Midterm 1 solutions.

1. Let A, B and C be sets such that $P(A) = 0.55$, $P(B) = 0.50$, $P(C) = 0.32$, $P(A \cup B) = 0.75$, $P(A \cup C) = 0.70$, $P(B \cup C) = 0.72$, $P(A \cap B \cap C) = 0.05$.

(a) Compute $P(A \cap B)$;

(b) Compute $P(A|B)$;

(c) Compute the probability that a point belongs to exactly one of the sets A, B, and C.

Solution. (a) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.55 + 0.50 - 0.75 = 0.30$.

(b) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.30}{0.50} = \frac{3}{5}$.

(c) Similarly to (a) $P(A \cap C) = P(A) + P(C) - P(A \cup C) = 0.55 + 0.32 - 0.70 = 0.17$ and $P(B \cap C) = P(B) + P(C) - P(B \cup C) = 0.50 + 0.32 - 0.72 = 0.10$. Next $P(A \cap B \cap C') = P(A \cap B) - P(A \cap B \cap C) = 0.25$, $P(A \cap B' \cap C) = P(A \cap C) - P(A \cap B \cap C) = 0.12$, $P(A' \cap B \cap C) = P(B \cap C) - P(A \cap B \cap C) = 0.05$. Thus $P(A \cap B' \cap C') = P(A) - P(A \cap B' \cap C) - P(A \cap B \cap C') - P(A \cap B \cap C) = 0.13$, $P(A' \cap B \cap C') = P(B) - P(A' \cap B \cap C) - P(A \cap B \cap C') - P(A \cap B \cap C) = 0.15$, $P(A' \cap B' \cap C) = P(C) - P(A' \cap B \cap C) - P(A \cap B' \cap C) - P(A \cap B \cap C) = 0.10$ so $P(\text{Exactly one event}) = 0.13 + 0.15 + 0.10 = 0.38$.

2. A class has 20 students.

(a) How many ways are there to choose 3 students for math competition?

(b) How many ways are there to chooose 3 students for science competition so that one student will participate in physics competition, another will participate in chemestry competition and the third will participate in biology competition?

(c) If the math and science competitions above are held on the same day so that no students can participate in both how many ways are there to choose the participants for both competitions?

(d) If the math and science competitions above are held on the different days so that students can participate in both how many ways are there to choose the participants for both competitions?

Solution. (a) Since the students will compete in the same event the answer is $\binom{20}{3} = 1140$.

(b) Since the students will compete different events the answer is $P_{20,3} = \frac{20!}{3!} = 6840$.

(c) We need to divide the students into 5 groups: math competition participants, physics competition participants, chemestry competition participants and non-competing students. So the answer is $\binom{20}{3, 1, 1, 1, 14} = \frac{20!}{3!1!1!1!14!} = 4651200$.

(d) Due to parts (a) and (b) and the product rule the answer is $1140 \times 6840 = 7797600$.

3. In a certain state 40% of adults are democrats, 40% are independents and 20% are republicans. 30% of democrats own guns, 80% of independents own guns and 60% of republicans own guns.

(a) Compute the probability that a randomly chosen adult is democrat and owns a gun.

(b) Compute the probability that a randomly chosen adult owns a gun.

(c) Compute the probability that a randomly chosen gun owner is a democrat.

Solution. Let D, I, R, G denote the event that a randomly chosen adult is a democrat, an independent, a republican, or a owner.

(a) $P(DG) = P(D)P(G|D) = 0.4 \times 0.3 = 0.12$.

(b) $P(G) = P(D)P(G|D) + P(I)P(G|I) + P(R)P(G|R) = 0.4 \times 0.3 + 0.4 \times 0.8 + 0.2 \times 0.6 = 0.56$.

(c) $P(D|G) = \frac{P(DG)}{P(G)} = \frac{12}{56} \approx 0.21$.

4. Next semester John takes 2 classes: math and dancing. He recons that he has 30% chance to get A in math and 70% chance to get A in dancing. Let X be the number of As John will get.

(a) Find the probability mass function of X.

(b) Find $E(X)$.

(c) Find $V(X)$.

Solution. Let M and D denote the probability that John gets A in math and dancing. Then

(a) $P(X = 0) = P(M'D') = 0.7 \times 0.3 = 0.21$, $P(X = 2) = P(MD) = 0.3 \times 0.7 = 0.21$, $P(X = 1) = 1 - P(X = 0) - P(X = 2) = 0.58$.

(b) $EX = 0.21 \times 0 + 0.58 \times 1 + 0.21 \times 2 = 1$.

(c) $EX^2 = 0.21 \times 0 + 0.58 \times 1 + 0.21 \times 4 = 1.42$. $VX = 1.42 - 1 = 0.42$.