ON SMALL GAPS IN THE LENGTH SPECTRUM

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Abstract. We discuss upper and lower bounds for the size of gaps in the length spectrum of negatively curved manifolds. For manifolds with algebraic generators for the fundamental group, we establish the existence of exponential lower bounds for the gaps. On the other hand, we show that the existence of arbitrary small gaps is topologically generic: this is established both for surfaces of constant negative curvature (Theorem 3.1), and for the space of negatively curved metrics (Theorem 4.1). While arbitrary small gaps are topologically generic, it is plausible that the gaps are not too small for almost every metric. One result in this direction is presented in Section 5.

1. Introduction: geodesic length separation in negative curvature

On negatively curved manifolds, the number of closed geodesics of length $\leq T$ grows exponentially in $T$. (We refer the reader to [Mar04, P-P, P-S] for a comprehensive discussion about the growth and distribution of closed geodesics).

The abundance of closed geodesics leads to the natural question about the sizes of gaps in the length spectrum. In the current note we present a number of results related to this question. In some situations we are able to control the gaps from below, while in other we show that such control is not possible in general.

We note that a presence of exponentially large multiplicities in the length spectrum of a Riemannian manifold (which can be considered as a limiting case of small gaps) changes the level spacings distribution of Laplace eigenvalues on that manifold, see e.g. [L-S].

For generic Riemannian metrics, the length spectrum is simple [Abr, A2], so for any closed geodesic $\gamma$, only $\gamma^{-1}$ will have the same length. So, by the Dirichlet box principle, there exist exponentially small gaps between the lengths of different geodesics.

Accordingly, it seems interesting to investigate manifolds where the gaps between the lengths of different geodesics have exponential lower bound: there exist constants $C, \beta > 0$, such that for any $l_1 \neq l_2 \in \text{Lsp}(M)$ (length spectrum of the negatively curved manifold $M$), we have

$$|l_2 - l_1| > Ce^{-\beta \max(l_1, l_2)}.$$ (1.1)

This assumption is satisfied for arithmetic hyperbolic groups by the trace separation criterion (cf. [Tak] and [Hej, §18]). In Section 2 we explain (see Theorem 2.6) why the assumption (1.1) holds for hyperbolic manifolds whose fundamental group has algebraic elements.

In particular, the surfaces satisfying (1.1) form a dense set in the corresponding Teichmüller space. On the other hand the existence of arbitrary small gaps is topologically generic as is shown in Theorem 3.1 for surfaces of constant negative curvature and in Theorem 4.1 for the space of negatively curved metrics endowed with $C^r$-topology, for any $r > 0$.

While arbitrary small gaps are topologically generic, it is plausible that the gaps are not too small for almost every metric. One result in this direction is presented in Section 5 there we obtain an explicit lower bound for the gaps valid for almost every hyperbolic surface.

Length separation between closed geodesics is relevant for the study of wave trace formulas on negatively-curved manifolds: to accurately study contributions from exponentially many closed
geodesics to the wave trace formula, it is necessary to separate contributions from geodesics which differ either on the length axis, or in phase space. We remark that a suitable version of (1.1) always holds in phase space: small tubular neighbourhoods of closed geodesics in phase space are disjoint, as shown in [JPT]. Since there exist metrics for which the size of the length gaps cannot be controlled (Theorem 4.1), the authors in [JPT] established microlocal wave trace formula, and used the separation of closed trajectories in phase space in the proof.

2. Diophantine results for hyperbolic manifolds.

2.1. Distances between algebraic numbers. In this section we consider gaps in the length spectrum for manifolds whose fundamental group admits algebraic generators. But first we provide a few general results about the algebraic numbers.

**Lemma 2.1.** If \( \alpha \) is a root of \( P(x) = x^D + a_{D-1}x^{D-1} + \cdots + a_0 \) then
\[
|\alpha| \geq \frac{|a_0|}{\left(1 + \sum_{j=0}^{D-1} |a_j| \right)^{D-1}}.
\]

**Proof.** Let \( \alpha_j \) be the roots of \( P \) counted with multiplicities. We claim that
\[
|\alpha_j| \leq R := 1 + \sum_{j=0}^{D-1} |a_j|.
\]
Indeed if \( |x| > R \) then since \( R > 1 \) we get
\[
|P(x)| \geq |x|^D - \sum_{j=0}^{D-1} |a_j||x|^j \geq |x|^D - \sum_{j=0}^{D-1} |a_j| > 0.
\]
The result follows since \( \prod_j |\alpha_j| = |a_0| \).

Given a field \( K \) which is an extension of \( \mathbb{Q} \) of degree \( d \) let \( H(L,N,p) \) be the set of all elements of \( K \) of the form \( \beta L^d \) where \( \beta \in O_K \) and for each automorphism \( \sigma_j \) of \( K \) we have \( |\sigma_j(\beta)| \leq L \).

