argument in Section 4 since generic environments provide a useful comparison with Liouvillian environments. In particular, similar ideas will be used to prove analogous results for Liouvillian walks later in the paper.

We note several interesting consequences of Theorem 2.4.

We say that the walk satisfies a limit theorem if there exist sequences b_n and σ_n and a proper (that is, not concentrated on a single point) distribution $\mathcal{D}(\cdot)$ such that

$$\lim_{n\to\infty} \mathbb{P}(Z_n - b_n < \sigma_n z) = \mathcal{D}(z).$$

Corollary 2.5. *(a)* $\liminf_{T\to\infty} \frac{\ln|\mathbb{E}(Z_T)|}{\ln T} = 0$, $\limsup_{T\to\infty} \frac{\ln|\mathbb{E}(Z_T)|}{\ln T} = 1$;

$$\liminf_{T\to\infty} \frac{\ln(\operatorname{Var}_x(Z_T))}{\ln T} = 0, \quad \limsup_{T\to\infty} \frac{\ln(\operatorname{Var}_x(Z_T))}{\ln T} = 2.$$

(b) Z_T does **not** *satisfy a limit theorem.*

Proof. (a) Follows from parts (b) and (d) of Theorem 2.4.

To prove part (b) we observe that by Theorem 2.4(b), if a limit theorem holds, the limit should be normal. On the other hand, let t_k be as in Theorem 2.4(d). In this case the distance between $-b_k$ and b_k grows linearly in T. Hence to get a limit distribution, the normalizing factor should be of order T. But then the limit distribution should give a probability larger than 0.1 to two intervals, each one having size at most ε_k. This

implies that any limit point along the sequence t_k has non-trivial atoms, so it cannot be normal, giving a contradiction. □

2.4. **The Liouville case. Main results.** Theorems 2.1, 2.2 and 2.3 naturally raise the question of what would be the behavior of a quasi-periodic walk when the driving frequency α is Liouville. The following statements show that their behavior can indeed be very different from the Diophantine case, and is rather very similar to that of generic walks discussed in §2.3.

Theorem A. *For any Liouville α there exists a dense G_δ set $S \subset \mathcal{P}$ with the following property: for any $\mathfrak{p} \in S$, for almost every $x \in \mathbb{T}$, there are strictly increasing sequences $\{r_k\}$, $\{s_k\}$, $\{t_k\}$ such that for any $\varepsilon > 0$ and for k sufficiently large*

(a) **(Localization)** *For $T = r_k$*

$$\mathbb{P}\left(\max_{t\le T} |Z_T| > 16(\ln T)^2\right) < T^{-2} \text{ and } \operatorname{Var}(Z_T) < 300(\ln T)^4. \tag{2.9}$$

(b) **(One-sided drift)** *For $T = s_k$ and some $\mu_k > s_k^{1-\varepsilon}$,*

$$\left|\mathbb{P}_x\left(\frac{Z_T - \mu_k}{\sigma_k} < z\right) - \Phi(z)\right| < \varepsilon, \quad \left|\frac{\ln \sigma_k}{\ln T} - \frac{1}{2}\right| < \varepsilon \tag{2.10}$$

where $\sigma_k = \sqrt{\operatorname{Var}_x(Z_{s_k})}$.

(c) **(Two-sided drift)** *For $T = t_k$ there exist $b_k, b'_k \in [0.1T^{1/6}, 0.5T^{1/6}]$ and $\varepsilon_k \to 0$ such that*

$$\mathbb{P}_x\left(|Z_T - b_k| < \varepsilon_k T^{1/6}\right) > 0.1, \quad \mathbb{P}_x\left(|Z_T + b'_k| < \varepsilon_k T^{1/6}\right) > 0.1. \tag{2.11}$$

In fact, the proof of Theorem A will provide a valuable additional information which we record below.

Proposition 2.6. *The sequences r_k, s_k, t_k in Theorem A can be chosen in such a way that the inequalities (2.9), (2.10), and (2.11) hold for x in sets $\mathcal{J}_{k,j} \in \mathbb{T}, j = a, b, c$, each of which has measure $1/100$. Moreover, the conclusion of Theorem A(c) holds for x in $\mathcal{J}_{k,c}$, with the same ε_k, and with $b_k(x)$ and $b'_k(x)$ whose oscillations on $J_{k,c}$ are less than $\frac{T^{1/6}}{100}$.*

In addition, we can also find a sequence s'_k as in part (b), but with the one sided drift $\mu'_k < -s'_k$.

As a byproduct of our analysis, we will show that the existence of an absolutely continuous invariant measure is incompatible with the erratic behavior of Theorem A.

Corollary B. *If α is Liouville, then there is a dense G_δ subset of $\mathcal{P}$ such that the corresponding walk has no absolutely continuous invariant measure.*

The proof of Corollary B will be given in §9.2.

As in §2.3, Theorem A implies that the walk does not satisfy any limit theorems. Let us give more precise statements.

We say that the walk satisfies a quenched limit theorem at a point x if there exist sequences $b_n(x)$ and $\sigma_n(x)$ and a proper distribution $\mathcal{D}_x(\cdot)$ such that

$$\lim_{n\to\infty} \mathbb{P}_x(Z_n - b_n(x) < \sigma_n(x)z) = \mathcal{D}_x(z).$$

We say that the walk satisfies an annealed limit theorem if there does exist sequences b_n and σ_n and a proper distribution $\mathcal{D}(\cdot)$ such that

$$\lim_{n\to\infty} \mathbb{P}_{\mathrm{Leb}}(Z_n - b_n < \sigma_n z) = \mathcal{D}(z).$$

Corollary C. *If α is Liouville, then there is a dense G_δ subset of $\mathcal{P}$, such that the walk has no annealed limit theorem. Moreover for x in a set of full measure the walk has no quenched limit theorem at x.*

Proof. The proof is the same as the proof of Corollary 2.5 except that we use Theorem A and Proposition 2.6 instead of Theorem 2.4. We leave the details to the reader. □

Our next result shows that for a generic Liouville frequency, the behavior of the corresponding walk "simulates" that of a Diophantine one (described by Theorem 2.2) for long periods of time.

Theorem D. *There exists a dense G_δ set $\mathcal{D} \subset \mathbb{R}$ such that for any $(\alpha, \mathfrak{p}) \in \mathcal{D} \times \mathcal{P}$ there exist sequences T_n and σ_n such that for any $x \in \mathbb{T}$*

$$\left|\mathbb{P}_x(Z_t < \sigma_n\sqrt{t}z) - \Phi(z)\right| < \frac{1}{n} \quad \textit{for all } \ n,\ T_n \le t \le e^{T_n},$$

where $\Phi(z)$ is as in Theorem 2.2.

Note that when we consider t of the order of T_n we see that σ_n is necessarily much larger than $1/\sqrt{T_n}$ since the distribution of the walk on the unit scale is discrete, and hence, not normal. Therefore, when t is of order e^{T_n}, we see that the variance is almost linear in t.

We now turn to the asymmetric quasi-periodic walks. Recall that by Theorem 2.3 a quenched CLT (2.5) holds with some function $b_n(\cdot)$. Moreover, in the Diophantine case $b_n(x) \equiv n\mathbf{v}$, and the annealed limit theorem (2.4) holds. To show that no annealed limit theorem holds in the Liouville case, it suffices to show that the drift function b_n fluctuates much more than $\sqrt{n}$ when we vary x.

Theorem E. *For any Liouville α there exists a dense G_δ set $\mathcal{S} \subset \mathcal{P}^c$ with the following property: for any $\mathfrak{p} \in \mathcal{S}$ there exists sequences of integers $t_n, s_n \to \infty$, $b_n, b'_n > 0$ and of measurable sets $\mathcal{A}_n, \mathcal{B}_n, \mathcal{A}'_n, \mathcal{B}'_n \subset \mathbb{T}$ with*

- *(a)* $\mu(\mathcal{A}_n) > 0.8$ *and* $\mu(\mathcal{B}_n) > 0.1$.
- *(b) For* $x \in \mathcal{A}_n$, $|b_{t_n}(x) - b_n| < {t_n}^{1/4}$, *and for* $x \in \mathcal{B}_n$, $b_{t_n}(x) > b_n + {t_n}^{0.9}$.
- *(a′)* $\mu(\mathcal{A}'_n) > 0.4$ *and* $\mu(\mathcal{B}'_n) > 0.4$.
- *(b′) For* $x \in \mathcal{A}'_n$, $|b_{s_n}(x) - b'_n| < {s_n}^{1/4}$, *and for* $x \in \mathcal{B}'_n$, $b_{s_n}(x) > b'_n + {s_n}^{0.9}$.

As a consequence we get:

Corollary F. *For any Liouville α there exists a dense G_δ set $S \subset \mathcal{P}^c$ such that the walk has no annealed limit.*

Proof. As in the proof of Corollary C, (a) and (b) imply that if the annealed limit would exist, it would necessarily require that $\sigma_{t_n} > t_n^{0.9}$. But then we get that the limit distribution must give a mass larger than 0.8 to some point on the line. However, (a') and (b') show that the limit distribution must not give mass larger than 0.6 to any point, a contradiction. □

2.5. **Open questions.** We close Section 2 with two open questions about the Liouvillian one dimensional walks.

Comparing Theorem 2.1 with Corollary C leads to the following natural question.

Question 1. *Suppose that the walk is symmetric and α is Liouville. By compactness, the walk has at least one stationary measure. Is the stationary measure unique? Are ergodic stationary measures mixing?*

We also note that the maximal growth exponent for the variance of a generic walk obtained in Corollary 2.5(a) is optimal, since the variance of Z_T is at least $\mathcal{O}(1)$ and at most T^2. However for Liouville walks we can only show that the variance grows along a subsequence at a rate that is not slower than $T^{1-\varepsilon}$, see the discussion after Theorem D. We believe that the optimal result should be of the same order as for the generic walks. Thus we formulate

Conjecture 2. *For generic quasi-periodic symmetric walks for almost every x*

$$\limsup_{T\to\infty} \frac{\ln \mathrm{Var}_x(Z_T)}{\ln T} = 2.$$

3. Preliminaries

3.1. **Invariant measures.** Here we comment on the relation between our findings and the question of the existence of absolutely continuous invariant measures for the quasi-periodic walks. The walk given by the pair $(\mathfrak{p}, \alpha)$ has an absolutely continuous invariant measure with density $\rho(x)$ iff

$$\rho(x) = \mathfrak{q}(x-\alpha)\rho(x-\alpha) + \mathfrak{p}(x+\alpha)\rho(x+\alpha). \tag{3.1}$$

A direct computation shows that the flux

$$f(x) = \mathfrak{p}(x)\rho(x) - \mathfrak{q}(x+\alpha)\rho(x+\alpha)$$

is constant along the orbit of the rotation and by ergodicity $f(x) \equiv f$. Now there are two cases

(I) The walk is not symmetric. We may assume (applying a reflection if necessary) that

$$\int \ln \mathfrak{p}(x)dx > \int \ln \mathfrak{q}(x)dx.$$

In this case we can take (rescaling ρ if necessary) $f = 1$ so that

$$\rho(x) = \frac{1}{\mathfrak{p}(x)} + \frac{\mathfrak{q}(x+\alpha)}{\mathfrak{p}(x)}\rho(x+\alpha).$$

Iterating further we obtain a solution

$$\rho(x) = \frac{1}{\mathfrak{p}(x)} \sum_{k=0}^{\infty} \left(\prod_{j=1}^{k} \frac{\mathfrak{q}(x+j\alpha)}{\mathfrak{p}(x+j\alpha)} \right). \tag{3.2}$$

(II) The walk is symmetric. In this case using recursive analysis similar to case (I) we see that there are no solutions with $f \neq 0$. In case $f = 0$ our equation reduces to

$$\mathfrak{p}(x)\rho(x) = \mathfrak{q}(x+\alpha)\rho(x+\alpha). \tag{3.3}$$

Introducing

$$g(x) = \mathfrak{q}(x)\rho(x),$$

we see that (3.3) reduces to

$$\frac{\mathfrak{q}(x)}{\mathfrak{p}(x)} = \frac{g(x)}{g(x+\alpha)}. \tag{3.4}$$

We denote by $\mathcal{B}_\alpha \subset \mathcal{P}$ the set of functions $\mathfrak{p}(x)$ such that (3.4) has a smooth solution g. Such $\mathfrak{p}$ is called *(multiplicative) coboundary above* α, and g its corresponding *transfer function.*

We summarize the foregoing discussion as follows.

