SAMPLE PATH PROPERTIES OF RANDOM TRANSFORMATIONS.

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1. Models.

Let $M$ be a smooth compact manifold of dimension $N$ and $X_0, X_1 \ldots X_d$, $d \geq 2$, be smooth vectorfields on $M$.

(I) Let $\{w_j\}_{j=-\infty}^{+\infty}$ be a sequence of independent random variables such that $w_j$ is a pair $w_j = (\xi_j, \eta_j)$ uniformly distributed on the set $[-1, 1] \times \{1 \ldots d\}$. Let $\phi_k(t)$ denote time $t$ map of the flow generated by $X_k$ and let

$$f_j = \phi_{\eta_j}(\xi_j), \quad F_{m,n} = f_{n-1} \circ \ldots \circ f_{m+1} \circ f_m, \quad F_n = F_{0,n}.$$ 

(II) Let $w_1(t) \ldots w_k(t)$ be independent Brownian motions. Consider Stratanovich differential equation

$$dx_t = X_0(x)dt + \sum_{k=1}^{d} X_k(x) \circ dw_k(t).$$

Let $F_{s,t}$ be the flow of diffeomorphisms generated by (1) and $F_t = F_{0,t}$.

Definition. We say that either (I) or (II) on satisfy condition (H) on $M$ if the Lie algebra generated by $X_1 \ldots X_d$ is $TM$.

We assume the following condition

(A) Systems induced by either (I) or (II) on both $M \times M \times M - \text{diag}$ and on Grassmann bundles over $M$ satisfy (H).

Let $\lambda_1 \geq \lambda_2 \geq \lambda_N$ be Lyapunov exponents of our system (given $x$ Lyapunov exponents exist for almost all $w$ and are independent of $x$ see e.g [7]).

Theorem 1. ([11, 4]) (a) Either all exponents coincide or all exponents are different. In the former case $F_t$ preserve a smooth Riemannian metric.

(b) $\sum_{k=1}^{N} \lambda_k \leq 0$ and if $\sum_{k=1}^{N} \lambda_k = 0$ then $F_t$ preserve a smooth volume form.

We impose the second restriction

(B) $\lambda_1 \neq 0$. 
Below we consider only the systems satisfying conditions (A) and (B).

Remark. If \( \{X_j\} \) preserve volume then, by Theorem 1, (B) holds except on a set of infinite codimension. It seems that (B) holds generically also in the dissipative setting but I am not aware of the proof.

2. SRB measures.

Let \( m \) denote Lebesgue measure on \( M \).

**Theorem 2.** (a) The following limits exist almost surely
\[
\nu_n = \lim_{k \to -\infty} F_{k,n} m.
\]
\( \{\nu_n\} \) are invariant in the sense that \( F_{k,n} \nu_k = \nu_n \).

(b) ([8]) Given \( \alpha > 0 \) there exists \( \theta < 1, C = C(w) \) such that
\[
\forall A \in C^\alpha(M) \quad \left| \int A(F_n(x)) dm(x) - \nu_n(A) \right| \leq C(w) ||A||_{C^\alpha(M)} \theta^n.
\]

(c) ([13]) If \( \lambda_1 < 0 \) then there is a random point \( x = x(w) \) such that \( \nu_n = \delta_{F_n x(w)} \). Moreover for Lebesgue almost all \( x \) \( d(F_n x, F_n x(w)) \to 0 \) exponentially fast.

(d) ([15]) If \( \lambda_1 > 0 \) then \( \lambda_1 > 0 \) then \( \nu_n \) has positive Hausdorff dimension. Namely let \( \Lambda_1 = \sum_{k=1}^t \lambda_k \). Let \( K \) be the largest number such that \( \Lambda_k \geq 0 \). Let \( D \) be equal to \( N \) if \( K = N \) and \( D = K - (\Lambda_K/\lambda_{k+1}) \) otherwise. Then \( \text{HD}(\nu_n) = D \).

(e) ([8]) If \( \lambda_1 > 0 \) then for all \( A \in C^3(M) \) there exists \( D(A) \) such that for almost all \( w \)
\[
m \left\{ \frac{\sum_{j=0}^{n-1} [A(F_j x) - \nu_j(A)]}{D \sqrt{n}} < s \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-\xi^2/2} d\xi.
\]

**Question 1.** What happens if \( \lambda_1 = 0 \)? (see [5] for partial results).

**Question 2.** Can the formula of part (d) be generalized to escape problem from nice domains?

3. Non-typical points.

Let \( \mathcal{E}(A,w) = \{ x : \frac{1}{n} \sum_{j=0}^{n-1} A(F_j x) - \nu_j(A) \neq 0 \} \).

**Theorem 3.** ([10]) If \( A \neq \text{Const} \) then
\begin{enumerate}
\item if \( \lambda_1 > 0 \) then \( \text{HD}(\mathcal{E}(A)) = N \);
\item if \( \lambda_1 < 0 \) then \( \text{HD}(\mathcal{E}(A)) < N \).
\end{enumerate}
Question 3. If \( \lambda_1 > 0 \) what can be said about

\[
\delta_r = \text{HD} \left \{ x : \frac{1}{n} \sum_{j=0}^{n-1} A(F_j x) - \nu_j(A) > r \right \}
\]

Let \( \mathcal{E} = \bigcap_{A \in C(M)} \mathcal{E}(A) \).

Question 4. What can be said about \( \text{HD}(\mathcal{E}) \) in case \( \lambda_1 < 0 \)?