**Lemma 2.2.** If \( 0 \neq \alpha \in H(L,N,p) \) then
\[
|\alpha| \geq \frac{1}{L^{d-1}N^d}.
\]

**Proof.** Indeed \( |\beta|L^d \geq 1 \) because \( \prod_{j=1}^{d} |\sigma_j(\beta)| \geq 1 \).

Let \( J(L,N,p,D) \) be the set of numbers which satisfy
\[
\alpha^E + a_{E-1}\alpha^{E-1} + \cdots + a_0 = 0
\]
where \( E \leq D \) and \( a_j \in H(L,N,p) \).

**Corollary 2.3.** If \( 0 \neq \alpha \in J(L,N,p,D) \) then
\[
|\alpha| \geq \frac{1}{L^{d-1}N^d(DL + 1)^{D-1}}.
\]

**Proof.** Since \( \alpha \neq 0 \) we can assume after possibly reducing the degree of the polynomial that \( a_0 \neq 0 \). Then the result follows by combining Lemmas 2.1 and 2.2.

**Proposition 2.4.** (see e.g. [vW], Section 17.2) There exists constants \( C \) and \( q \) such that if \( \alpha_1, \alpha_2 \in J(L,N,p,D) \) then \( \alpha_1 + \alpha_2 \) and \( \alpha_1 - \alpha_2 \) are in \( J(CL^q, N, pq, D^2) \).

Combining Proposition 2.4 with Corollary 2.3 we obtain

**Corollary 2.5.** If \( \alpha_1, \alpha_2 \in J(L,N,p,D) \) then either \( \alpha_1 = \alpha_2 \) or
\[
|\alpha_1 - \alpha_2| \geq \frac{c}{L^q(d-1)N^pqLD^2}.
\]
2.2. Manifolds with algebraic generators of \(\pi_1\). We now formulate the main result of this section.

**Theorem 2.6.** Let \(X\) be a hyperbolic manifold such that the generators of \(\pi_1(X)\) belong to \(\text{PSO}_{n,1}(\mathbb{Q})\). Then (1.1) holds.

We remark that in dimension 2 groups satisfying the assumptions of Theorem 2.6 form a dense set in the corresponding Teichmüller space \(T_g\). This can be established, for example, by the arguments of Section 5.

If \(n \geq 3\) then \([G-R, \text{Theorem 0.11}]\) building on earlier results of Selberg \([\text{Sel}]\) and Mostow \([\text{Most}]\) shows the conditions of Theorem 2.6 are satisfied for all finite volume hyperbolic manifolds. Hence we obtain

**Corollary 2.7.** (1.1) holds for finite volume hyperbolic manifolds of dimension \(n \geq 3\).

2.3. Proof of Theorem 2.6

**Proof.** Let \(\gamma_1\) and \(\gamma_2\) be two closed geodesics. Let \(l_j\) be the length of \(\gamma_j\), \(W_j\) be the word fixing \(\gamma_j\), \(B_j\) be the matrix corresponding to \(W_j\), \(m_j\) be the word length of \(W_j\) and \(r_j = l_j/2\). To establish (1.1) it suffices to show that

\[
|e^{r_1} - e^{r_2}| \geq C e^{-\bar{c} \max(r_1, r_2)}.
\]

Without a loss of generality we assume that \(m_j \gg 1\). By ([Miln, Lemma 2]) we know that

\[
l_j / C \leq m_j \leq Cl_j
\]

so (1.1) if trivial unless \(m_1\) and \(m_2\) are comparable. Let us assume to fix our ideas that \(m_2 \geq m_1\).

By assumption there is a finite extension \(K\) of \(\mathbb{Q}\) and numbers \(L\) and \(N\) such that all entries of the generators belong to \(\mathcal{H}(L, N, 1)\). Accordingly the entries of \(B_j\) belong to \(\mathcal{H}(L(n+1)^{m_j}, N, m_j)\).

Closed geodesics on \(X\) correspond to loxodromic elements of \(\pi_1(X) \subset \text{PSO}_{n,1}\) (also called boosts) that fix no points in \(\mathcal{H}^n\) and fix two points in \(\partial \mathcal{H}^n\). It is shown in the proof of \([\text{F-LJ, Thm. I.5.1}]\) that \(B_j\) has precisely two positive real eigenvalues \(\alpha_{1,j} = e^{r_j}\) and \(\alpha_{2,j} = e^{-r_j}\); all other eigenvalues of \(B_j\) have modulus one. Since the coefficients of the characteristic polynomial of \(B_j\) are the sums of minors we have

\[
e^{r_j} \in \mathcal{J}((L(n+1))^{(n+1)m_j}(n+1)!, N, m_j(n+1)).
\]

Reducing to the common denominator we see that both \(e^{r_1}\) and \(e^{r_2}\) belong to

\[
\mathcal{J}((L(n+1))^{(n+1)m_2}N^{m_2-m_1}(n+1)!, N, m_2(n+1)).
\]

Now (2.8) follows by Corollary 2.5 and (2.9). \(\square\)

**Remark 2.10.** In dimension two the proof can be simplified slightly by remarking that \(2 \cosh(l_j/2) = \text{tr}B_j \in K\). An alternative proof of Theorem 2.6 could proceed by using explicit formulas for the lengths of closed geodesics on hyperbolic manifolds (see e.g. \([P-R, (3), p. 246]\)) and the estimates for linear forms in logarithms (see e.g. \([B-W, \text{Chapter 2}]\)). The proof we give is more elementary, using only basic facts about algebraic numbers and matrix eigenvalues; and fairly concrete.
3. Small gaps for surface of constant negative curvature.