Proposition 3.1. *(*[4, Theorem 3.1], [2, Theorem 1.8]*). The Markov chain* (1.1) *defined by* $(\alpha, \mathfrak{p})$ *has an invariant measure which is absolutely continuous with respect to the Lebesgue measure iff either the walk is asymmetric or it is symmetric and* $\mathfrak{p}$ *is a coboundary above* α*.*

For every $x \in \mathbb{T}$, denote $\Sigma_x(0) = 0$,

$$\Sigma_x(n) = \begin{cases} \sum_{j=1}^{n} \ln \mathfrak{q}(x+j\alpha) - \ln \mathfrak{p}(x+j\alpha), & n \geq 1, \\ \sum_{j=n+1}^{0} \ln \mathfrak{p}(x+j\alpha) - \ln \mathfrak{q}(x+j\alpha), & n \leq -1. \end{cases}$$

Notice that if $\mathfrak{p} \in \mathcal{B}_\alpha$, then $\Sigma(x,n)$ has an easy expression

$$\Sigma_x(n) = \ln g(x+(n+1)\alpha) - \ln g(x+\alpha),$$

where g is as in (3.4). This behavior of $\Sigma_x(n)$, as we will see in §3.2, renders the walk very similar to the simple random walk.

In the symmetric case of Theorem 2.2, a crucial observation is that for α Diophantine, any smooth $\mathfrak{p} \in \mathcal{P}$ is a coboundary above α.

A contrario, obtaining various specific behaviors for the sums $\Sigma_x(n)$ of a generic function $\mathfrak{p} \in \mathcal{P}$ when α is Liouville underlies all our findings. Displaying very different behaviors of $\Sigma_x(n)$ at different time scales n and different initial condition x is the key behind the phenomena described in Theorems A, D, and E and Corollaries B, C and F.

3.2. **Birkhoff sums and a martingale.** For $\mathrm{p} \in \mathcal{E}$, let $\mathrm{q}(\cdot) = 1 - \mathrm{p}(\cdot)$. Denote $\Sigma(0) = 0$ and

$$\Sigma(n) = \begin{cases} \sum_{j=1}^{n} \ln \mathrm{q}(j) - \ln \mathrm{p}(j), & n \geq 1, \\ \sum_{j=n+1}^{0} \ln \mathrm{p}(j) - \ln \mathrm{q}(j), & n \leq -1. \end{cases} \tag{3.5}$$

For $j < k$, we use the notation

$$\Sigma(j,k) := \Sigma(k) - \Sigma(j).$$

Denote $M(0) = 0$, $M(1) = 1$,

$$M(n) = \begin{cases} 1 + \sum_{k=1}^{n-1} \prod_{j=1}^{k} \frac{\mathrm{q}(j)}{\mathrm{p}(j)}, & n \geq 2, \\ -\sum_{k=n+1}^{0} \prod_{j=k}^{0} \frac{\mathrm{p}(j)}{\mathrm{q}(j)}, & n \leq -1. \end{cases} \tag{3.6}$$

Notice that

$$M(n) = \begin{cases} \sum_{j=0}^{n-1} e^{\Sigma(j)}, & n \geq 1, \\ -\sum_{j=n}^{-1} e^{\Sigma(j)}, & n \leq -1. \end{cases}$$

The optional stopping theorem for the martingales implies that under $\mathbb{P}$, if Z starts from a position n where $a < n < b$, and τ_0 is the first time the walk reaches either a or b, then $M(Z_{\min|t,\tau_0|})$ is a martingale. In particular for any $a < n < b$

$$\mathbb{P}(Z_t \text{ reaches } b \text{ before } a | Z_0 = n) = \frac{M(n) - M(a)}{M(b) - M(a)}. \tag{3.7}$$

(see e.g. [9, Theorem 6.4.6]).

This formula provides a relation between the sums $\Sigma(j,k)$ and the behavior of the walk.

We note that (3.7) also holds if $a = -\infty$ or $b = +\infty$ (see e.g. [23, §VII.3]). In particular,

$$Z_t \text{ is recurrent } \Leftrightarrow \lim_{n \to -\infty} M(n) = -\infty \text{ and } \lim_{n \to +\infty} M(n) = +\infty, \tag{3.8}$$

and

$$\mathbb{P}(Z_t \to +\infty) = \frac{M_-}{M_- + M_+} \quad \text{where} \quad M_\pm = \lim_{n \to \pm\infty} |M(n)|. \tag{3.9}$$

4. Erratic behavior in a generic deterministic environment.

Here we deal with generic environments $\mathrm{p} \in \mathcal{E}$ and prove Theorem 2.4.

Proof. (a) By (3.8), the recurrence holds iff $M(n) \to \pm\infty$ as $n \to \pm\infty$. The result follows since for each R the condition that there is $n \in \mathbb{N}$ such that $M(n) > R$ and $M(-n) < -R$ is open and dense. Indeed, to obtain the density it is enough to modify any given p to $\tilde{\mathrm{p}}$ satisfying

$$\tilde{\mathrm{p}}(n) = \begin{cases} \frac{1}{3} & \text{for } n > K \\ \frac{2}{3} & \text{for } n < -K. \end{cases} \tag{4.1}$$

To prove (b), we also consider the environment given by (4.1). Note that for this environment there are constants $C_1 = C_1(K)$, and $C_2 = C_2(K)$ such that

$$|M(n)| \geq C_1 e^{C_2|n|}.$$

Thus, for each T and r

$$\mathbb{P}(|Z_T| \geq r) \leq \frac{1}{C_1 e^{C_2 r}}.$$

It follows that for large T, (2.6) is satisfied, showing the density of this condition. The openness is also clear.

To prove (c), it is sufficient to show that for each ε the set of environments such that for some T

$$\sup_z \left| \mathbb{P}\left(\frac{Z_T - \frac{T}{3}}{\sqrt{\frac{8T}{9}}} \leq z \right) - \Phi(z) \right| < \varepsilon$$

is dense. We now modify any given environment so that $\tilde{\mathrm{p}}(n) = \frac{2}{3}$ for $|n| > K$. Then the walk spends a finite time to the left of K. It follows that

$$\mathbb{P}\left(\frac{Z_T - \frac{T}{3}}{\sqrt{\frac{8T}{9}}} \leq z \right) \to \Phi(z)$$

uniformly in z as needed.

To prove part (d), we modify a given environment outside $[-K, K]$ in three steps. First we take $K_1 \gg K$ and modify p on $[-K_1, K_1] \setminus [-K, K]$ to achieve that

$$\sum_{j=1}^{n} \ln \tilde{\mathfrak{q}}(j) - \ln \tilde{\mathfrak{p}}(j) = \sum_{j=-n+1}^{0} \ln \tilde{\mathfrak{p}}(j) - \ln \tilde{\mathfrak{q}}(j),$$

where $\tilde{\mathrm{q}}(j) = 1 - \tilde{\mathrm{p}}(j)$. Next we take $K_2 \gg K_1$ and let $\tilde{\mathrm{p}}(n) = \frac{1}{2}$ if $|n| \in [K_1 + 1, K_2]$. Finally, we let $\tilde{\mathrm{p}}(n) = \frac{1}{3}$ if $n < -K_2$ and $\tilde{\mathrm{p}}(n) = \frac{2}{3}$ if $n > K_2$. It is easy to see that, given $\varepsilon > 0$, we can make K_1 and K_2 so large that

$$1 - \varepsilon < \frac{|M_-|}{|M_+|} < 1 + \varepsilon, \tag{4.2}$$

where M_+ and M_- are defined in (3.9). Then (3.9) shows that

$$\mathbb{P}\left(\lim_{t\to\infty} Z_T = +\infty\right) = \frac{|M_+|}{|M_+|+|M_-|} \in \left[\frac{1}{2+\varepsilon}, \frac{1}{2-\varepsilon}\right].$$

The same holds for $\mathbb{P}\left(\lim_{t\to\infty} Z_T = -\infty\right)$.

On the other hand, it is easy to see that

$$\mathbb{P}\left(\frac{Z_T - \frac{T}{3}}{\sqrt{\frac{8T}{9}}} \le z \,\middle|\, \lim_{t\to\infty} Z_t = +\infty\right) = \Phi(z)$$

and

$$\mathbb{P}\left(\frac{Z_T + \frac{T}{3}}{\sqrt{\frac{8T}{9}}} \le z \,\middle|\, \lim_{t\to\infty} Z_t = -\infty\right) = \Phi(z).$$

It follows that for T sufficiently large (2.8) is satisfied with $b(T) = \frac{T}{3}$, $\varepsilon(T) = T^{-1/3}$ proving the density of this condition. □

The main idea of the above proof is the following. If we want to speed up the walk, we modify p by adding a drift away from the origin, while to slow it down we increase the drift towards the origin.

The remaining part of the paper is devoted to the proof of Theorem A and its implications. The main idea is the same. However, a significant effort needs to be applied in order to realize the desired modification as a set of values of a smooth function along orbits of a rotation (note that the results of §2.2 show that this is only possible if the frequency is Liouville).

5. Random walks in deterministic aperiodic medium. Diffusion and localization *via* optional stopping

Consider the quasi-periodic random walk Z_t on $\mathbb{Z}$ defined by the probability function $\mathrm{p} \in \mathcal{E}$:

$$\mathbb{P}(Z_{t+1} = j+1 | Z_t = j) = \mathrm{p}(j), \quad \mathbb{P}(Z_{t+1} = j-1 | Z_t = j) = \mathrm{q}(j). \tag{5.1}$$

In this section we present criteria involving the sums of $\ln \mathrm{q} - \ln \mathrm{p}$, denoted by $\Sigma(n)$ in (3.5), that guarantee the different behaviours in Theorem A.

Proposition 5.1 below gives a criterion for localization, Proposition 5.2 for one-sided drift, and Proposition 5.3 for two-sided drift.

5.1. Localization criterion.

We say that p **satisfies condition** $\mathcal{C}_1(N)$ if

$$\Sigma(N) > N^{1/2} \tag{$\mathcal{C}_1 a_+$}$$

$$\Sigma(-N) > N^{1/2}. \tag{$\mathcal{C}_1 a_-$}$$

Proposition 5.1. *If* p *satisfies condition* $\mathcal{C}_1(N)$*, then for* $T = e^{\sqrt{N}/4}$ *we have*

$$(\textbf{Localization}) \qquad \mathbb{P}\left(\max_{t\le T}|Z_T| > 16(\ln T)^2\right) < T^{-2} \ \textit{and} \ \ \mathrm{Var}(Z_T) < 300(\ln T)^4.$$

Condition $(\mathcal{C}_1)$ means that the origin is a sharp local minimum of the potential, and as explained in the introduction, it implies that the walker spends a lot of time near the origin.

