(1) Let $x_1, x_2, \ldots, x_{100}$ be numbers such that $\sum_{j=1}^{100} x_j = 100$. Let $X_1, X_2, \ldots, X_{10}$ be numbers taken from the above sample without replacement. Let $S_n = \sum_{j=1}^{n} X_j$. Compute $\mathbb{E}(S_5 | S_{10})$.

By symmetry all $\mathbb{E}(X_j | S_{10})$ are equal, $j = 1, \ldots, 10$. Since $\sum_{j=1}^{10} \mathbb{E}(X_j | S_{10}) = \mathbb{E}(S_{10} | S_{10}) = S_{10}$ it follows that $\mathbb{E}(X_j | S_{10}) = S_{10}/10$. Hence

$$
\mathbb{E}(S_5 | S_{10}) = \sum_{j=1}^{5} \mathbb{E}(S_5 | S_{10}) = \frac{S_{10}}{2}.
$$

(2) Let $X_{jk}$ be independent random variables such that $X_{jk} \Rightarrow X_j$ as $k \to \infty$. Let $N$ be a random variable independent of $X$’s. Show that $S_k = \sum_{j=1}^{N} X_{jk}$, $S = \sum_{j=1}^{N} X_j$. Then

$$
\phi_{S_k}(t) = \sum_{n=1}^{N} P(N = n) \prod_{j=1}^{n} \phi_{X_{jk}}(t).
$$

For fixed $n, t$ the continuity theorem implies

$$
\prod_{j=1}^{n} \phi_{X_{jk}}(t) \to \prod_{j=1}^{n} \phi_{X_j}(t).
$$

Since $|\phi_{X_{jk}}(t)| \leq 1$ we can the dominated convergence theorem implies $\phi_{S_k}(t) \to \Phi_S(t)$ as needed.

(3) Let $X_1, X_2, \ldots, X_n$ de iid taking values ±1 with probability $\frac{1}{2}$. Let $S_n = \sum_{j=1}^{n} X_j$.

(a) Show that $S_n^2 - n$ is a martingale.

(b) Find $a, b$ such that $S_n^4 - 6S_n^2n - an - bn^2$ is a martingale.

(c) Let $\tau_N$ be the first time when $|S_n| = N$. Compute $\mathbb{E}(\tau)$ and $\mathbb{E}(\tau^2)$.

(You do not need to justify your calculations in part (c).)

(a) $S_{n+1}^2 - (n+1) - (S_n^2 - n) = 2S_nX_n$, and $2\mathbb{E}(S_nX_n | \mathcal{F}_n) = 2S_n\mathbb{E}(X_n) = 0$.

(b) $M_{n+1} - M_n$

$$
= 4S_n^3X_n + 6S_n^2X_n^2 + 6S_nX_n^3 + X_n^4 - 12S_nX_n^2 + 6nX_n^2 - 6S_n^2 - 12S_nX_n - 6X_n^2 - 2bn - a - b.
$$

Hence

$$
\mathbb{E}(M_{n+1} - M_n | \mathcal{F}_n) = -6n - bn - 5 - a - b.
$$
Hence $\mathbb{E}(M_{n+1} - M_n\mid F_n) = 0$ iff $-6n - bn - 5 - a - b = 0$ that is $2b = -6 \ a - b = -5$. Solving these equations we get $b = -3$, $a = -2$.

(c) Observe that $S^2_\tau = N^2$. Thus part (a) and optional stopping theorem give $\mathbb{E}(\tau) = \mathbb{E}(\tau^2) = N^2$. Now part (b) and optional stopping theorem give

$$N^4 - 6N^2\mathbb{E}(\tau) + 3\mathbb{E}(\tau^2) + 2\mathbb{E}(\tau) = 0,$$

Plugging $\mathbb{E}(\tau)$ from the previous line we obtain

$$\mathbb{E}(\tau^2) = \frac{5N^4 - 2N^2}{3}.$$

(4) Let $U_1, U_2, \ldots, U_n, \ldots$ be iid uniformly distributed on $[0, 1]$. Let $F_n = \sigma(U_1, U_2 \ldots U_n)$. Let $(M_n, F_n)$ be martingale such that $M_3 = U_1U_2 + U_1U_3 + U_2U_3$. Find $M_1$ and $M_2$.

$$M_2 = E(U_1U_2 + U_1U_3 + U_2U_3\mid F_2) = U_1U_2 + \frac{U_1}{2} + \frac{U_2}{2},$$

$$M_1 = E(M_2\mid F_1) = \frac{U_1}{2} + \frac{U_1}{2} + \frac{1}{4} = U_1 + \frac{1}{4}.$$

(5) Let $X_1, X_2 \ldots X_n$ de iid taking values $\pm 1$ with probability $\frac{1}{2}$. Let $S_n = \sum_{j=1}^n X_j$. Show that

$$\sum_{n=2}^{\infty} \frac{S_n}{(n \ln n)^{3/2}}$$

converges with probability 1.

By Law of iterated logarithm for large $n$

$$\left| \frac{S_n}{(n \ln n)^{3/2}} \right| \leq 2\frac{\sqrt{n \ln \ln n}}{(n \ln n)^{3/2}} \leq \frac{1}{n(\ln n)^{5/4}}.$$

(6) Let $X_1, X_2 \ldots X_n, \ldots$ be independent random variables such that $X_j$ is uniformly distributed on $[-j^2, j^2]$. Denote $S_n = \sum_{j=1}^n X_j$. Find $a_n, b_n$ such that $\frac{S_n}{a_n b_n}$ converges to a non-degenerate distribution and identify this distribution.

$\mathbb{E}(X_j) = 0$, $\text{Var}(X_j) = \frac{i^2}{3}$ so $\sum_{j=1}^n \text{Var}(X_j) \sim \int_1^n \frac{x^4}{3} \sim \frac{n^5}{15}$. Since

$$\sum_{j=1}^n \mathbb{E}(|X_j|^3) = \sum_{j=1}^n \frac{j^6}{4} \sim \int_1^n x^6 \frac{4}{4} \sim \frac{n^7}{28} \ll \left(\frac{n^5}{15}\right)^{3/2}$$

the Lyapunov condition holds so $\frac{S_n a_n}{b_n} \Rightarrow N(0, \frac{1}{15})$. 