Let
\[ G_\delta = \{(A_1, \ldots, A_{2g}) \in (SL_2(\mathbb{R}))^{2g} : [A_1, A_2][A_3, A_4] \ldots [A_{2g-1}, A_{2g}] = I\}. \]

**Theorem 3.1.** The set of tuples \((A_1, A_2 \ldots A_{2g}) \in G_\delta\) where \([1,1]\) fails is topologically generic.

**Proof.** Let \(\gamma_A\) denote the closed geodesic whose lift to the fundamental cover joins \(q\) and \(Aq\). Let \(L\) denote the length spectrum of the geodesics \(\gamma_A\) where \(A\) belongs to a subgroup generated by \(A_1\) and \(A_2\). Note that for a dense set of tuples it holds that for each \(\delta\) there exists \(L\) such that for \(l > L\) the set \([l, l + \delta]\) intersects \(L\). One way to see this is to consider the geodesics \(\gamma_{A_1}, \gamma_{A_2}\). Their length have asymptotics
\[ \kappa(A_1, A_2)k\lambda_1 + m\lambda_2 \]
where \(c^{\lambda_j}\) is the leading eigenvalue of \(A_j\). Note that for typical \(A_1, A_2\) we have \(\kappa(A_1, A_2) \neq 0\) and \(\lambda_1\) and \(\lambda_2\) are non commensurable. Consider a geodesic \(\tilde{\gamma} = \gamma_{A_3A_1}\) where \(n\) is very large. By the foregoing discussion there exists \(l \in \mathbb{L}\) such that \(|l - L_{\tilde{\gamma}}| < \delta\). Now consider the perturbations of \(A_3\) of the form \(A_3(\eta) = \left( \begin{array}{cc} 1 & \eta \\ 0 & 1 \end{array} \right) A_3\). Assume that \(A_3A_1^\eta = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)\). After applying a small perturbation if necessary we can assume that all entries of this matrix have the same order as its trace. Then
\[ \text{tr}(A_3(\eta)A_1^\eta) = \text{tr}(A_3A_1^\eta) + \eta c, \]
so by a small perturbation we can make \(L_{\gamma_{A_3(\eta)A_1}}\) as close to \(l\) as we wish. Now the result follows by a standard Baire category argument (cf. Section 4).

4. Constructing metrics with small gaps in the length spectrum

This section is devoted to the proof of the following fact.

**Theorem 4.1.** For any \(r > 3\) for any negatively curved \(C^r\) metric \(g\), for any function \(F(t)\) (which we assume is monotone and fast decreasing), and a number \(\delta > 0\), there exists a metric \(\tilde{g}\), such that \(|\tilde{g} - g|_{C^r} < \delta\) and there exists an infinite sequence of pairs of closed \(\tilde{g}\)-geodesics \(\gamma_{1,j}, \gamma_{2,j}\) with \(L_{\tilde{g}}(\gamma_{1,j}) \to \infty\) as \(j \to \infty\), and
\[ |L_{\tilde{g}}(\gamma_{1,j}) - L_{\tilde{g}}(\gamma_{2,j})| < \min\{F(L_{\tilde{g}}(\gamma_{1,j})), F(L_{\tilde{g}}(\gamma_{2,j}))\}. \]

This shows that, in general, one cannot obtain good lower bounds for gaps in the length spectrum for a \(C^r\) open set of negatively curved metrics.

Theorem 4.1 follows from the lemma below by a standard Baire category argument.

**Lemma 4.3.** Given a metric \(g\) and numbers \(L\) and \(\delta\) there is a metric \(\tilde{g}\) such that \(|g - \tilde{g}|_{C^r} \leq \delta\) and there are two \(\tilde{g}\)-geodesics \(\gamma_1\) and \(\gamma_2\) such that
\[ L_{\tilde{g}}(\gamma_1) = L_{\tilde{g}}(\gamma_2) > L. \]

We also need the following fact

**Lemma 4.4.** Let \(g\) and \(\tilde{g}\) be two negatively curved metrics such that \(|g - \tilde{g}|_{C^0} \leq \delta\) and \(\gamma\) and \(\tilde{\gamma}\) be two closed geodesics for \(g\) and \(\tilde{g}\) respectively of lengths \(L\) and \(\tilde{L}\). If \(\gamma\) and \(\tilde{\gamma}\) are homotopic then
\[ \frac{L}{1 + \delta} \leq \tilde{L} \leq L(1 + \delta). \]