5.2. **One-sided drift criterion.** We say that p **satisfies condition** $\mathcal{C}_2(N,\varepsilon)$ for $\varepsilon > 0$ if there exist $A > 100$ and L satisfying $e^{e^A} < L \le N^{\varepsilon^2}$ such that the following conditions hold:

$$(\mathcal{C}_2a) \qquad \Sigma(-L) > \sqrt{L};$$

$$(\mathcal{C}_2b) \qquad \Sigma(k,k') < A \ \text{ for all } k,k' \in [-N,N], \quad k \le k';$$

$$(\mathcal{C}_2c) \qquad |\mathrm{p}(k+L) - \mathrm{p}(k)| < L^{-1/\varepsilon^3} \ \text{ for all } k \in [-N,N].$$

Proposition 5.2. *If* p *satisfies condition* $\mathcal{C}_2(N,\varepsilon)$ *for some* $\varepsilon > 0$*, then for* $T = N$*, there exist* $\mu > T^{1-\varepsilon}$ *and* σ *with* $\left|\frac{\ln\sigma}{\ln T} - \frac{1}{2}\right| < \varepsilon$ *such that*

$$(\textbf{One-sided drift}) \qquad \left|\mathbb{P}\left(\frac{Z_T-\mu}{\sigma} < z\right) - \Phi(z)\right| < \varepsilon.$$

Conditions $(\mathcal{C}_2(a))$ and $(\mathcal{C}_2(c))$ imply that on a large segment around the origin, the potential Σ is decreasing on scale L, while $(\mathcal{C}_2(b))$ means that there are no deep wells of the potential (also known as traps) on smaller scales. Thus, Proposition 5.2 confirms the heuristic arguments described in the introduction after (1.3).

5.3. **Two-sided drift criterion.** We say that p **satisfies condition** $\mathcal{C}_3(N,\varepsilon)$ for $\varepsilon > 0$ if there exist $A > 100$, $e^{e^A} < Q < N^{1/2}$ and numbers $u, v, w_\pm, u', v', w'_\pm$ such that $v, v' \in [0.3, 0.4]$,

$$0.225 < u < v-\varepsilon < w_- < v < w_+ < v+\varepsilon < 0.5,$$
$$0.225 < u' < v'-\varepsilon < w'_- < v' < w'_+ < v'+\varepsilon < 0.5,$$

and

$$(\mathcal{C}_3a) \qquad \Sigma(vN, w_+N) > N^{1/2}, \quad \Sigma(vN, w_-N) > N^{1/2},$$
$$\Sigma(-v'N, -w'_-N) > N^{1/2}, \quad \Sigma(-v'N, -w'_+N) > N^{1/2},$$

$$(\mathcal{C}_3b) \qquad \Sigma(k,k') < A \quad \text{for} \quad k,k' \in [-u'N, vN], \quad k' \ge k,$$
$$\Sigma(k,k') < A \quad \text{for} \quad k,k' \in [-v'N, uN], \quad k' \le k,$$

$$(\mathcal{C}_3c) \qquad \Sigma(k) = \bar\Sigma(k) + B(k), \quad k \in [-v'N, vN],$$

where $\bar\Sigma$ and B satisfy

$$(5.2) \qquad |\bar\Sigma(k) - \bar\Sigma(k+lQ)| < Q^{-1/2} \ \text{ for } \ k \in [0,Q], \ l \in [-v'N/Q, vN/Q]$$

and

$$B(k)\begin{cases} = 0 & \text{for } \ k \in [-u'N, uN], \\ \leq 0 & \text{for } k \in [-v'N, vN]. \end{cases} \tag{5.3}$$

Figure 5.3 illustrates the behavior of $\Sigma(k)$ for $k \in (-(v'+\varepsilon)N, (v+\varepsilon)N)$ due to $(\mathcal{C}_3)$. In the figure we assumed $\bar{\Sigma} \equiv 0$ since (5.2) and (5.3) imply that the effect of $\bar{\Sigma}$ is not important in the behavior of $\Sigma(k)$.

We note that condition $(\mathcal{C}_3 a)$ implies localization of the walk around the points $-v'N$ and vN (compare with conditions of Proposition 5.1).

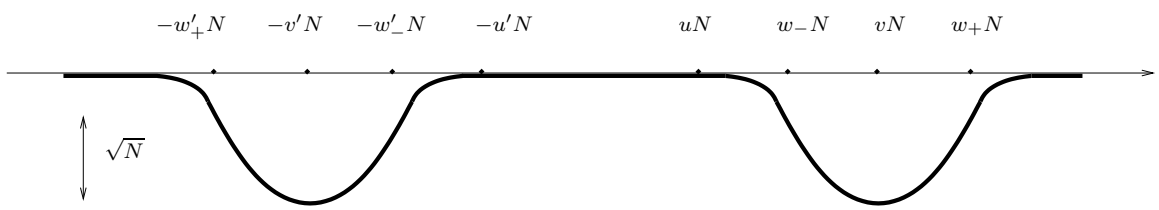

FIGURE 1. A sketch of the graph of $\Sigma(k)$ under condition $\mathcal{C}_3(N,\varepsilon)$, in which it is assumed that $\bar{\Sigma} \equiv 0$.

Conditions $(\mathcal{C}_3 b_\pm)$ imply that the random walk starting at zero exits the interval $[-v'N, vN]$ before time N^5 with probability almost one (see Lemma 6.2 below).

Condition $(\mathcal{C}_3 c)$ compares the walk on $[-v'N, vN]$ to a Q-periodic walk. This condition, combined with $(\mathcal{C}_3 b_\pm)$, makes sure that for the random walk starting at zero, both the probability of reaching $-v'N$ before time N^5, and the probability of reaching vN before time N^5, are not too small. Since $(\mathcal{C}_3 a)$ implies localization around $-v'N$ and vN we get the following.

Proposition 5.3. *If* $\mathrm{p} \in \mathcal{C}_3(N,\varepsilon)$*, then for any* $T \in [N^5, e^{N^{1/4}}]$ *we have:*

$$\text{(\textbf{Two-sided drift})} \qquad \begin{cases} \mathbb{P}\left(Z_T \in [w_-N, w_+N]\right) > 0.1, \\ \mathbb{P}\left(Z_T \in [-w'_+N, -w'_-N]\right) > 0.1. \end{cases}$$

6. Exit time estimates

In this section we derive the key estimates used in the proof of the above propositions for $\mathrm{p} \in \mathcal{E}$. We recall again the definitions of $\Sigma(n)$ and $M(n)$ given in (3.5) and (3.6).

6.1. Traps.

Lemma 6.1. *Suppose that for some* N *condition* $(\mathcal{C}a_+)$ *holds, i.e.,*

$$(\mathcal{C}a_+) \qquad \Sigma(N) > \sqrt{N}.$$

Then for $T = e^{\sqrt{N}/2}$ *we have*

$$\mathbb{P}(\max_{t \leq T} Z_t > N) < \exp(-\sqrt{N}/2).$$

In the same way, if

$$(\mathcal{C}a_-) \qquad \Sigma(-N) > \sqrt{N},$$

then for $T = e^{\sqrt{N}/2}$ *we have*

$$\mathbb{P}(\max_{t\le T} Z_t < -N) < \exp(-\sqrt{N}/2).$$

Moreover, if for some N *both* $(\mathcal{C}a_+)$ *and* $(\mathcal{C}a_-)$ *hold, then for* $T_1 = e^{\sqrt{N}/4}$ *we have:*

$$\mathrm{Var}(Z_{T_1}) < 300(\ln T_1)^4.$$

Proof. Recall the notations and the background material from §3.2. If $(\mathcal{C}a_+)$ holds, then for N large enough we have $M(N) \ge 2\exp(\Sigma(N))$, and (3.7) implies that

$$\begin{aligned}\mathbb{P}(Z_t \text{ visits } N \text{ before } 0|Z_0 = 1) &= \frac{M(1) - M(0)}{M(N) - M(0)} = \frac{1}{M(N)} \\ &\le \frac{1}{2}\exp(-\Sigma(N)) \le \frac{1}{2}\exp\left(-\sqrt{N}\right).\end{aligned}$$

Hence, for $L = o(e^{\sqrt{N}})$ we have:

$$\mathbb{P}(Z_t \text{ visits } N \quad \text{before visiting } 0 \;\; L \text{ times}) \le L\exp\left(-\sqrt{N}\right).$$

Choosing $L = \exp(\sqrt{N}/2)$, we obtain:

$$\mathbb{P}\left(\max_{t\le \exp(\sqrt{N}/2)} Z_t > N\right) < \exp(-\sqrt{N}/2).$$

Likewise, if $(\mathcal{C}a_-)$ holds, we have:

$$\mathbb{P}\left(\max_{t\le \exp(\sqrt{N}/2)} -Z_t > N\right) < \exp(-\sqrt{N}/2).$$

To estimate the variance, assume that both $(\mathcal{C}a_+)$ and $(\mathcal{C}a_-)$ hold. Then for $T_1 = \exp(\sqrt{N}/4)$ we have: $N = 16(\ln T_1)^2$, and
$\mathrm{Var}(Z_{T_1}) \le \mathbb{E}(Z_{T_1}^2) \le N^2 + cT^2\exp(-\sqrt{N}/2) < N^2 + c < 300(\ln T_1)^4.$ □

6.2. **Exit time in the absence of traps.** Let $L \in \mathbb{N}$. For an arbitrary choice of $k_0 \in (-L, L)$, let τ be the first time the walk that starts at k_0 hits L or $-L$.

Lemma 6.2. *Suppose that there exist* $A > 100$ *and* L *satisfying* $e^{e^A} < L$ *such that for each* $k \in [-L, L]$ *either* $(\mathcal{C}b_+)$ *or* $(\mathcal{C}b_-)$ *holds:*

$$(\mathcal{C}b_+) \qquad \Sigma(k, k') < A \text{ for all } k' \in [-L, L], \quad k' \ge k;.$$

$$(\mathcal{C}b_-) \qquad \Sigma(k, k') < A \text{ for all } k' \in [-L, L], \quad k' \le k.$$

Then there is a constant $c > 0$ *such that for* $s \in \{1, 2, 3\}$ *we have*

$$\mathbb{E}(\tau^s) \le ce^{sA}L^{2s+1}. \tag{6.1}$$

Moreover $\mathbb{E}(\tau) \ge L$ *and* $\mathrm{Var}(\tau) \ge 1$.

Proof. For every $k \in I = [-L+1, L-1]$, let η_k be the total time the walker (starting at 0) spends at site k before reaching $-L$ or L. Then $\tau \leq \sum_{k\in I} \eta_k$. Hence, for any $s \in \mathbb{N}$

$$\tau^s \leq L^s \sum_{k\in I} \eta_k^s.$$

Thus, it suffices to show that for $s \in \{1, 2, 3\}$ and for any $k \in I$

$$\mathbb{E}(\eta_k^s) \leq ce^{sA} L^s.$$

For $k \in I$, let $\bar{\eta}_k$ be the total time a walker starting at site k spends at site k before reaching $-L$ or L.

