

AMSC 667 – Numerical Analysis II

Spring Term 2005

Instructor: Georg Dolzmann

Homework set #12

Problem 1: [Qualifying exam, August 2004]

Consider the boundary value problem

$$(1) \quad -u''(x) + u(x) = f(x) \quad \text{on } 0 \leq x \leq 1$$

subject to the boundary conditions

$$u'(0) = 2, \quad u(1) = 0.$$

(a) Derive a variational formulation for (1).

(b) What do we mean by the Finite Element Approximation u_h to u ?

(c) State and prove an estimate for

$$\|u - u_h\|_1 \left(\int_0^1 |u(x) - u_h(x)|^2 dx + \int_0^1 |u'(x) - u'_h(x)|^2 dx \right)^{1/2}.$$

(d) Prove the formula

$$\|u - u_h\|_1^2 = \|u\|_1^2 - \|u_h\|_1^2.$$

Problem 2: [Qualifying Exam, January 2004]

(a) Consider the boundary value problem

$$(2) \quad -u''(x) + b(x)u(x) = f(x) \quad \text{on } 0 \leq x \leq 1$$

subject to the boundary conditions

$$u'(0) = u_\ell, \quad u(1) = u_r$$

where $b(x), f(x) \in C([0, 1])$, and $b(x) \geq 0$. Formulate a difference method for the approximate solution of (2) on a uniform mesh of size h . Explain how $u''(x)$ is approximated by a difference quotient.

(b) Suppose $b(x) = 0$ and $u_\ell = u_r = 0$ in (2). Formulate a finite element method for the approximate solution of (2) in this special case, again on a uniform mesh. Using the standard “hat functions” basis for the finite element space, write out the finite element equations explicitly. Show that if an appropriate quadrature formula is used on the right-hand side of the finite element equations, they (the finite element equations) are the same as the finite difference equations.

(c) Show that the matrix in (b) is nonsingular.

Problem 3: Consider the boundary value problem

$$(3) \quad -u''(x) + u(x) = f(x), \quad x \in (0, 1)$$

with periodic boundary conditions

$$(4) \quad u(0) = u(1), \quad u'(0) = u'(1).$$

Consider the function space $V = \{v \in H^1(0, 1) : v(0) = v(1)\}$ and the variational formulation: Find $v \in V$ such that for all $v \in V$

$$(5) \quad \int_0^1 (u'(x)v'(x) + u(x)v(x)) \, dx = \int_0^1 f(x)v(x) \, dx.$$

(a) Let $u \in C^2([0, 1])$ be a solution of (5). Show that u is a solution of (3) and (4).

(b) Let $h = 1/N$ with $N \in \mathbb{N}$. We partition $[0, 1]$ into N subintervals of length h . Let V_h be the *subspace* of the space V consisting of piecewise linear functions on this partition. What is the dimension of V_h ? Give a basis of V_h consisting of functions of smallest support (nodal basis functions).

(c) Give the finite element formulation corresponding to (5). Let \mathbf{U} be the coefficient vector of u with respect to the basis. Determine the stiffness matrix A and the right-hand side vector \mathbf{F} so that the finite element solution satisfies $A\mathbf{U} = \mathbf{F}$. Give explicit values (in terms of N) for all entries of the stiffness matrix A .