**Proof.** Recall that for negatively curved there exists a unique geodesic in each homotopy class and this geodesic is length minimizing. The second inequality follows since the length of \(\gamma\) with respect to \(\tilde{g}\) is at most \(L(1 + \delta)\) and \(\tilde{\gamma}\) is shorter. The first inequality follows from the second by interchanging the roles of \(g\) and \(\tilde{g}\). \(\square\)
Proof of Theorem 4.1. We claim that given metric $g$ and numbers $k \in \mathbb{N}$ and $\delta > 0$ there exists a metric $\tilde{g}$ such that $\|\tilde{g} - g\|_{C^r} < \delta$ and for each $j = 1 \ldots k$ there are geodesics $\gamma_{1,j}, \gamma_{2,j}$ such that

$$L_{\tilde{g}}(\gamma_{i,j}) > j,$$

$$|L_{\tilde{g}}(\gamma_{1,j}) - L_{\tilde{g}}(\gamma_{2,j})| \leq F(\max(L_{\tilde{g}}(\gamma_{1,j}), L_{\tilde{g}}(\gamma_{2,j}))).$$

It follows that the space of metrics satisfying (4.2) is topologically generic and hence dense.

It remains to construct $\tilde{g}$ satisfying (4.5).

By Lemma 4.3 we can find $g_1$ such that $\|g - g_1\|_{C^r} < \frac{\delta}{2}$ and there are two geodesics $\gamma_{1,1}$ and $\gamma_{2,1}$ such that

$$L_{g_1}(\gamma_{1,1}) > 1 \text{ and } L_{g_1}(\gamma_{1,j}) = L_{g_1}(\gamma_{2,1}).$$

For $j \geq 1$ we apply Lemma 4.3 to find $g_j$ such that

$$\|g_j - g_{j-1}\|_{C^r} \leq \min \left( \frac{\delta}{2}, \min_{l=1}^{j-1} \frac{F(L_{g_l}(\gamma_{1,l})) + 1}{L_{g_l}(\gamma_{1,l})2^{j-l+1}} \right)$$

and there are two geodesics $\gamma_{1,j}, \gamma_{2,j}$ such that

$$L_{g_j}(\gamma_{1,j}) = L_{g_j}(\gamma_{2,j}) > j.$$ Then $g_k$ satisfies the required properties since, by Lemma 4.4, the lengths of $g_k, l$ have changed by less than $F(L_{g_l}(\gamma_{1,l}) + 1)/2$ in the process of making consecutive inductive steps. \hfill \Box

Remark 4.6. In particular if we continue the above procedure for the infinite number of steps then the limiting metric will satisfy the conditions of Theorem 4.1.

The proof of Lemma 4.3 relies on two facts. If $\gamma$ is a closed geodesic let $\nu_\gamma$ denote the invariant measure for the geodesic flow supported on $\gamma$. Let $h$ denote the topological entropy of the geodesic flow. Let $\mu$ denote the Bowen-Margulis measure. Recall \cite{P-P} that $\mu$ the measure of maximal entropy for the geodesic flow. It has a full support in the unit tangent bundle $SM$.

Lemma 4.7. \cite{P-P}. Theorem 6.9 and Proposition 7.2] $Lh e^{-Lh} \sum L(\gamma) \leq L \nu_\gamma$ converges as $L \to \infty$ to $\mu$.

Lemma 4.8. For each $q_0 \in M$ there exists $\varepsilon$ such that for each $L$ there is periodic geodesic $\gamma$ such that $L(\gamma) > L$ and $\gamma$ does not visit an $\varepsilon$ neighborhood of $q_0$.

Proof of Lemma 4.3. Pick a small $\tilde{\delta}$ and large $L$. By Lemma 4.8 there exists a closed geodesic $\gamma_1$ such that $L_{g}(\gamma_1) > L$ and $d(g(\gamma_1(t)), q_0) > \varepsilon$. Let $\gamma_2$ be a closed geodesic such that $L_{g}(\gamma_1) < L_{g}(\gamma_2) < L_{g}(\gamma_1) + \tilde{\delta}$ and $\gamma$ spends at least time $\mu(B(q_0, \varepsilon/2)/2)L_{g}(\gamma_1)$ inside $B(q_0, \varepsilon/2)$ (the existence of such a geodesic follows from Lemma 4.7). Take $\tilde{g}^n = (1 - \eta z(q))g$ where $z(q) = 1$ on $B(q_0, \varepsilon/2)$ and $z(q) = 0$ outside $B(q_0, \varepsilon)$. We can choose $z$ so that $\|z\|_{C^r} = O(\varepsilon^{-r})$. Then $\|g - \tilde{g}^n\|_{C^r} = O(\eta/\varepsilon^r)$.