Note that $\bar{\eta}_k$ has geometric distribution with parameter

$$r_k = \mathbb{P}(Z \text{ starting at } k \text{ does not return to } k \text{ before exiting } I).$$

Since $\mathbb{E}(\eta_k^s) \leq \mathbb{E}(\bar{\eta}_k^s)$, we finish the proof of (6.1) once we prove the following

Claim. If either $(\mathcal{C}b_+)$ or $(\mathcal{C}b_-)$ holds, we have for every $k \in I$

$$r_k \geq \frac{c}{Le^A}.$$

Proof of the Claim. Fix $k \in [-L, L]$. Observe that

$$\begin{aligned} r_k \geq c \max\{&\mathbb{P}(Z \text{ visits } L \text{ before } k | Z_0 = k+1), \\ &\mathbb{P}(Z \text{ visits } (-L) \text{ before } k | Z_0 = k-1)\}. \end{aligned}$$

Now, if $(\mathcal{C}b_+)$ holds, then (3.7) implies

$$\begin{aligned} \mathbb{P}(Z \text{ visits } L \text{ before } k | Z_0 = k+1) =& \frac{M(k+1) - M(k)}{M(L) - M(k)} = \frac{e^{\Sigma(k+1)}}{\sum_{j=k+1}^{L-1} e^{\Sigma(j)}} \\ =& \frac{1}{1 + \sum_{j=k+2}^{L-1} e^{\Sigma(k+1,j)}} > \frac{1}{Le^A}. \end{aligned}$$

In the same way, if $(\mathcal{C}b_+)$ holds, then

$$\mathbb{P}(Z \text{ visits } (-L) \text{ before } k | Z_0 = k-1) > \frac{1}{Le^A},$$

and the claim is proved. □

Since the walker moves one step at a time, $\mathbb{E}(\tau) \geq L$. The lower bound on the variance of τ is obvious due to the ellipticity condition on the walk. □

7. Proofs of the criteria

In this section, we prove Propositions 5.1–5.3.

Proposition 5.1 follows directly from Lemma 6.1.

Proof of Proposition 5.2. To make the argument easier to follow, we first consider the periodic case, i.e., we assume that the medium is periodic, namely,

$$\mathrm{p}(k+L) = \mathrm{p}(k) \text{ for any } k \in \mathbb{Z}. \tag{7.1}$$

If we run the walk starting from 0 and stop it at the time τ when it hits either L or $-L$, we get two random variables: τ and $U = Z_\tau$ (thus, U takes value L or $-L$). Let us consider iid copies (τ_i, U_i) of such pairs. Denote

$$\hat{\mu} = \mathbb{E}(\tau_i), \quad \hat{V} = \mathbb{E}((\tau_i - \hat{\mu})^2), \quad \hat{\gamma} = \mathbb{E}(|\tau_i - \hat{\mu}|^3).$$

By Lemma 6.2 we have the following estimates:

$$L \le \hat{\mu} \le ce^A L^3, \quad 1 \le \hat{V} \le ce^{2A} L^5, \quad \hat{\gamma} \le ce^{3A} L^7. \tag{7.2}$$

Note that

$$\mathbb{P}(U = L) \ge 1 - e^{-0.5\sqrt{L}}$$

by condition $(\mathcal{C}_2 a)$ (cf. the proof of Lemma 6.1).

For $M \le N^2$, denote

$$\Theta_M := \sum_{i=1}^{M} \tau_i.$$

For such M we have that

$$Z_{\Theta_M} = \sum_{i=1}^{M} U_i = ML$$

with probability larger than $(1 - e^{-0.5\sqrt{L}})^{N^2} \ge 1 - e^{-0.1\sqrt{L}}$ if L is sufficiently large.

Define the stopping time M_N as the first integer such that $\Theta_{M_N} \ge N$. By [14, Theorem 1], we have that the "residual lifetime" or "excess over the boundary", $\Theta_{M_N} - N$ has expectation less than $\hat{V}/\hat{\mu}$.

Hence, (7.2) implies that with probability larger than $1 - 1/L$,

$$\Theta_{M_N} \in [N, N + L^{10}].$$

Thus, with the same probability,

$$Z_N = M_N L + O(L^{10}).$$

By the Berry-Esseen theorem for renewal counting processes [13, Theorem 2.7.1] we have

$$\left| \mathbb{P}\left(\frac{M_N - \frac{N}{\hat{\mu}}}{\sqrt{\frac{\hat{V}}{\hat{\mu}^3} N}} < z \right) - \Phi(z) \right| < 4 \left(\frac{\hat{\gamma}}{\sqrt{\hat{V}}} \right)^3 \sqrt{\frac{\hat{\mu}}{N}} < \frac{1}{L}$$

if L is sufficiently large. Hence

$$\left|\mathbb{P}\left(\frac{Z_N - \frac{LN}{\hat{\mu}}}{L\sqrt{\frac{\hat{V}}{\hat{\mu}^3}N}} < z\right) - \Phi(z)\right| < \frac{1}{\sqrt{L}}.$$

Since $L \leq N^{\varepsilon^2}$, (7.2) implies that

$$\mu := \frac{LN}{\hat{\mu}} > \frac{N}{ce^A L^2} > N^{1-\varepsilon},$$

and

$$\sigma := L\sqrt{\frac{\hat{V}}{\hat{\mu}^3}N}$$

satisfies $|\ln\sigma/\ln N - 1/2| < \varepsilon$.

This completes the proof in the periodic case (7.1). Now let the periodicity assumption (7.1) be replaced by a weaker condition $(\mathcal{C}_2 c)$. In this case we consider a new periodic environment $\bar{\mathrm{p}}_n$, where $\bar{\mathrm{p}}_n = \mathrm{p}_n$ for each $n \in [0, L]$, and $\bar{\mathrm{p}}_n$ is periodic with period L. Let $\overline{\mathbb{P}}$ denote the corresponding probability for the walk. By $(\mathcal{C}_2 c)$, for any trajectory $\xi : [0, N] \to \mathbb{Z}$ of length N we have:

$$\left|\frac{\overline{\mathbb{P}}(\xi)}{\mathbb{P}(\xi)} - 1\right| < CNL^{-1/\varepsilon^3}.$$

Hence, the general case follows from the periodic one. □

Proof of Proposition 5.3. We divide the proof into three steps.

Step 1. For an arbitrary choice of $k \in (-v'N, vN)$, denote by τ the exit time from $(-v'N, vN)$ (while starting at k). We need to estimate

$$\mathbb{P}(Z_t \text{ reaches } -v'N \text{ or } vN \text{ before time } N^5 | Z_0 = k) = \mathbb{P}(\tau < N^5).$$

By Lemma 6.2, under condition $(\mathcal{C}_3 b)$ there exists $c > 0$ such that for any $k \in (-v'N, vN)$ we have: $\mathbb{E}(\tau) < ce^A N^3$. Then $\mathbb{P}(\tau > N^4)N^4 < \mathbb{E}(\tau) < ce^A N^3$, so

$$\mathbb{P}(\tau > N^4) < ce^A/N.$$

This implies

$$\mathbb{P}(\tau > N^5) < (ce^A/N)^N < e^{-N}.$$

Hence,

$$\mathbb{P}(Z_t \text{ reaches } -v'N \text{ or } vN \text{ before time } N^5) > 1 - e^{-N}.$$

Step 2. We have the following two inequalities:

$$\mathbb{P}(Z_t \text{ visits } -v'N \text{ before visiting } vN) \leq 0.89,$$
$$\mathbb{P}(Z_t \text{ visits } vN \text{ before visiting } -v'N) \leq 0.89.$$

We prove the first estimate, the second one can be proved similarly. By (3.7)

$$\mathbb{P}(Z \text{ visits } (-v'N) \text{ before visiting } vN) = \frac{M(vN)}{M(vN) - M(-v'N)}.$$

Using $(\mathcal{C}_3 c)$, we get:

$$M(vN) = \sum_{j=1}^{vN} e^{\Sigma(j)} \leq \sum_{j=1}^{vN} e^{\bar{\Sigma}(j)} = \sum_{l=1}^{vN/Q} \sum_{j=1}^{Q} e^{\bar{\Sigma}(j+(l-1)Q)} = \frac{vN}{Q} M(Q)(1 + O(Q^{-1/2})).$$

In the same way,

$$M(uN) = \sum_{j=1}^{uN} e^{\Sigma(j)} = \sum_{j=1}^{uN} e^{\bar{\Sigma}(j)} = \sum_{l=1}^{uN/Q} \sum_{j=1}^{Q} e^{\bar{\Sigma}(j+(l-1)Q)} = \frac{uN}{Q} M(Q)(1 + O(Q^{-1/2})).$$

Hence

$$M(vN) \geq M(uN) = \frac{uN}{Q} M(Q)(1 + O(Q^{-1/2})).$$

Similarly

$$M(-v'N) \leq M(-u'N) = -\frac{u'N}{Q} M(Q)(1 + O(Q^{-1/2})).$$

Hence,

$$\frac{M(vN) - M(x)}{M(vN) - M(-v'N)} < \frac{v}{u + u'} + o(1) < \frac{0.4}{0.45} + o(1) < 0.89.$$

Step 3. By Step 1, with probability $1 - e^{-N}$, the walk starting at 0 reaches either vN or $-vN'$ before time N^5. By Step 2, it reaches vN before time N^5 with probability larger than 0.11. The first part of $(\mathcal{C}_3 a)$ states that $\Sigma(vN, w_+N) > N^{1/2}$ and $\Sigma(vN, w_-N) > N^{1/2}$. Under this condition, Lemma 6.1 implies that the walk starting at vN satisfies

$$\mathbb{P}\left(Z_T \in [w_-N, w_+N]\right) > 0.1$$

for all $T \in [N^5, e^{N^{1/4}}]$, which implies the desired result. The same argument holds for $-v'N$. □

8. Quasi-periodic environments

In this section we fix a Liouville number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and for $\mathfrak{p} \in \mathcal{P}$, consider the walks X_n and Z_n defined by (1.1). For any given initial condition $x \in \mathbb{T}$, we introduce the environment defined by x:

$$\mathrm{p}(j) := \mathfrak{p}(x + j\alpha).$$

It will be convenient to reformulate the main conditions $\mathcal{C}_j$, $j = 1, 2, 3$, in this new context. First of all, notice that condition $\mathcal{C}_1 = \mathcal{C}_1(N)$ does not depend on ε, while the other two conditions do. For the uniformity of notations, we formally include an ε in all the three conditions. We say that

$$x \in \mathcal{C}_j(\mathfrak{p}, N, \varepsilon) \text{ if and only if } \mathrm{p} \in \mathcal{C}_j(N, \varepsilon), \quad j = 1, 2, 3.$$

The goal of this section is to prove the following statement.

Theorem 8.1. *For any Liouville α there exists a dense G_δ set $S \subset \mathcal{P}$ with the following property: for any $\mathfrak{p} \in S$, for almost every $x \in \mathbb{T}$, there are strictly increasing sequences of numbers $N_{j,n}$, such that for all $j = 1, 2, 3$, $n \in \mathbb{N}$ we have*

$$x \in \mathcal{C}_j(\mathfrak{p}, N_{j,n}, 1/n).$$

By the results of Propositions 5.1–5.3, this will suffice, to prove Theorem A, see §9.1 for details.

