Review problems.

(1) How many fields there exist on a 5 point space?

(2) $3n$ books comprising $n$ three volume treatises are put randomly on the shelf. Find the probability that at least one treatise is kept together in the correct order.

(3) Show that the closed discs generate the Borel sigma-field on the plane.

(4) Let $X_1, X_2 \ldots X_n$ be iid and $N$ be an $\mathbb{N}$-valued random variable independent of $X_i$’s. Show that if $X_1$ and $N$ are integrable then

$$E \left( \sum_{j=1}^{N} X_j \right) = E(N)E(X_1).$$

(5) Given a sequence $p_n$ such that $0 < p_n < 1$ and $p_n \to 0$ show that there are sets $A_n$ defined on some probability space such that $P(A_n) = p_n$ and $P(\lim\inf A_n) = 0$.

(6) Let $X_1, X_2 \ldots X_n$ be iid integer valued random variables. Assume that the distribution of $X_1$ is symmetric in the sense that $P(X_1 = m) = P(X_1 = -m)$. Let $S_n = \sum_{j=1}^{n} X_j$. Show that

$$P(\lim_{n \to \infty} S_n = +\infty) = 0$$

(7) Find all measures on $(0, \infty)$ which are finite on compact sets and are invariant with respect to dilations.

(8) Let $f_1, f_2 \ldots f_n \ldots$ be measurable functions. Let

$$S_n(\omega) = \text{Card}(i < j < k \leq n : f_i < f_j + f_k).$$

Let $C = \{ \frac{S_n}{n^3} \text{ converges} \}$. Show that

$$f(\omega) = 1_C \times \lim_{n \to \infty} \frac{S_n}{n^3}$$

is measurable.

(9) Let $\Omega = \{0, 1, 2\}^\mathbb{N}$. Let $T : [0,1] \to \Omega$ map $x$ to its ternary representation. That is if $T(x) = \{\omega_j\}$ then $x = \sum_{j=1}^{\infty} \frac{\omega_j}{3^j}$. Let $P$ be the measure on $\Omega$ such that $\omega_j$ are independent and $P(\omega_j = 0) = \frac{1}{2}$, $P(\omega_j = 1) = \frac{1}{3}$, $P(\omega_j = 2) = \frac{1}{6}$. Find $PT^{-1}([0,1/2])$.

(10) Let $F$ and $G$ be distribution functions. Show that $\frac{F+G}{2}$ is a distribution function.
(11) Let $X_1, X_2, \ldots, X_n, \ldots, Y_1, Y_2, \ldots, Y_n, \ldots,$ be independent $X_i$ taking values 1 and 0 with probabilities $p$ and $1-p$ respectively and $Y_i$ be uniformly distributed on $[0,1]$. Let $Z_i = X_i - Y_i$ and $M_n = \max(Z_1, Z_2, \ldots, Z_n)$. Find the limiting distribution of $n(1-M_n)$.

(12) Let $X_1, X_2, \ldots, X_n$ be iid with continuous distribution function $F(x)$. Let $Y_1 < Y_2 < \ldots < Y_n$ be $X_j$’s rearranged from the smallest to largest. Find the distribution of $Y_k$.

(13) Let $B(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1}dx$. $(u, v > 0)$ We say that $X \sim B(u, v)$ if $X$ has density $B(u, v)$ for $0 < x < 1$ and zero elsewhere. Compute $E(X^k)$.

**Hint.** Use identity $B(u, w) = \frac{\Gamma(u)\Gamma(w)}{\Gamma(u+w)}$, proven while computing the convolution of Gamma distributions.

(14) Let $(X_1, X_2, \ldots, X_n)$ be uniformly distributed in the ball $X_1^2 + X_2^2 + \cdots + X_n^2 \leq n$. Prove that the joint distribution of $(X_1, X_2, X_3)$ converges to a three dimensional Gaussian distribution.

(15) (a) Suppose that $S_n$ are random variables such that $F_{\sqrt{n}S_n}(x) \Rightarrow F(x)$. Find the limiting distribution of $n^{3/2}(\sin S_n - S_n)$.

(b) Give an example of $X_n, Y_n, X, Y$ such that $F_{X_n} \Rightarrow F_X$, $F_{Y_n} \Rightarrow F_Y$ but $F_{X_n+Y_n} \not\Rightarrow F_{X+Y}$.

(c) Show that if $X_n \rightarrow X$ in probability and $Y_n \rightarrow Y$ in probability then $X_n + Y_n \rightarrow X + Y$ in probability.

