Sample final problems.

(1) Consider a random walk \( S_n = X_1 + X_2 + \ldots X_n \) where \( X_j \) are iid taking values \(-1\) and \(2\) each with probability \(1/2\).
(a) Find \( \theta \) such that \( \theta S_n \) is a martingale.
(b) Find expected number to visits to \(-1\).

(2) Consider a Markov chain with states \(\{1, 2, 3, 4\}\) and transition matrix
\[
\begin{pmatrix}
0 & 1/2 & 0 & 1/2 \\
1/3 & 0 & 2/3 & 0 \\
0 & 2/3 & 0 & 1/3 \\
1/2 & 0 & 1/3 & 0
\end{pmatrix}.
\]
Find the expected time till the first visit to \(3\) given that the state starts at \(1\).

(3) Let \( W(t) \) be a standard Brownian Motion. Find the best linear prediction of \( W(2) \) based on \( W(1) \) and \( W(3) \).

(4) Let \( X_n \) be a weakly stationary sequence having twice differentiable spectral density. Show that the series
\[
\sum_{n=1}^{\infty} \frac{X_n}{n}
\]
converges in the mean square sense.

(5) Alice and Bob arrive to the bus stop at 6PM. Alice waits for bus A while Bob waits for bus B. Both buses start running at 6AM. Interarrival times for bus A have uniform distribution on \([0, 10]\) min and Interarrival times for bus B have exponential distribution on with mean 10 min. Suppose that interarrival times are iid independent from each other. Compute (approximately) the probability that Alice will leave before Bob.

(6) A coin is tossed \(n\) times. By a run of heads we mean a maximal uninterrupted sequence of heads. Let \( R_n \) be the number of runs of heads up to time \(n\). Find \(a_n\) and \(b_n\) such that \( R_n - a_n/b_n \) has a non-trivial limiting distribution and compute that distribution.

(7) Consider \( M(1 - \varepsilon)/M(1)/1\)-queue. Let \( L_\varepsilon \) be the queue length at equilibrium. Find \(a_\varepsilon\) and \(b_\varepsilon\) such that \( L_\varepsilon - a_\varepsilon/b_\varepsilon \) has a non-trivial limiting distribution and compute that distribution.

(8) Consider a queue with interarrival times \(X_n\) being Uniform(0,1) and service times being exponential with parameter 1. Find the conditional distribution of \(X_n\) given that no customer was served during that time.

(9) Let \( W(t) \) be a standard Brownian Motion. Let \( M(t) = \max_{[0,t]} W(s) \).
(a) Find \( P(M(t) > s | W(t) = u) \).
(b) Find the distribution of \( M(t) - W(t) \).

(10) Consider a diffusion process \( X_t \) with drift \( a(x) = |x| \) and diffusion coefficient \( b(x) = x^2 + 1 \).
(a) Find a non-constant function \( \phi \) such that \( \phi(X_t) \) is a martingale.
(b) Let \( M = \min_{[0,\infty)} X_t \). Find the distribution of \(M\).