8.1. **The G^δ argument.** In §3.1 we denoted by $\mathcal{B}_\alpha \subset \mathcal{P}$ the set of (multiplicative) coboundaries, i.e., functions $\bar{\mathfrak{p}}(x)$ such that (3.4) has a solution with a smooth transfer function g. We noticed that if $\bar{\mathfrak{p}} \in \mathcal{B}_\alpha$, then

$$\Sigma_x(n) = \ln g(x + \alpha) - \ln g(x + (n + 1)\alpha).$$

In particular, for all n, $|\Sigma_x(n)|$ is bounded by a constant independent of n, hence none of the criteria from the previous section holds for $\bar{\mathfrak{p}} \in \mathcal{B}_\alpha$. The advantage of coboundaries however, is that it is reasonable to try to perturb a coboundary $\bar{\mathfrak{p}}$ into $\mathfrak{p}$ for which we can check, for some good choices of $x \in \mathbb{T}$, the criteria involving $\Sigma_x(n)$ that lead to the erratic behavior of Theorem A.

Thus, we start by proving that the set of coboundaries are dense in $\mathcal{P}$.

Lemma 8.2. *For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the set $\mathcal{B}_\alpha$ of smooth multiplicative coboundaries is dense in $\mathcal{P}$ for the C^∞ topology.*

Proof. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. By truncating the Fourier series of $\ln(\mathfrak{p}/\mathfrak{q})$ it is possible to approximate it in the C^∞ topology by coboundaries of the form $\psi(\cdot) - \psi(\cdot + \alpha)$ where $\psi \in C^\infty(\mathbb{T}, \mathbb{R})$. Hence $F(\cdot) = g(\cdot)/g(\cdot + \alpha)$ where $g = e^\psi$ will approach $\mathfrak{p}/\mathfrak{q}$. Now define $\bar{\mathfrak{p}} = F/(1 + F)$ and observe that $\bar{\mathfrak{p}}$ approaches $\mathfrak{p}$ while $\bar{\mathfrak{p}} \in \mathcal{B}_\alpha$. □

To prove Theorem 8.1, we will construct explicit sequences of functions e_n, numbers $N_{j,n}$ and sets $U_{j,n}$ such that the following statement holds true:

Proposition 8.3. *For any Liouville number α there exists*

- *A strictly increasing sequence of integers q_n,*
- *An explicit sequence of C^∞ functions e_n satisfying $|e_n|_{C^{n-1}} < 1/n$ (see §8.2.3),*
- *For every $j \in \{1, 2, 3\}$, a sequence of numbers $N_{j,n}$,*
- *For every $j \in \{1, 2, 3\}$, a sequence of sets $U_{j,n} = \bigcup_{i \in [0, q_n - 1]} (I_{j,n} + i/q_n) \subset \mathbb{T}$, where $I_{j,n}$ is an interval in $[0, 1/q_n]$ of size $|I_{j,n}| > \frac{0.01}{q_n}$*

with the following property. For every $\bar{\mathfrak{p}} \in \mathcal{B}_\alpha$, for every sufficiently large n

$$U_{j,n} \subset \mathcal{C}_j\left(\bar{\mathfrak{p}} + e_n, N_{j,n}, \frac{1}{n}\right).$$

Before proving Proposition 8.3 we show how it implies Theorem 8.1.

Proof of Theorem 8.1. Fix a Liouville $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Fix any $j \in \{1, 2, 3\}$. Let $U_{j,n}$ be the sets from Proposition 8.3. Denote

$$\mathcal{G}_{j,n} = \{\mathfrak{p} \in \mathcal{P} \mid U_{j,n} \subset \mathcal{C}_j(\mathfrak{p}, N_{j,n}, 1/n)\}.$$

By the definitions of the conditions $\mathcal{C}_j$, the sets $\mathcal{G}_{j,n}$ are open. Lemma 8.2 and Proposition 8.3 show that for any $m \in \mathbb{N}$ the set $\bigcup_{n\geq m} \mathcal{G}_{j,n}$ is dense. Hence the set

$$\mathcal{G}_j = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \mathcal{G}_{j,n}$$

is a dense G_δ set (in C^r topology for any $r \in \mathbb{N}$).

Observe now that for $\mathfrak{p} \in \mathcal{G}_j$ we have that there exists a strictly increasing sequence l_n such that $\mathfrak{p} \in \mathcal{G}_{j,l_n}$. Recall that U_{j,l_n} has Lebesgue measure larger than 0.01 for every l_n. Moreover, up to extracting a subsequence we may assume that $q_{l_{n+1}} \gg q_{l_n}$, so that

$$\lambda\left(U_{j,l_n} \cap \bigcap_{i=1}^{n-1} U_{j,l_i}^c\right) \geq \frac{1}{2}\lambda(U_{j,l_n}) \cdot \lambda\left(\bigcap_{i=1}^{n-1} U_{j,l_i}^c\right).$$

Now, an enhanced version of the Borel-Cantelli Lemma (see [15, Chapter IV]) states that if events C_n are such that for each $k \geq 1$

$$\sum_{n=k}^{\infty} \mathbb{P}\left(C_n \middle| \bigcap_{j=k}^{n-1} C_j^c\right) = +\infty,$$

then with probability 1, infinitely many of those events occur. We thus conclude that a.e. x belongs to infinitely many U_{j,l_n}, thus to infinitely many $\mathcal{C}_j(\mathfrak{p}, N_{j,l_n}, 1/n)$. In conclusion, the set $S = \bigcap_{j=1}^3 \mathcal{G}_j$ satisfies the property required in Theorem 8.1 □

The rest of Section 8 is devoted to the proof of Proposition 8.3.

8.2. **Perturbation of a coboundary. The Main construction.**

8.2.1. **Coboundaries.** Given a coboundary $\bar{\mathfrak{p}} \in \mathcal{B}_\alpha$ with a transfer function $g(x)$, for $M \in \mathbb{N}$, let $\bar{\Sigma}_x(M)$ be the ergodic sum of $\bar{\mathfrak{p}}$ defined by formula (3.5) with $\mathfrak{p}$ exchanged by $\bar{\mathfrak{p}}$. Thus, for all $k \leq k'$ and all $M \in \mathbb{N}$ we denote:

$$\begin{aligned}
&\bar{\Sigma}_x(M) = \ln g(x+\alpha) - \ln g(x+(M+1)\alpha),\\
&\bar{\Sigma}_x(k,k') = \ln g(x+(k+1)\alpha) - \ln g(x+(k'+1)\alpha),\\
&A := \ln\|g\| + \ln\|1/g\|.
\end{aligned}$$

Then,

$$\forall x \in \mathbb{T}, \forall M \in \mathbb{N} : |\bar{\Sigma}_x(M)| \leq A, \quad |\bar{\Sigma}_x(k,k')| \leq A,$$

Moreover, for any coboundary $\bar{\mathfrak{p}}$, there exists $0 < \kappa \leq 1/2$ such that

$$\kappa \leq \bar{\mathfrak{p}}(x) \leq 1-\kappa \quad \text{for all} \quad x \in \mathbb{T}.$$

Define

$$K(x) = \frac{1}{\bar{\mathfrak{p}}(x)} + \frac{1}{\bar{\mathfrak{q}}(x)},$$

and observe that

$$K(x) \in (2, 2/\kappa].$$

8.2.2. **The sequences q_n and $N_{j,n}$.** Given a Liouville number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let q_n be a sequence of integers satisfying

$$\eta_n := |q_n \alpha| < q_n^{-n^6}, \tag{8.1}$$

where $|\cdot|$ denotes the closest distance to integers. Moreover, for each n we will need to choose q_n sufficiently large for our arguments to hold.

Denote the integer closest to $q_n\alpha$ by s_n. In the constructions below we assume that $q_n\alpha > s_n$ for all n. If $q_n\alpha < s_n$, the arguments are the same up to a suitable change of signs.

Denote

$$\begin{aligned} &N_n := \left[(q_n\eta_n)^{-1}\right] q_n, \\ &N_{1,n} := [N_n/20], \quad N_{2,n} := q_n^{n^5}, \quad N_{3,n} := N_n, \end{aligned} \tag{8.2}$$

were $[a]$ stands for the integer part of a.

We make the following useful observation on the combinatorics of the irrational rotation R_α on the circle. The orbit of any fixed point x of the circle under the rotation by α on $\mathbb{T}$ is essentially distributed in the following way. The points $x, x+\alpha, x+2\alpha, \dots x+(q_n-1)\alpha$ are very close (closer than η_n) to $x, x+\frac{p_n}{q_n}, x+2\frac{p_n}{q_n}, \dots x+(q_n-1)\frac{p_n}{q_n}$. Hence, there will be one point of this q_n piece of orbit in each *basic interval* $[k/q_n, (k+1)/q_n]$, $k = 0, \dots q_n - 1$. Moreover, the first return of x to its basic interval will be shifted by η_n. The next return will thus be shifted by one more η_n. Finally, the orbit $x, x+\alpha, x+2\alpha, \dots, x+N_n\alpha$ will form an η_n-grid inside each basic interval, and thus in the whole circle.

8.2.3. **The functions e_n.** In this section, the names of functions with the shortest period 1 are marked with a tilde, while $\frac{1}{q_n}$-periodic functions have no tilde in their name.

Let $\widetilde{e}_n(x) \in C^\infty$ be a 1-periodic function satisfying $\int_{\mathbb{T}} \widetilde{e}_n(x)dx = 0$ and such that

$$\widetilde{e}_n(x) = \begin{cases} \sin 8\pi x & \text{for } x \in [-\frac{1}{2}+\frac{1}{n^2}, -\frac{3}{8}-\frac{1}{n^2}] \cup [-\frac{3}{8}+\frac{1}{n^2}, -\frac{1}{4}-\frac{1}{n^2}] \cup \\ & \qquad [\frac{1}{4}+\frac{1}{n^2}, \frac{3}{8}-\frac{1}{n^2}] \cup [\frac{3}{8}+\frac{1}{n^2}, \frac{1}{2}-\frac{1}{n^2}], \\ 0 & \text{for } x \in [-\frac{1}{4}, \frac{1}{4}], \\ \text{increasing} & \text{on the intervals } [-\frac{1}{2}, -\frac{1}{2}+\frac{1}{n^2}], [-\frac{1}{4}-\frac{1}{n^2}, -\frac{1}{4}], [\frac{1}{4}, \frac{1}{4}+\frac{1}{n^2}], [\frac{1}{2}-\frac{1}{n^2}, \frac{1}{2}], \\ \text{decreasing} & \text{on the intervals } [-\frac{3}{8}-\frac{1}{n^2}, -\frac{3}{8}+\frac{1}{n^2}], [\frac{3}{8}-\frac{1}{n^2}, \frac{3}{8}+\frac{1}{n^2}], \\ 0 \text{ and } \infty\text{-flat} & \text{at } x = \pm\frac{3}{8}. \end{cases}$$

Observe that $\widetilde{e}_n$ is also flat at $\pm\frac{1}{4}$ since it is smooth. Figure 2 represents the function $\widetilde{e}_n$.

The idea is to perturb a given coboundary $\bar{\mathfrak{p}}$ by a function of the form $q_n^{-n}\widetilde{e}_n(q_n x)$ to produce the desired behavior of the walk. For each n we will choose q_n satisfying (8.1) and sufficiently large. In particular, although the C^k norms of $\widetilde{e}_n$ may grow fast as n grows, we can still guarantee that $\|q_n^{-n}\widetilde{e}_n(q_n x)\|_{C^{n-1}} < \frac{1}{n}$.