Let $\gamma_2^n$ be the closed geodesic for $\tilde{g}^n$ homotopic to $\gamma_2$. Note that $\gamma_1$ is a geodesic of $\tilde{g}^n$ for each $\eta$ and $\tilde{L}_{\tilde{g}^n}(\gamma_1) = L_{g}(\gamma_1)$. Also

$$\tilde{L}_{\tilde{g}^n}(\gamma_2) \leq L_{\tilde{g}^n}(\gamma_2) \leq L_{g}(\gamma_1) + \tilde{\delta} - \frac{\mu(B(q_0, \varepsilon/2)L_{g}(\gamma_1)\eta)}{2}.$$ 

Accordingly there exists $\eta < \frac{2\tilde{\delta}}{L\mu(B(q_0, \varepsilon/2))}$ such that $L_{\tilde{g}^n}(\gamma_1) = L_{\tilde{g}^n}(\gamma_2)$ as claimed. \hfill \Box

In the proof of Lemma 4.8 we need several facts about the dynamics of the geodesic flow which we call $\phi_t$. Recall \cite{A-L} that $\phi_t$ is uniformly hyperbolic. In particular, there is a cone field $\mathcal{K}(x)$ and $\lambda > 0$ such that for $u \in \mathcal{K}$, $\|d\phi_t(u)\| \geq e^{\lambda t}\|u\|$. Moreover the cone field $\mathcal{K}$ can be chosen in such a way that if $x = (q, v)$ and $u = (\delta q, \delta v) \in \mathcal{K}(x)$ then

$$\|\delta g\| \geq c\|\delta v\| \text{ and } \angle(\delta q, \delta v) \geq \frac{\pi}{4}$$

We call a curve $\sigma$ unstable if $\sigma \in \mathcal{K}$. By the foregoing discussion if $\sigma$ is un unstable curve then the length of the projection of $\phi_t(\sigma)$ on $M$ is longer than $ce^{\lambda t}$.
Proof of Lemma 4.8. We first show how to construct a not necessary closed geodesic avoiding $B(q_0, \varepsilon)$ and then upgrade the result to get the existence of a closed geodesics.

The first part of the argument is similar to [B-S, D]. Pick a small $\kappa > 0$. Take an unstable curve $\sigma$ of small length $\kappa$. We show that if $\kappa$ and $\varepsilon$ are sufficiently small then $\sigma$ contains a point such that the corresponding geodesic avoids $B(q_0, \varepsilon)$. Let $T_1$ be a number such that $|\phi_{T_1}(\sigma)| = 1$ where $\phi$ denotes the geodesic flow. Note that $T_1 = O(|\ln \kappa|)$. Also observe that due to (4.9) there exists a number $r_0$ such that if $\sigma$ is an unstable curve and $x \in \sigma$ is such that $d(q(x), q_0) < \varepsilon$ then for all $y \in \sigma$ such that $C\varepsilon \leq d(y, x) \leq r_0$ we have

$$d(q(\phi_t y), q_0) > \varepsilon$$

for $|t| < r_0$ where $d$ denotes the distance in the phase space (just take $r_0$ much smaller than the injectivity radius of $q_0$).

Thus the set

$$\{ y \in \phi_{T_1}(\sigma) : d(q(\phi_{-t} y), q_0) \leq \varepsilon \text{ for some } 0 \leq t \leq T_1 \}$$

is a union of $O(|\ln \kappa|/r_0)$ components of length $O(\varepsilon/\kappa^a)$ for some $a > 0$. Therefore if $\kappa \ll 1$ and $\varepsilon \ll \kappa$ then the average distance between the components is much larger than $\kappa$. So we can find $\sigma_1 \subset \phi_{T_1}\sigma$ such that $|\sigma_1| = \kappa$, and if $y \in \sigma_1$ then $d(q(\phi_{-t} y), q_0) > \varepsilon$ for each $0 \leq t \leq T_1$. Take $T_2$ such that $|\phi_{T_2}\sigma_1| = 1$. Then we can find $\sigma_2 \subset \phi_{T_2}\sigma_1$ such that $|\sigma_2| = \kappa$, and if $y \in \sigma_2$ then $d(q(\phi_{-t} y), q_0) > \varepsilon$ for each $0 \leq t \leq T_2$. We continue this procedure inductively to construct arcs $\sigma_j$ for all $j \in \mathbb{N}$. Taking

$$x = \bigcap_{j=1}^{\infty} \phi_{-(T_1+T_2+\ldots+T_j)} \sigma_j$$

we obtain a geodesic avoiding $B(q_0, \varepsilon)$. To complete the proof we need

Lemma 4.10. (Anosov Closing Lemma) (see [H-K, Section 18]) Given $\eta > 0$ there exists $\delta > 0$ such that if for some $t_1, t_2$ such that $|t_2 - t_1|$ is sufficiently large we have $d(\gamma(t_1), \gamma(t_2)) < \delta$ then there exists a closed geodesic $\tilde{\gamma}$ such that $|L(\tilde{\gamma}) - |t_2 - t_1|| < \eta$ and for each $t \in [t_1, t_2]$ there exists $s$ such that $d(\gamma(t), \tilde{\gamma}(s)) < \eta$.

