Math 462 - Partial Differential Equations for Scientists and Engineers

Spring Term 2005

Instructor: Georg Dolzmann

Homework set #7

Problem 1: [Strauss, Problem 2.1.1]

Solve the wave equation on the whole line, $u_{tt} = c^2 u_{xx}$, with initial conditions $u(x,0) = e^x$ and $u_t(x,0) = \sin x$.

Problem 2: [Strauss, Problem 2.1.3]

The midpoint of a piano string of tension T, density ρ, and length ℓ is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $\ell/4$ from one end (assume that $a < \ell/4$; otherwise, poor flea!). How long does it take for the disturbance to reach the flea?

Problem 3: [Strauss, Problem 2.1.5]

(The hammer blow.) Let $\Phi(x) = 0$ and $\Psi(x) = 1$ for $|x| < a$ and $\Psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c$, a/c, $3a/2c$, $2a/c$, and $5a/c$.

Hint: Calculate

$$\begin{aligned} u(x,t) &= \frac{1}{2}\big(\Phi(x-ct) + \Phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+xt} \Psi(y)\,\mathrm{d}y \\ &= \frac{1}{2c}\big(\text{length of } (x-ct, x+ct) \cap (-a,a)\big). \end{aligned}$$

Then

$$u(x, a/2c) = \frac{1}{2c}\big(\text{length of } (x-a/2, x+a/2) \cap (-a,a)\big).$$

This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.

Problem 4: [Strauss, Problem 2.1.6]

In Problem 3, find the greatest displacement, $\max_x u(x,t)$ as a function of t.

Problem 5: [Cooper, Problem 5.5.1] Solve the IBVP

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in } 0 < x < L, t \in \mathbb{R},$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0 \quad \text{for } t \in \mathbb{R},$$

and the initial conditions

$$\begin{aligned} u(x,0) &= f(x) = 7\sin(2\pi x/L) - 2\sin(5\pi x/L) \\ u_x(x,0) &= g(x) = -4\sin(3\pi x/L) \end{aligned}$$

for $0 < x < L$. Think!! You will not need the full expansion of the solution. What frequencies are present in the solution? What is the period of the motion?

Problem 6: [Cooper, Problem 5.5.2] Solve the IBVP

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in } 0 < x < L, t \in \mathbb{R},$$

with homogeneous Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$. Take $\Phi(x) = 0$ and $\Psi(x) = x$.