A small problem is that the perturbed function $\mathfrak{p}(x) = \bar{\mathfrak{p}}(x) + q_n^{-n}\widetilde{e}_n(q_n x)$ may not satisfy the symmetry condition (1.2). Below we modify $\widetilde{e}_n(x)$ in order to assure condition

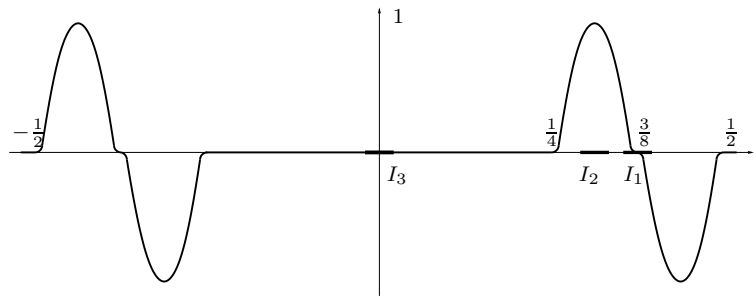

FIGURE 2. Graph of $\widetilde{e}_n(x)$. The intervals I_1, I_2 and I_3 are the sets for which the conditions $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ hold (Lemmas 8.6–8.8).

(1.2) for $\mathfrak{p}$. Let $\widetilde{e}_n^+(x)$ and $\widetilde{e}_n^-(x)$ be the positive and negative parts of $\widetilde{e}_n(x)$:

$$\widetilde{e}_n^+(x) = \begin{cases} \widetilde{e}_n(x) & \text{if } \widetilde{e}_n(x) \geq 0, \\ 0, & \text{otherwise,} \end{cases} \qquad \widetilde{e}_n^-(x) = \begin{cases} -\widetilde{e}_n(x) & \text{if } \widetilde{e}_n(x) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $\delta \in [-1, 1]$, define $\widetilde{e}_{n,\delta}(x)$:

$$\widetilde{e}_{n,\delta}(x) = \begin{cases} \widetilde{e}_n(x) + \delta \widetilde{e}_n^+(x) \text{ if } \delta \in [0, 1], \\ \widetilde{e}_n(x) + \delta \widetilde{e}_n^-(x) \text{ if } \delta \in [-1, 0). \end{cases} \tag{8.3}$$

Note that, since $\widetilde{e}_n$ is flat at $\pm\frac{3}{8}$ and $\pm\frac{1}{4}$, where it actually changes the sign, the functions $\widetilde{e}_{n,\delta}$ are also smooth. This is the only reason why we need $\widetilde{e}_n$ to be flat at those points.

The following lemma introduces the function e_n that will be the main building block of our construction.

Lemma 8.4. *Given any $\bar{\mathfrak{p}} \in \mathcal{B}_\alpha$, if q_n is sufficiently large, there exists $\delta_n \in [-\frac{1}{n}, \frac{1}{n}]$ satisfying*

$$\mathfrak{p}_n(x) := \bar{\mathfrak{p}}(x) + e_n(x) \in \mathcal{P} \quad \textit{with} \quad e_n(x) = q_n^{-n} \widetilde{e}_{n,\delta_n}(q_n x). \tag{8.4}$$

Proof. In this proof, we will use the notation

$$e_{n,\delta}(x) := q_n^{-n} \widetilde{e}_{n,\delta}(q_n x).$$

We are looking for $\delta \in [-1/n, 1/n]$ such that $\mathfrak{p}_n(x)$ satisfies the symmetry condition (1.2), i.e.,

$$I_{n,\delta} := \int_{\mathbb{T}} \ln(\mathfrak{p}_n(x)) - \ln(1 - \mathfrak{p}_n(x)) dx = \int_{\mathbb{T}} \ln\left(1 + \frac{e_{n,\delta}(x)}{\bar{\mathfrak{p}}(x)}\right) - \ln\left(1 - \frac{e_{n,\delta}(x)}{1 - \bar{\mathfrak{p}}(x)}\right) dx = 0$$

We will approximate $I_{n,\delta}$ with

$$J_{n,\delta} := \int_{\mathbb{T}} e_{n,\delta}(x) K(x) dx.$$

Claim. There exists a constant $c > 0$ that does not depend on n or δ such that

$$|I_{n,\delta} - J_{n,\delta}| < cq_n^{-2n} \tag{8.5}$$

$$|J_{n,0}| < c\frac{1}{n^2}q_n^{-n} \tag{8.6}$$

$$\text{For } \delta > 0, \quad J_{n,\delta} - J_{n,0} > c\delta q_n^{-n} \tag{8.7}$$

$$\text{For } \delta < 0, \quad J_{n,\delta} - J_{n,0} < -c\delta q_n^{-n} \tag{8.8}$$

From the continuity of $I_{n,\delta}$ and $J_{n,\delta}$ in δ, it follows directly from the claim that there exists $\delta \in (-1/n, 1/n)$ such that $I_{n,\delta} = 0$.

Proof of the Claim. (8.5) follows from the fact that $\max |e_{n,\delta}(x)| \le 2q_n^{-n}$. (8.6) follows from the fact that the average of $\widetilde{e}_n$ is zero and from the fact that K is almost constant on intervals of size $1/q_n$. As for (8.7), it follows from the fact that $K > 2$, and that the average of $\widetilde{e}_n^+$ is larger than some positive constant independent of n. (8.8) is proved similarly. □

Lemma 8.4 is thus proved. □

8.2.4. **The sets $U_{j,n}$.** Consider the following subintervals of $[0, 1]$:

$$I := \left[-\frac{1}{200}, \frac{1}{200}\right],$$

$$I_1 := \frac{3}{8} + I, \quad I_2 := \frac{5}{16} + I, \quad I_3 := I,$$

$$I_1' := -\frac{3}{8} + I, \quad I_2' := -\frac{7}{16} + I,$$

and let $I_{j,n} = I_j/q_n$, $I'_{j,n} = I'_j/q_n$ for $j = 1, 2, 3$,

$$U_{j,n} = \bigcup_{k=0}^{q_n-1} \left(I_{j,n} \cup I'_{j,n} + \frac{k}{q_n}\right), \quad j = 1, 2, \quad U_{3,n} = \bigcup_{k=0}^{q_n-1} I_{3,n} + \frac{k}{q_n}. \tag{8.9}$$

Notice the total measure of $U_{j,n}$: $|U_{j,n}| = 0.02$, and $|U_{3,n}| = 0.01$.

8.3. **Estimates of ergodic sums.** In the rest of Section 8 we shall use the notation $\mathcal{O}(\cdot)$ and $o(\cdot)$ as a shorthand for $\mathcal{O}_{n\to\infty}(\cdot)$ and $o_{n\to\infty}(\cdot)$.

Recall that $\mathfrak{p}(x) = \bar{\mathfrak{p}}(x) + e_n(x)$ (see (8.4)), and that $N_{j,n}$ is defined by (8.2).

Proposition 8.5 (Main technical lemma)**.** *Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and a coboundary $\bar{\mathfrak{p}} \in \mathcal{B}_\alpha$, let $\bar{\Sigma}_x(M), A, \kappa$, and $K(x)$ be as in Section 8.2.1 and let $\widehat{K} = \int_{\mathbb{T}} K(x)dx$. We have for all $x \in \mathbb{T}$ and all $M \in [0, N_n]$:*

$$\Sigma_x(M) = -\sum_{m=1}^{M} e_n(x + m\alpha)K(x + m\alpha) + \bar{\Sigma}_x(M) + R(x, M), \tag{8.10}$$

$$\Sigma_x(-M) = \sum_{m=-M+1}^{0} e_n(x + m\alpha)K(x + m\alpha) + \bar{\Sigma}_x(-M) + R'(x, M), \tag{8.11}$$

$$\sum_{m=1}^{M} e_n(x+m\alpha)K(x+m\alpha) = \widehat{K} q_n^{-n} N_n \int_{q_n x}^{q_n x + M/N_n} \widetilde{e}_{n,\delta_n}(t)\,dt + o\left(M/q_n^n\right), \tag{8.12}$$

$$\sum_{m=-M+1}^{0} e_n(x+m\alpha)K(x+m\alpha) = \widehat{K} q_n^{-n} N_n \int_{q_n x - M/N_n}^{q_n x} \widetilde{e}_{n,\delta_n}(t)\,dt + o\left(M/q_n^n\right), \tag{8.13}$$

$$\|R\|, \|R'\| \le 4\kappa^{-2} M q_n^{-2n}. \tag{8.14}$$

Moreover,

$$e_n(x+m\alpha) \ge 0 \ \ \textit{for all } m = k,\dots,k' \ \Rightarrow \Sigma_x(k,k') \le \bar{\Sigma}_x(k,k') \le A \tag{8.15}$$

$$e_n(x+m\alpha) \le 0 \ \ \textit{for all } m = k',\dots,k \ \Rightarrow \Sigma_x(k,k') \le \bar{\Sigma}_x(k,k') \le A. \tag{8.16}$$

The statement of this lemma covers several different situations that will be useful in checking all the conditions $\mathcal{C}_j$.

Proof. We start by estimating $\Sigma_x(M)$ for any $x \in \mathbb{T}$ and any $M > 0$:

$$\Sigma_x(M) = \sum_{m=1}^{M} \ln(\mathfrak{q}_n(x+m\alpha)) - \ln(\mathfrak{p}_n(x+m\alpha)) = \tag{8.17}$$

$$\sum_{m=1}^{M} \ln(\bar{\mathfrak{q}}(x+m\alpha)) - \ln(\bar{\mathfrak{p}}(x+m\alpha)) - \sum_{m=1}^{M} e_n(x+m\alpha)K(x+m\alpha) + R(x,M)$$

and

$$|R(x,M)| \le \left| \sum_{m=0}^{M} e_n^2(x+m\alpha)\left(\bar{\mathfrak{q}}^{-2}(x+m\alpha) + \bar{\mathfrak{p}}^{-2}(x+m\alpha)\right) \right| \le 4\kappa^{-2} M q_n^{-2n},$$

since $\|e_n\| \le 2q_n^{-n}$ and $\bar{\mathfrak{p}}, \bar{\mathfrak{q}} \in [\kappa, 1-\kappa]$. This gives (8.10) and (8.14). The proof of (8.11) is similar.

Let us prove (8.12). By (8.1) and (8.2), $\eta_n = |q_n\alpha| \approx N_n^{-1}$. Due to the $1/q_n$-periodicity of e_n for any $x \in \mathbb{T}$,

$$\widetilde{e}_n(q_n(x+m\alpha)) = \widetilde{e}_n(q_n x + m(q_n\alpha)) = \widetilde{e}_n(q_n x) + \mathcal{O}\left(q_n \eta_m \|\widetilde{e}_n\|_{C^1}\right) = \widetilde{e}_n(q_n x) + o\left(q_n^{-n}\right).$$

Therefore,

$$\begin{aligned}\sum_{m=1}^{q_n} e_n(x+m\alpha)K(x+m\alpha) &= \left(e_n(x) + o\left(q_n^{-n}\right)\right) \sum_{m=1}^{q_n} K(x+m\alpha) \\ &= \widehat{K} q_n \left(e_n(x) + o\left(q_n^{-n}\right)\right).\end{aligned}$$

Hence, for $M \gg q_n$ we have

$$\sum_{m=1}^{M} e_n(x+m\alpha)K(x+m\alpha) = \widehat{K} q_n \left(\sum_{m=1}^{M/q_n} e_n(x+mq_n\alpha) + o(M/q_n^{n+1}) \right)$$

$$= \widehat{K} q_n N_n \sum_{m=1}^{M/q_n} e_n(x+m|q_n\alpha|)\eta_n + o\left(M/q_n^n\right) = \widehat{K} q_n N_n \int_x^{x+M/(q_nN_n)} e_n(t)dt + o\left(M/q_n^n\right)$$

$$= \widehat{K} q_n^{-n} N_n \int_{q_nx}^{q_nx+M/N_n} \widetilde{e}_{n,\delta_n}(t)dt + o\left(M/q_n^n\right).$$

The proof of (8.13) is similar.