Take $\delta$ corresponding to $\eta = \varepsilon/2$. Consider points $\gamma(jL)$ where $j = 1 \ldots K$. By pigeonhole principle if $K$ is sufficiently large we can find $j_1, j_2$ such that $d(\gamma(j_1L), \gamma(j_2L)) < \delta$ and so there exists a closed geodesic $\tilde{\gamma}$ avoiding $B(q_0, \varepsilon/2)$. Since $\varepsilon$ is arbitrary, Lemma 4.8 follows. \qed

Suppose now that dim($M$) = 2. Let $\mathcal{H}_r(M)$ denote the space of $C^r$ metrics with positive topological entropy. This set is $C^r$ open ([K]) and dense. (If genus($M$) $\geq 2$ then every metric has positive topological entropy [K].) For torus the density of $\mathcal{H}_r(M)$ follows from [Ban] and for sphere it follows from [K-W]).

**Theorem 4.11.** The set of metrics satisfying (4.2) is topologically generic in $\mathcal{H}(M)$.

**Corollary 4.12.** The set of metrics satisfying (4.2) is topologically generic in the space of all metrics on $M$.

**Proof of Theorem 4.11.** By [K] if $g \in \mathcal{H}_r(M)$ then there is a hyperbolic basic set $\Lambda$ for the geodesic flow. Since Lemmas 4.3, 4.7, 4.8 and 4.10 remain valid in the setting of hyperbolic sets the proof is similar to the proof of Theorem 4.2. (In the proof of Lemma 4.8 we need to take $\sigma_1$ so that it crosses completely an element of some Markov partition $\Pi$ such that all elements of $\Pi$ have unstable length between $\kappa$ and $C\kappa$. The number of eligible segments now is not $O(1/\kappa)$ but $O(1/\kappa^a)$ for some $a > 0$ but this is still much larger than $|\ln \kappa|$.) \qed
5. Small gaps for hyperbolic surfaces, continued

Here we show that for Lebesgue-typical hyperbolic surface the gaps in the length spectrum cannot be too small. Our argument in similar to [K-R]. Related results are obtained in [Var].

5.1. Small values of polynomials.

Proposition 5.1. (see e.g. [M-H] Section 3.2) Consider a degree $D$ polynomial $P(x) = a_D x^D + a_{D-1} x^{D-1} + \ldots + a_0$. Then

$$\sup_{[-1,1]} |P(x)| \geq \frac{|a_D|}{2^D}.$$ 

Corollary 5.2. Let $0 \neq P \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, $\deg(P) = D$ then

$$\sup_{[-1,1]^n} |P(x)| \geq \frac{1}{2^{D-1}}.$$ 

Proof. By induction. For $n = 0$ or $1$ the result follows from Proposition 5.1. Next, suppose the statement is proven for polynomials of $n-1$ variables. If $P$ does not depend on $x_n$ then we are done. Otherwise let $k > 0$ be the degree of $P$ with respect to $x_n$. Then

$$P(x) = a_k(x_1, \ldots, x_{n-1}) x_n^k + a_{k-1}(x_1, \ldots, x_{n-1}) x_n^{k-1} + \ldots + a_0(x_1, \ldots, x_{n-1})$$

where $a_k$ is the polynomial with integer coefficients of degree $D-k$. Let

$$(\bar{x}_1, \ldots, \bar{x}_{n-1}) = \arg \max_{[a_{-1],[a_{n-1}]} |a_k(x_1, \ldots, x_{n-1})|).$$

Then

$$\sup_{[-1,1]^n} |P(x_1, \ldots, x_{n-1}, x_n)| \geq \max_{x_n \in [-1,1]} |P(\bar{x}_1, \ldots, \bar{x}_{n-1}, x_n)| \geq |a(\bar{x}_1, \ldots, \bar{x}_{n-1})| 2^{1-k} \geq 2^{1+k-D} 2^{1-k} = 2^{2-D}$$

completing the proof.

Proposition 5.3. (Remez inequality) (see [B-G] or [Yom, Theorem 1.1]) Let $B$ be a convex set in $\mathbb{R}^n$, $\Omega \subset B$, and $P$ be a polynomial of degree $D$. Then

$$\sup_B |P| \leq C_B \text{mes}^{-D}(\Omega) \sup_{\Omega} |P|.$$ 

Corollary 5.4. Under the conditions of Proposition 5.3

$$\text{mes}(x \in B : |P(x)| \leq \varepsilon) \leq \left( \frac{C_B \varepsilon}{\sup_B |P|} \right)^{1/D}.$$ 

Proof. Apply Proposition 5.3 with $\Omega = \{ x \in B : |P(x)| \leq \varepsilon \}$. 

Corollary 5.5. If $P_N \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ are polynomials of degree $D_N$ and $\varepsilon_N$ is a sequence such that $\sum_{N} \varepsilon_N^{1/D_N} < \infty$ then $|P(x_1, \ldots, x_n)| < \varepsilon_N$ has only finitely many solutions for almost every $(x_1, \ldots, x_n) \in \mathbb{R}^n$. 