(16) Let $|Z| < 1$ show that

$E(\ln(1 + Z)) = -\sum_{j=1}^{\infty} \frac{(-1)^j E(Z^j)}{j}$

(this limit maybe $-\infty$ that is it is possible that $E([\ln(1 + Z)]^+) < \infty$ and $E([\ln(1 + Z)]^-) = \infty$ and the same happens with the right hand side).

(17) Let $X$ be a positive random variable such that $X^z$ be integrable for $0 \leq z < R$. Show that $\phi(z) = E(X^z)$ is differentiable and find its derivative.

(18) Show that if $X_n \rightarrow X$ and $EX_n^2 \rightarrow EX^2$ then $EX_n \rightarrow EX$. 
(19) Find all distribution on the plane having smooth positive density and such that for all \( \theta \), \( X \cos \theta + Y \sin \theta \) and \( -X \sin \theta + Y \cos \theta \) are independent (that is the marginals are independent and the same is true after applying an arbitrary rotation).

(20) Show that there is a constant \( c \) such that if \( X \) is a positive random variable with finite second moment and distribution function \( F \) then
\[
EX^2 = c \int_0^\infty x(1 - F(x))dx.
\]

(21) Let \( X \) be a positive random variable with moment generating function \( M_X(s) \) and \( Y \) be a random variable independent of \( X \) and having \( \text{Exp}(1) \) distribution. Compute \( P(Y > X) \).

(22) Show that if \( F \) has no atoms then for any \( G \) the convolution \( F \ast G \) has no atoms.

(23) Let \( X_1, X_2 \ldots X_n \) be iid having uniform distribution on \([0, 1] \). Show that
\[
\sup_{\alpha \in [1, \infty)} \left| \frac{1}{n} \sum_{j=1}^n X_j^\alpha - \frac{1}{\alpha + 1} \right| = 0
\]
almost everywhere.

(24) Compute an expected number of records in a random permutation of length \( n \).

(25) Each cereal box contains one of \( n \) pictures independently of one another. Joe buys boxes one-by-one. Let \( T_j \) be the first time he gets \( j \) different pictures.

(a) Show that \( T_{j+1} - T_j \) are independent and have geometric distribution with parameter \( \frac{n-j}{n} \).

(b) Prove that there exist limits
\[
c_1 = \lim_{n \to \infty} \frac{ET_n}{n \ln n}, \quad c_2 = \lim_{n \to \infty} \frac{\text{Var}(T_n)}{n^2}
\]
(c) Show that \( \frac{T_n}{n \ln n} \) converge to \( c_1 \) in probability.

(26) Let \( \phi(x) \) be a function on \( \mathbb{R} \) such that \( \phi'' \geq 2c \). Show that
\[
E(\phi(X)) \geq \phi(E(X)) + c\text{Var}(X).
\]

(27) Show that \( |E(X^3)| \leq (EX^8)^{1/4}(EX^{4/3})^{3/4} \).

(28) \( n \) couples and \( n \) single people are seated randomly at a round table. Let \( Z_n \) be a number of couples seating together. Find \( E(Z_n) \), \( \text{Var}(Z_n) \).
(29) Let $X_1, X_2 \ldots X_n$ be iid and $N$ be an $\mathbb{N}$-valued random variable independent of $X_i$’s. Assume that the moment generating functions $M_X(s)$ and $M_N(x)$ are defined for all $s \in \mathbb{R}$. Let $S = \sum_{j=1}^{N} X_j$. Show that the moment generating function of $S$ is $M_S(s) = M_N(\ln M_X(s))$.

(30) Let $X_j$ be iid with $E|X_1| < \infty$. Given a sequence $m_n$ such that $m_n \to \infty$ let

$$A_n = \frac{1}{m_n} \sum_{j=n}^{n+m_n-1} X_j.$$

(a) Show that $A_n \to E(X_1)$ in probability.
(b) Demonstrate by an example that $A_n$ need not converge to $E(X_1)$ almost everywhere.

(31) Suppose that $i$th light bulb burns for time $X_i$ and then stays burned out for time $Y_i$ before being replaced. Let $R_t$ be the amount of time in $[0, t]$ we have a working bulb. Assuming that $(X_i, Y_i)$ are iid and integrable show that

$$\frac{R_t}{t} \to \frac{E(X_1)}{E(X_1) + E(Y_1)}$$

almost everywhere.