To show (8.15), notice that under the assumption $e_n(x+m\alpha) \geq 0$ for all $m = k, \dots, k'$ we have for these m that $\mathfrak{p}(x+m\alpha) \geq \bar{\mathfrak{p}}(x+m\alpha)$, hence

$$\Sigma_x(k,k') = \sum_{m=k}^{k'} \ln \frac{\mathfrak{q}(x+m\alpha)}{\mathfrak{p}(x+m\alpha)} \leq \sum_{m=k}^{k'} \ln \frac{\bar{\mathfrak{q}}(x+m\alpha)}{\bar{\mathfrak{p}}(x+m\alpha)} = \bar{\Sigma}_x(M) \leq A.$$

Estimate (8.16) is proved in the same way. □

8.4. **Proof of Proposition 8.3.** To shorten the notations, we shall sometimes write $h_1(N_n) \approx h_2(N_n)$ if $h_1(N_n) = h_2(N_n)(1+\sigma(N_n))$, where $\sigma(N_n) \to 0$ when $N_n \to \infty$.

Lemma 8.6. *For n sufficiently large, we have:*

$$U_{1,n} \subset \mathcal{C}_1(\mathfrak{p}, N_{1,n}).$$

Proof. Fix $x \in I_{1,n}$ (the same argument holds for all $x \in U_{1,n}$). Then q_nx lies in an interval of size 0.01 around the point 3/8. Since $\widetilde{e}_{n,\delta_n}$ is smaller or equal to $\sin(8\pi x)$ for most of the above interval, we have that for large n

$$\int_{q_nx}^{q_nx+1/20} \widetilde{e}_n(t) \leq \int_{3/8-0.01}^{3/8+0.04} \sin 8\pi t < -0.001.$$

By (8.12) with $M = N_{1,n} = N_n/20$, we have

$$\sum_{m=1}^{N_{1,n}} e_n(x+m\alpha)K(x+m\alpha) \approx \widehat{K} q_n^{-n} N_{1,n} \int_{q_nx}^{q_nx+1/20} \widetilde{e}_{n,\delta_n}(t)dt$$

$$< -0.001\widehat{K} q_n^{-n} N_{1,n}$$

Since $|R(x,N_{1,n})| \leq 4\kappa^{-2}N_{1,n}q_n^{-2n}$, and κ and A are independent of N_n, we get from (8.10) for any n sufficiently large:

$$\Sigma_x(N_{1,n}) > 0.001\widehat{K}N_{1,n}q_n^{-n}.$$

Recall that $N_{1,n} \geq q_n^{n^6}/40 \geq q_n^{6n}$. Hence,

$$\Sigma_x(N_{1,n}) > N_{1,n}^{1/2}.$$

Likewise, $\Sigma_x(-N_{1,n}) > N_{1,n}^{1/2}$. □

Lemma 8.7. *For n sufficiently large, we have:*

$$U_{2,n} \subset \mathcal{C}_2(\mathfrak{p}, N_{2,n}, 1/n).$$

Proof. We choose q_n and $N_{2,n}$ satisfying (8.1) and (8.2). Let $\bar{\Sigma}_x(M)$ and A be as in (8.2.1), recall that A only depends on $\bar{\mathfrak{p}}$. Assuming that q_n is sufficiently large, we define

$$L := q_n^{n^2} > e^{e^A}.$$

Since $N_{2,n} = q_n^{n^5}$ by (8.2), we have

$$N_{2,n} = L^{n^3} > L^{n^2},$$

as required in $\mathcal{C}_2(\mathfrak{p}, N_{2,n}, \frac{1}{n})$.

Let $x \in I_{2,n}$ (the same argument holds for all $x \in U_{2,n}$). Then

$$q_n x \in [5/16 - 0.01, 5/16 + 0.01].$$

By the definition of $\widetilde{e}_{n,\delta_n}$, for any $t \in [q_n x - 0.001, q_n x + 0.001]$ it holds that $\widetilde{e}_{n,\delta_n}(t) \geq 0.5$. Since by (8.2) we have $N_{2,n}/N_n < 0.0001$, we get from (8.13) with $M = L < N_{2,n}$

$$\sum_{m=-L+1}^{0} e_n(x + m\alpha)K(x + m\alpha) \approx \widehat{K} q_n^{-n} N_n \int_{q_n x - L/N_n}^{q_n x} \widetilde{e}_{n,\delta_n}(t)dt \geq \widehat{K} 0.5 L q_n^{-n}.$$

Then, since $\widehat{K} > 2$, we conclude from (8.11) and (8.14) that

$$\Sigma_x(-L) > 0.1 L q_n^{-n} > \sqrt{L}.$$

This gives $(\mathcal{C}_2 a)$.

To verify $(\mathcal{C}_2 b)$, notice that for $x \in I_{2,n}$ and any $m \in [-N_{2,n}, N_{2,n}]$ we have $q_n x + m q_n \alpha \in [5/16 - 0.02, 5/16 + 0.02]$. Then

$$e_n(x + m\alpha) = q_n^{-n} \widetilde{e}_{n,\delta_n}(q_n x + m q_n \alpha) \geq 0.$$

By (8.15), we have $(\mathcal{C}_2 b)$, i.e.,

$$\Sigma_x(k, k') \leq A \quad \text{for all} \quad -N_{2,n} \leq k \leq k' \leq N_{2,n}.$$

To verify $(\mathcal{C}_2 c)$, notice that for L as above we have

$$|L\alpha| = \frac{L}{q_n}|q_n \alpha| < \frac{L}{q_n} q_n^{-n^6} < L^{-n^3 - 1}.$$

Hence, for any $k \in [-N_{2,n}, N_{2,n}]$ we have

$$|\mathfrak{p}(x + k\alpha + L\alpha) - \mathfrak{p}(x + k\alpha)| \leq \|\mathfrak{p}\|_1 |L\alpha| \leq \|\mathfrak{p}\|_1 L^{-n^3 - 1} < L^{-n^3}.$$

□

Lemma 8.8. *For any n sufficiently large, we have:*

$$U_{3,n} \subset \mathcal{C}_3(\mathfrak{p}, N_{3,n}, 1/n).$$

Proof. Let $x \in I_{3,n}$ be fixed (the same argument holds for all $x \in U_{3,n}$). Define $Q = q_n$, and take for the numbers $u, v, w_\pm, u', v', w'_\pm$ ($v, v' \in [0.3, 0.4]$) the following choice

$$u = \frac{1}{4} - xq_n, \quad v = \frac{3}{8} - xq_n, \quad u' = \frac{1}{4} + xq_n, \quad v' = \frac{3}{8} + xq_n,$$
$$w_\pm = v \pm \varepsilon, \quad w'_\pm = v' \pm \varepsilon,$$

where we fix $\varepsilon = \frac{1}{n}$. Assume without loss of generality that for each of the numbers introduced above, its product with N_n is an integer that is a multiple of q_n. Let $A > 0$ be as in Proposition 8.5, and assume that q_n is sufficiently large to satisfy

$$e^{e^A} < Q < N_n^{1/2}.$$

The proof of $(\mathcal{C}_3 a)$ is almost the same as the proof of $(\mathcal{C}_1)$ in Lemma 8.6. Namely, we have $x + vN_n\alpha = x + v/q_n + \mathcal{O}(N_n^{-1}) = 3/(8q_n) + \mathcal{O}(N_n^{-1})$;

$$\Sigma_x(vN_n, w_+N_n) = \sum_{m=1}^{(w_+-v)N_n} \ln \mathfrak{q}\left(x + (vN_n + m)\alpha\right) - \ln \mathfrak{p}\left(x + (vN_n + m)\alpha\right) \approx$$
$$\sum_{m=1}^{\varepsilon N_n} \ln \mathfrak{q}\left(3/(8q_n) + m\alpha\right) - \ln \mathfrak{p}\left(3/(8q_n) + m\alpha\right) = \Sigma_{3/(8q_n)}(\varepsilon N_n).$$

Notice that $3/(8q_n) \in I_{1,n}$, so the analysis of the latter sum is analogous to that of Lemma 8.6. Let us repeat the argument. The sum above is estimated using formula (8.12). Since $1/n^2 \ll \varepsilon$, it follows from the definition of $\widetilde{e}_{n,\delta_n}$ that it is negative on most of the interval of integration $[\frac{3}{8}, \frac{3}{8} + \varepsilon]$. Moreover, on all the interval, if $\widetilde{e}_{n,\delta_n}(t) < 0$ then $\widetilde{e}_{n,\delta_n}(t) \le \sin 8\pi t$. Hence

$$\sum_{m=1}^{\varepsilon N_n} K(3/(8q_n) + m\alpha) e_n(3/(8q_n) + m\alpha) \approx \widehat{K} q_n^{-n} N_n \int_{3/8}^{3/8+\varepsilon} \widetilde{e}_{n,\delta_n}(t)\, dt$$
$$< \frac{\widehat{K}}{2} q_n^{-n} N_n \int_{3/8+\varepsilon/2}^{3/8+\varepsilon} \sin(8\pi t)\, dt < -0.001 \widehat{K} q_n^{-n} N_n \varepsilon^2,$$

$|R(x, \varepsilon N_n)| \le 4\kappa^{-2}\varepsilon N_n q_n^{-2n}$, and A is independent of N_n. Hence, by (8.10), and since $\widehat{K} \ge 2$

$$\Sigma_x(vN_n, w_+N_n) > 0.002 q_n^{-n} N_n \varepsilon^2 - A - 4\kappa^{-2}\varepsilon N_n q_n^{-2n} > N_n^{1/2}$$

for N_n sufficiently large. The remaining three estimates of this item are proved in the same way.

To verify $(\mathcal{C}_3 b)$, notice that, by the definition of u' and v we have:

$$x - u'N_n\alpha = x - u'/q_n + \mathcal{O}(N_n^{-1}) = -1/(4q_n) + \mathcal{O}(N_n^{-1}),$$
$$x + vN_n\alpha = x + v/q_n + \mathcal{O}(N_n^{-1}) = 3/(8q_n) + \mathcal{O}(N_n^{-1}).$$

Hence, for all $m \in [-u'N_n, vN_n]$ we have $e_n(x + m\alpha) \ge 0$. By (8.15), we have the first part of $(\mathcal{C}_3 b)$:

$$\Sigma_x(k, k') \le A \quad \text{for all} \quad -u'N_n \le k \le k' \le vN_n.$$

The second part of $(\mathcal{C}_3 b)$ is verified in the same way using formula (8.16).

It remains to verify $(\mathcal{C}_3 c)$. For $k \in [-v'N_n/Q, v'N_n/Q]$, take for $\bar{\Sigma}(k)$ the sums $\bar{\Sigma}_x(k)$ and let $B_x(M) := \Sigma_x(M) - \bar{\Sigma}_x(M)$. To verify (5.2) notice that for each $l \in [-N_n/Q, N_n/Q]$ we have

$$|lQ\alpha| < \frac{1}{Q}.$$

Therefore, since $\bar{\Sigma}_x(M) = \ln g(x+\alpha) - \ln g(x+(M+1)\alpha)$

$$|\bar{\Sigma}_x(M) - \bar{\Sigma}_x(M+lQ)| \leq 2\frac{\|\ln g\|_{C^1}}{Q} < Q^{-1/2}$$

if Q is sufficiently large.