Proof. It suffices to show this for a fixed cube $B$ with side 2. Then Corollaries 5.2 and 5.4 give

$$\text{mes}(x \in B : |P_N(x)| \leq \varepsilon_N) \leq (C 2^{D_N} \varepsilon_N)^{1/D_N} = C \varepsilon_N^{1/D_N},$$

so the statement follows from Borel-Cantelli Lemma.
5.2. Polynomial maps on \( SL_2(\mathbb{R}) \).

**Corollary 5.6.** Let \( m \) be a fixed number.

(a) Let \( P_N \in \mathbb{Z}((a_1, b_1, c_1, d_1), \ldots, (a_m, b_m, c_m, d_m)) \) be polynomials of degree \( D_N \). For \( A_1, \ldots, A_m \in SL_2(\mathbb{R}) \) with \( A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \) let
\[
H_N(A_1, \ldots, A_m) = P_N((a_1, b_1, c_1, d_1), \ldots, (a_m, b_m, c_m, d_m)).
\]
If \( \sum_N \varepsilon_N^{1/(m+2)D_N} < \infty \) then \( |H_N(A_1 \ldots A_m)| < \varepsilon_N \) for only finitely many \( N \) for almost every \( (A_1, \ldots, A_m) \in (SL_2(\mathbb{R}))^m \).

(b) Given \( g \in \mathbb{N} \) let
\[
G_g = \{(A_1, \ldots, A_{2g}) \in (SL_2(\mathbb{R}))^{2g} : [A_1, A_2][A_3, A_4] \ldots [A_{2g-1}, A_{2g}] = I\}.
\]
Let \( P_N \in \mathbb{Z}((a_1, b_1, c_1, d_1), \ldots, (a_{2m}, b_{2m}, c_{2m}, d_{2m})) \) be polynomials of degree \( D_N \). Let
\[
H_N(A_1, \ldots, A_{2g}) = P_N((a_1, b_1, c_1, d_1), \ldots, (a_{2g}, b_{2g}, c_{2g}, d_{2g})).
\]
Assume that \( H_N \) is not identically equal to 0 on \( G_g \). If \( \sum_N \varepsilon_N^{\delta N} < \infty \) where \( \delta_N = \frac{1}{(4g-2)(g+2)D_N} \) then
\[
|H_N(A_1 \ldots A_{2g})| < \varepsilon_N \text{ for only finitely many } N \text{ for almost every } (A_1, \ldots, A_{2g}) \in G_g.
\]

**Proof.** (a) It suffices to prove the statement under the assumption that \( |a_j| > \delta \) for some fixed \( \delta > 0 \). Then \( d_j = \frac{1+b_j c_j}{a_j^2} \) and so
\[
H(A_1, \ldots, A_m) = \tilde{P}_N((a_1, b_1, c_1, d_1), \ldots, (a_m, b_m, c_m, d_m))
\]
where \( \tilde{P}_N \) is a polynomial of degree \( \tilde{d}_n \leq (m+2)d_N \). Thus if \( |P_N| \leq \varepsilon_N \) then \( |\tilde{P}_N| \leq \varepsilon_N := \frac{\varepsilon_N}{\delta^{(m+2)D_N}} \). Since
\[
\sum_N \varepsilon_N^{1/d_N} \leq \frac{1}{\delta} \sum_N \varepsilon_N^{1/(m+2)D_N} < \infty
\]
the result follows from Corollary 5.5.

(b) Rewriting the equations defining \( G_g \) in the form
\[
[A_1, A_2] \ldots [A_{2g-3}, A_{2g-2}]A_{2g-1}A_{2g}A_{2g-1}^{-1} = A_{2g}
\]
we can express the entries of \( A_{2g} \) as rational functions of the entries of the other matrices. Arguing as in part (a) we can reduce the inequality \( |P_N(A_1 \ldots A_{2g-1}, A_{2g})| < \varepsilon \) to \( |\tilde{P}_N(A_1, \ldots, A_{2g-1})| < \varepsilon_N \) where \( \tilde{P}_N \) is the polynomial of degree \((4g-2)D_N\). Now the result follows from part (a).

**Corollary 5.7.** For each \( \eta > 0 \) for almost every \( A_1, \ldots, A_m \in SL_2(\mathbb{R}) \) the inequality
\[
||W(A_1, \ldots, A_m) - I|| > (2m-1)^{-|W|^2(m+2+\eta)}
\]
holds for all except for finitely many words \( W \).

