STAT400 Midterm 2.

1. *An urn contains 4 red and 4 blue balls.*

(a) If 5 balls are chosen randomly without replacement find the probability that 3 balls will be red and 2 blue.

(b) If 5 balls are chosen randomly with replacement find the probability that 3 balls will be red and 2 blue.

(c) A class contains 10 students. If each student independently choses 5 balls with replacement find the probability that exactly 3 will pick 3 red balls and 2 blue balls.

Solution. (a) The number of red balls has hypergeometric distribution so the answer is $\dfrac{\binom{4}{3}\binom{4}{2}}{\binom{8}{5}} = \dfrac{3}{7}$.

(b) The number of red balls has binomial distribution with probability of success equal to $\frac{1}{2}$ so the answer is $\binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5}{16}$.

(c) Due to part (c) the number of students picking a 3 red balls has binomial distribution with probability of success $\frac{5}{16}$ so the answer is $\binom{10}{3}\left(\frac{5}{16}\right)^3\left(\frac{11}{16}\right)^7$.

2. *Let X be a random variable with density function equal to $3x^2$ if $0 \le x \le 1$ and equal to 0 otherwise.*

(a) Find the cumulative distribution function of X.

(b) Find the median of X.

(c) Find EX and VX.

(d) Let $Y = X^3$. Find the cumulative distribution function of Y.

Solution. (a) Since X only assumes values between 0 and 1 $F_X(x) = 0$ for $x \le 0$ and $F_X(x) = 1$ for $x \ge 1$. If $x \in [0,1]$ then

$$F_X(x) = \int_0^x 3s^2 ds = x^3.$$

(b) By part (a) the median satisfies $m^3 = \frac{1}{2}$ so $m = \left(\frac{1}{2}\right)^{1/3}$. (c) $EX = \int_0^1 3x^2 * x dx = \frac{3}{4}$, $EX^2 = \int_0^1 3x^2 * x^2 dx = \frac{3}{5}$ so $V(X) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$.

(d) As in part (a) $F_Y(y) = 0$ for $y \le 0$ and $F_Y(y) = 1$ for $y \ge 1$. If $y \in [0,1]$ then

$$F_Y(y) = P(X^3 \le y) = P(X \le y^{1/3}) = (y^{1/3})^3 = y.$$

Thus Y is uniformly distributed on $[0,1]$.

3. *Let (X,Y) have joint density fundtion*

$$f(x,y)=\begin{cases} xy+\frac{3}{4} & \textit{if } 0\le x\le 1 \textit{ and } 0\le y\le 1 \\ 0 & \textit{otherwise} \end{cases}.$$

(a) Find the marginal density of X.
(b) Find the conditional density of Y given that $X=\frac{1}{2}$.
(c) Compute $P(X>Y)$.

Solution.

$$(a)\quad f_X(x)=\int_0^1\left(xy+\frac{3}{4}\right)dy=\frac{x}{2}+\frac{3}{4}.$$

$$(b)\quad F_{Y|X}\left(y\Big|\frac{1}{2}\right)=\frac{\frac{y}{2}+\frac{3}{4}}{\frac{1}{4}+\frac{3}{4}}=\frac{y}{2}+\frac{3}{4}.$$

(c) **Solution 1.**

$$P(X>Y)=\int_0^1\left(\int_0^x\left(xy+\frac{3}{4}\right)dy\right)dx=\int_0^1\left(\frac{x^3}{2}+\frac{3x}{4}\right)dx=\frac{1}{8}+\frac{3}{8}=\frac{1}{2}.$$

Solution 2. Since (X,Y) is continuous $P(X=Y)=0$. Therefore $P(X>Y)=1-P(X<Y)$. On the other hand by symmetry $P(X<Y)=P(X>Y)$. hence $P(X>Y)=\frac{1}{2}$.

4. *Let $S=X_1+X_2+\dots X_{100}$, where X_j are independent random variables having exponential distribution with parameter 1.*
(a) Compute $E(S)$.
(b) Compute $V(S)$.
(c) Compute approximately the probability that $S>105$.

Solution. (a) $ES=E(X_1)+E(X_2)+\dots E(X_{100})=100*1=100$.
(b) $VS=V(X_1)+V(X_2)+\dots V(X_{100})=100*1=100$. Hence $\sigma_S=\sqrt{100}+10$.
(c) By the Central Limit Theorem $S\approx 100+10Z$ where $Z\sim N(0,1)$. Hence

$$P(S>105)\approx P(100+10Z>105)=P(10Z>5)=P(Z>0.5)\approx 0.31.$$