Next, we prove (5.3). For each $m \in [-u'N_n, uN_n]$ we have: $x + m\alpha \in [-\frac{1}{4q_n}, \frac{1}{4q_n}]$, and hence $e_n(x+m\alpha) = 0$. This implies that $\mathfrak{p}(x+m\alpha) = \bar{\mathfrak{p}}(x+m\alpha)$, and

$$\Sigma_x(M) = \bar{\Sigma}_x(M) \quad \text{for all} \quad M \in [-u'N_n, uN_n].$$

For $m \in [-u'N_n, vN_n]$ we have: $x+m\alpha \in [-\frac{1}{4q_n}, \frac{3}{8q_n}]$, and hence $e_n(x+m\alpha) \geq 0$. Then (8.15) implies, in particular, that that for $M \in [0, vN_n]$ we have

$$\Sigma_x(M) \leq \bar{\Sigma}_x(M).$$

For $m \in [-v'N_n, uN_n]$ we have $e_n(x+m\alpha) \leq 0$, which implies the second part of (5.3) by (8.16). This completes the proof of $(\mathcal{C}_3 c)$. □

Proof of Proposition 8.3. Putting together Lemmas 8.6, 8.7, 8.8 immediately yields Proposition 8.3. □

9. Proofs of the main Theorems

9.1. **Proof of Theorem A.** By Theorem 8.1, for any $\mathfrak{p} \in S$, for almost every $x \in \mathbb{T}$, there are strictly increasing sequences of numbers $N_{j,n}$, such that for all $j = 1, 2, 3$, $n \in \mathbb{N}$ we have

$$x \in \mathcal{C}_j(\mathfrak{p}, N_{j,n}, 1/n).$$

For $j = 1$, we have that $\mathrm{p}_i = \mathfrak{p}(x+i\alpha)$ satisfies condition $\mathcal{C}_1(N_{1,n})$. Hence Proposition 5.1, implies Theorem A(a) for $T = e^{\sqrt{N_{1,n}/4}}$.

For $j = 2$, we have that $\mathrm{p}_i = \mathfrak{p}(x+i\alpha)$ satisfies condition $\mathcal{C}_2(N_{2,n}, 1/n)$.

Hence Proposition 5.2, implies Theorem A(b) for $T = N_{2,n}$ and $\varepsilon = 1/n$.

Let $j = 3$. The function $\mathrm{p}_i = \mathfrak{p}(x+i\alpha)$ satisfies $\mathcal{C}_3(N_{3,n}, 1/n)$. Let $T = N_{3,n}^4$. Then Proposition 5.3 implies that for some $v, v' \in [0.3, 0.4]$ it holds

$$\begin{cases} \mathbb{P}_x\left(Z_T \in [vN - \frac{N}{n}, vN + \frac{N}{n}]\right) > 0.1, \\ \mathbb{P}_x\left(Z_T \in [-v'N - \frac{N}{n}, v'N + \frac{N}{n}]\right) > 0.1. \end{cases}$$

which proves Theorem A(c) with $t_k = T = N_{3,n}^4$, $b_k = vN$, $b'_k = v'N$ and $\varepsilon_k = \frac{1}{n}$. □

9.2. **Proof of Corollary B.** If the walk had an absolutely continuous invariant measure, then, by Proposition 3.1 $\ln \mathfrak{q} - \ln \mathfrak{p}$ would be a coboundary and so the distribution of ergodic sums $\Sigma_x(N)$ would be tight as x is uniformly distributed on $\mathbb{T}$ and $N \to \infty$.

However, Lemma 8.6 shows that if α is Liouville, then for a dense G_δ set of functions $\mathfrak{p} \in \mathcal{P}$ there exists a sequence $\{N_j\}$ such that $|\Sigma_x(N_j)| > \sqrt{N_j}$ for a set of x of measure 0.01. Since such behavior is incompatible with tightness, the result follows. □

9.3. **Proof of Theorem D.** Define

$$\mathcal{A}_{m,n} = \Big\{ \alpha \in \mathbb{R} : \forall \mathfrak{p} \in \mathcal{P} \ \text{ with } \|\mathfrak{p}\|_4 \le n, \exists \sigma \text{ such that } \forall x \in \mathbb{T}, \forall z \in [-n, n], \\ \Big| \mathbb{P}_x(Z_t < \sigma\sqrt{t} z) - \Phi(z) \Big| < \frac{1}{n} \quad \text{for all } \ t \in [m, e^m] \Big\}.$$

The set $\mathcal{D} = \cap_{n \ge 1} \cup_{m \ge 1} \mathcal{A}_{m,n}$ satisfies the conclusion of the theorem. The sets $\mathcal{A}_{m,n}$ are open hence $\mathcal{D}$ is a G^δ set.

By [25], $\mathcal{D}$ contains the Diophantine numbers. Hence $\mathcal{D}$ is a G^δ-dense set. □

9.4. **Proof of Theorem E.** We prove (a) and (b). The proof of (a') and (b') is similar.

Let $\lambda(x) = \dfrac{\mathfrak{q}(x)}{\mathfrak{p}(x)}$. To fix our notation, we assume that $\int \ln \lambda(x) dx = -c < 0$ so that the walk tends to $+\infty$. The case $\int \ln \lambda(x) dx > 0$ then follows by replacing x by $-x$.

Let

$$u(x) = 1 + 2 \sum_{k=0}^{\infty} \prod_{j=0}^{k} \lambda(x - j\alpha).$$

The drift coefficient of an asymmetric walk in Theorem 2.3 is given by the first integer b_n such that

$$\sum_{k=0}^{b_n} u(x + k\alpha) \ge n \tag{9.1}$$

(see formula (1.6) and Theorem 4 of [11]).

Lemma 9.1. *Let $d, M > 0$ and q such that $q > e^{e^{d+M}}$. If V is a trigonometric polynomial of degree d, and all the coefficients of V are bounded by M, then for any $x \in \mathbb{T}$*

$$\left| \sum_{j=0}^{q-1} e^{V(x+j/q)} - q \int_{\mathbb{T}} e^{V(\theta)} d\theta \right| < e^{-q}.$$

Proof. First, expand $e^{V(\cdot)} = \displaystyle\sum_{k=0}^{N} \frac{V^k}{k!} + \varepsilon_N$, where $N := [2q/\ln q]$, so that the error ε_N is small compared to e^{-q}. On the other hand, the polynomials V^l that we keep are all of degree strictly less than q, hence $\displaystyle\sum_{j=0}^{q-1} V^l \left(x + \frac{j}{q} \right) = q \int V^l$. The lemma follows. □

Now suppose $\bar{\mathfrak{p}}$ is fixed such that

$$\ln \bar{\lambda} = \ln \bar{\mathfrak{q}} - \ln \bar{\mathfrak{p}} = -c + g(x) - g(x-\alpha) \tag{9.2}$$

where g a trigonometric polynomial. In a similar fashion as in the proof of Theorem A, Theorem 2.3 will follow from a G^δ argument if we prove that $\bar{\mathfrak{p}}$ can be perturbed into $\mathfrak{p}$ so that (i) and (ii) hold for $\mathfrak{p}$ for an arbitrarily large t_n with $\mathcal{A}_n$ and $\mathcal{B}_n$ a union of intervals.

Observe that if (9.2) holds then

$$\bar{u}(x) = 1 + 2\sum_{k=0}^{\infty} e^{-c(k+1)} e^{V_k(x)} \tag{9.3}$$

where $V_k(x) = g(x) - g(x-k\alpha)$.

Applying Lemma 9.1 to each term in (9.3) (note that the norm of V_k is bounded uniformly in k) we conclude that if q_n is sufficiently large then

$$\left| \sum_{j=0}^{q_n-1} \bar{u}\left(x + \frac{j}{q_n}\right) - q_n \int_{\mathbb{T}} \bar{u}(\theta) d\theta \right| < e^{-q_n}.$$

On the other hand, (8.1) tells us that there is an integer p_n such that

$$\left| \alpha - \frac{p_n}{q_n} \right| \le q_n^{-n^4}.$$

Thus

$$\left| \sum_{j=0}^{q_n-1} \bar{u}\left(x + j\alpha\right) - \sum_{j=0}^{q_n-1} \bar{u}\left(x + \frac{p_n j}{q_n}\right) \right| \le \|\bar{u}\|_{C^1} q_n^{1-n^4}.$$

Observe that

$$\sum_{j=0}^{q_n-1} \bar{u}\left(x + \frac{p_n j}{q_n}\right) = \sum_{j=0}^{q_n-1} \bar{u}\left(x + \frac{j}{q_n}\right),$$

since as j changes from 0 to q_{n-1} the set $p_n j$ goes over all possible residues mod q_n. Therefore for every x in $\mathbb{T}$ we have

$$\left| \sum_{j=0}^{q_n-1} \bar{u}(x + j\alpha) - q_n \int_{\mathbb{T}} \bar{u}(\theta) d\theta \right| < q_n^{-n^3}.$$

Dividing an orbit of length $q_n^{n^2}$ into pieces of length q_n, we obtain

$$\left| \sum_{j=0}^{q_n^{n^2}} \bar{u}(x + j\alpha) - q_n^{n^2} \int_{\mathbb{T}} \bar{u}(\theta) d\theta \right| < q_n^{-n^3/2}.$$

So taking

$$t_n := q_n^{n^2} \int_{\mathbb{T}} \bar{u}(\theta) d\theta,$$

we get that for every $x \in \mathbb{T}$

$$\bar{b}_{t_n}(x) = q_n^{n^2} + O(1).$$

Now we let $\mathfrak{p}(\theta) = \bar{\mathfrak{p}}(\theta) + e_n(\theta)$ with e_n satisfying

(a) $\|e_n\|_{C^n} \le 2^{-n}$;
(b) $e_n(\theta) = 0$ for $\{q_n\theta\} \in [0, 0.85]$;
(c) $e_n(\theta) = q_n^{-n-1}$ for $\{q_n\theta\} \in [0.86, 0.99]$.

Define

$$\mathcal{A}_n = \{x \in \mathbb{T} : \{q_n x\} \in [0, 0.84]\}, \quad \mathcal{B}_n = \{x \in \mathbb{T} : \{q_n x\} \in [0.86, 0.98]\}.$$

Note that for $j \in [0, 2q_n^{n^2}]$ we have $q_n(x + j\alpha) = q_n x + jp_n + O\left(q_n^{2-n^4}\right)$. Thus for $x \in \mathcal{A}_n$, $u(x + j\alpha) = \bar{u}(x + j\alpha)$ for every $j \in [0, 2q_n^{n^2}]$. Hence, letting $b_n = q_n^{n^2}$ we get $b_{t_n}(x) = \bar{b}_{t_n}(x) = b_n + O(1)$.

On the other hand, for $x \in \mathcal{B}_n$, we have that that $u(x + j\alpha) \le (1 - q_n^{-n-2})\bar{u}(x + j\alpha)$ for every $j \in [0, 2q_n^{n^2}]$. Hence

$$\sum_{j=0}^{b_n} u(x + j\alpha) \le (1 - q_n^{-n-2}) \left[\sum_{j=0}^{b_n} \bar{u}(x + j\alpha)\right] = t_n + O(1) - \frac{t_n}{q_n^{-n-2}} < t_n - t_n^{0.95}.$$

Therefore for such x

$$\sum_{j=b_n+1}^{b_n(x)} u(x + j\alpha) \ge t_n^{0.95}$$

and so $b_n(x) > b_n + \dfrac{t_n^{0.95}}{\max_\theta u(\theta)} > b_n + {t_n}^{0.9}$. □

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