**Proof.** If \( ||W(A_1, \ldots, A_m) - I|| \leq \varepsilon \) then all entries of \( W - I \) are \( \varepsilon \) close to \( I \). Considering for example, the condition \( W_{11}(A_1, \ldots, A_m) - 1 \) we get a polynomial of degree \( |W| \). Therefore, by Corollary 5.6 it suffices to check that
\[
\sum_{W}(2m-1)^{|W|^2(m+2+\eta)} < \infty
\]
but the above sum equals to
\[
\sum_{D}(2m-1)^{D}(2m-1)^{-D-D(\eta/(m+2))} = \sum_{D}(2m-1)^{-\eta D/(m+2)} < \infty.
\]
\( \square \)
Corollary 5.8. For $\mathcal{A} = (A_1 \ldots A_{2g}) \in G_g$ let $S_{\mathcal{A}}$ be the surface defined by $\mathcal{A}$. Given a word $W$ let $l(W, \mathcal{A})$ be the length of the closed geodesic in the homotopy class defined by $W$. Then for each $\eta > 0$ the following holds for almost all $\mathcal{A} \in G_g$

There exists a constant $K = K(\mathcal{A})$ such that for each pair $W_1, W_2$ either

$$l(W_1, \mathcal{A}) = l(W_2, \mathcal{A}) \text{ or}$$

$$|l(W_1, \mathcal{A}) - l(W_2, \mathcal{A})| \geq K(4g - 1)^{-\left\lfloor (2g+4)(4g-2)+\eta \right\rfloor \max^2(|W_1|, |W_2|)}.

(5.9)

Remark 5.10. Recall that [Ran] shows that for any hyperbolic surface the length spectrum has unbounded multiplicity so there are many pairs of non conjugated words there the first alternative of the corollary holds.

Remark 5.11. Note that (2.9) shows that $l(W_1, \mathcal{A})$ can be close to $l(W_2, \mathcal{A})$ only if the lengths of $W_1$ and $W_2$ are of the same order. Thus (5.9) implies that for almost every $\mathcal{A}$ there are constants $K, \tilde{R}$ such that

$$|l(W_1, \mathcal{A}) - l(W_2, \mathcal{A})| \geq K e^{-\tilde{R}(\mathcal{A})/l(W_1, \mathcal{A})}.$$  

Proof. Let $P_W(\mathcal{A}) = \text{tr}(W(\mathcal{A}))$. Since $P_W(\mathcal{A}) = 2 \cosh(l(W, \mathcal{A})/2)$, it follows that if $l(W_1(\mathcal{A}))$ is close to $l(W_2(\mathcal{A}))$ then

$$l(W_1, \mathcal{A}) - l(W_2, \mathcal{A})| \geq C|P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})|e^{-|W_1|D}.$$ 

Therefore it suffices to show that if $l(W_1, \mathcal{A}) \neq l(W_2, \mathcal{A})$ then

$$|P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})| \geq K e^{-|W_1|D}(4g - 1)^{-\left\lfloor (2g+4)(4g-2)+\eta \right\rfloor \max^2(|W_1|, |W_2|)}.$$ 

Since $\eta$ is arbitrary, we can actually check that

$$|P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})| \geq K(4g - 1)^{-\left\lfloor (2g+4)(4g-2)+\eta \right\rfloor \max^2(|W_1|, |W_2|)}.$$ 

To verify this we will show that for almost all $\mathcal{A} \in G_m$ the inequality

$$|P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})| < (4g - 1)^{-\left\lfloor (2g+4)(4g-2)+\eta \right\rfloor \max^2(|W_1|, |W_2|)}$$

has only finitely many solutions. Let $P_{W_1,W_2}(\mathcal{A}) = P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})$. It is a polynomial of degree $\max(|W_1|, |W_2|)$. So by Corollary 5.6b it suffices to check that

$$\sum_{W_1,W_2} (4g - 1)^{-\frac{\left\lfloor (2g+4)(4g-2)+\eta \right\rfloor \max^2(|W_1|, |W_2|)}{(4g-2)(4g+2)}} < \infty.$$ 

There are at most $(4g - 1)^{2k}$ pairs $(W_1, W_2)$ with $k = \max(W_1, W_2)$ so the last sum is estimated by

$$\sum_k (4g - 1)^{2k} (4g - 1)^{-\frac{\left\lfloor (2g+4)(4g-2)+\eta \right\rfloor k}{(4g-2)(4g+2)}} = \sum_k (4g - 1)^{-\frac{\eta k}{(4g-2)(4g+2)}} < \infty$$

proving the result. 

6. Open problems.

(1) A suitable version of Theorem 2.6 should hold for other symmetric spaces. In particular, recall that arithmetic manifolds appear as fundamental domains $G/\Gamma$ where $G$ is a connected semi-simple algebraic $\mathbb{R}$-group without compact factors of $\mathbb{R}$-rank $\geq 2$, and $\Gamma$ is a lattice in $G$ (cf. [Mar74, Mar75, Mar77]). Thus we expect that a version of Theorem 2.6 should hold in higher rank setting. Note however, that for higher rank symmetric spaces closed orbits are not isolated but appear in families.

(2) The proof of Theorem 4.1 relies on localized perturbations. Therefore it does not work in the analytic category. We expect that Theorem 4.1 is still valid for analytic metrics but the proof would require new ideas.
(3) It is likely that an explicit lower bound for the gaps in the length spectrum could also be obtained for prevalent set of negatively curved metrics (see [Kal] for related results) but we do not pursue this question here.

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References


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