Question 5. Let \( f \) be a volume preserving diffeomorphism. Suppose that \( \lambda_1 > 0 \). Is it true that \( \text{HD}(\mathcal{E}) = N \)? More generally, let \( T \) be an ergodic automorphism of a probability space \((\Omega, \mu)\) and

\[
F_n(\omega) = f(T^{n-1}(\omega)) \circ \cdots \circ f(T\omega) f(\omega)
\]

where \( f(\omega) \) preserve volume. Suppose that \( \lambda_1 > 0 \). Is it true that \( \text{HD}(\mathcal{E}) = N \)?

4. Stable lamination

If \( \nu \) is a measure on \( M \) let

\[
I_s(\nu) = \int \int \frac{d\nu(x)d\nu(y)}{d^s(x,y)}.
\]

Theorem 4. (a) ([8]) If \( \nu \) is a measure on \( M \) with \( I_s(\nu) < \infty \) for some \( s \) then for almost all \( w \) \( \int A(F_n x) d\nu(x) - \nu_n(A) \rightarrow 0 \) exponentially fast.

(b) Stable lamination is transitive on \( M \).

(c) ([9]) If \( M = \mathbb{T}^N \) and \( \lambda_j \neq 0 \) for all \( j \) then the lift of stable lamination to \( \mathbb{R}^N \) is transitive.

Question 6. For which over manifolds the lift of the stable lamination to the universal cover must be transitive?

5. Tools.

The following results play important role in the proofs.

5.1. Two-point motion. Let \( \delta > 0 \) be small. Denote

\[
\Omega = \{(x, y) \in M \times M : d(x, y) \geq \delta \}
\]

Let

\[
\tau(x, y) = \min \{ n \in \mathbb{N} : d(F_n(x), F_n(y)) \geq \delta \}.
\]

Theorem 5. (a) ([6]) If \( \lambda_1 < 0 \) then there exists \( \theta < 1 \) such that for all \( (x, y) \in M \times M \)

\[
\mathbb{P}\{\tau(x, y) < \infty \} < \theta.
\]

(b) ([6]) If \( \lambda_1 > 0 \) then there exists \( r > 1 \) such that for all \( |\xi| < r \) for all \( (x, y) \in \Omega \)

\[
\mathbb{E}(\xi^{\tau(x,y)}) \leq \text{Const}
\]
The return to $\Omega$ process $z_{\tau_n}$ is exponentially mixing in the sense that there exists a measure $\mu$ on $\Omega$ and a number $\theta < 1$ such that for all $A \in C(M)$ for all $(x, y) \in \Omega$

$$|\mathbb{E}_{(x,y)}(A(z_{\tau_n}) - \mu(A)\rho^n| \leq \text{Const}(\rho\theta)^n||A||C(M)).$$

Here $\rho = 1$ if $\lambda_1 > 0$ and $\rho < 1$ if $\lambda_1 < 0$.

5.2. Hyperbolic times. Given numbers $K, \alpha$ we call a curve $(K, \alpha)$-smooth if in the arclength parameterization the following inequality holds

$$\left|\frac{d\gamma}{ds}(s_1) - \frac{d\gamma}{ds}(s_2)\right| \leq K|s_2 - s_1|^{\alpha}.$$

Theorem 6. ([10]) Fix $\lambda < \lambda_1$ then $\exists r > 0, \alpha < 1, K > 0$ and $n_0 > 0$ such that for any $(K, \alpha)$-smooth $\gamma$ of length between $\frac{r}{100}$ and $100r$ the following holds. $\forall x \in \gamma$ there is a stopping time $\tau(x)$ such that

(a) $||dF_{\tau}[T\gamma]|| > 100$, $l(F_{\tau}\gamma) \geq r$;

Let $\tilde{\gamma}$ denote a ball of radius $r$ inside $F_{\tau}\gamma$ centered at $F_{\tau}(x)$. Then

(b) $\tilde{\gamma}$ is $(K, \alpha)$-smooth;

(c) $\forall k : 0 \leq k \leq \left[\frac{\tau}{n_0}\right] \forall y_1, y_2 \in \tilde{\gamma} d(F_{\tau,\tau-kn_0}y_1, F_{\tau,\tau-kn_0}y_2) \leq d(y_1, y_2)e^{-\lambda kn_0};$

(d) $\ln \left(||(dF_{\tau}^{-1}[T\gamma])||(y_1) - \ln \left(||(dF_{\tau}^{-1}[T\gamma])||(y_2)\right)\right| \leq \text{Const}d^n(y_1, y_2);$ 

(e) $\mathbb{E}(\tau(x)) \leq C_0; \mathbb{P}(\tau(x) > N) \leq C_1e^{-C_2N}$ where all constants do not depend on $\gamma$.

6. Further questions.

In the models described above the distribution of the point $x_n = F_n(x)$ has smooth component. By contrast in the deterministic case (if $F_n = f^n$) then $x_n$ has $\delta$-distribution. The results described above are either unknown or false for the generic deterministic systems.

Question 7. What can be said in the intermediate cases?

In other words where is the boundary between truly random and almost deterministic behavior? I believe that very little randomness is needed. For example consider the following model. Let $f_1 \ldots f_d$ be smooth diffeomorphisms of $M$ and apply the independently with probabilities $p_1 \ldots p_d$.

Conjecture. The above Markov process is ergodic for generic $f_1 \ldots f_d$ in the following cases

(a) $f_j$ preserve smooth volume;

(b) $f_j$ are close to a given diffeo $f$. 
7. Bibliographical comments.

Properties of SRB measures for random systems are discussed in [13, 14, 15, 18].
Properties of exceptional sets are discussed in [2, 3].
More general classes of random dynamical systems are studied in [1, 11, 16, 17, 18].

References