Parameters Estimation in Stochastic Process Model

A Quasi-Likelihood Approach

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Outline

The Big Model in Mind: Signal + Noise

In many physical phenomenon, the Response/Feedback can be decomposed as:

$$\textit{Response} = \textit{Signal} + \textit{Noise}$$

One way to represent this model is through the form of a SDE related to a semimartingale X_t as:

$$dX_t(\theta) = dA_t(\theta) + dM_t(\theta) \qquad t > 0$$
$$X_0 = x_0$$

where $A_t(\theta)$, a predictable finite variation(FV) process, and $M_t(\theta)$, a Càdlàg local martingale, are defined on a complete probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a right-continuous filtration.

The Big Model in Mind: Signal + Noise

OR, written in the form of a stochastic integral as:

$$X_t = X_0 + \int_0^t f_s(\theta) d\lambda_s + M_t(\theta), \qquad t \geq 0$$

where $\{\lambda_s\}$ is a real nondecreasing right-continuous predictable process with $\lambda_0 = 0$, $\{f_t\}$ is a predictable process, and $\{M_t(\theta)\}$ is an Càdlàg local martingale with $M_0(\theta) = 0$.

Note: we say a process X_t is **adapted** if X_t is $\mathcal{F}_t$ measurable; we say a process X_t is **predictable** if X_t is a left-continuous adapted process.

Model of Interest: Jump-Diffusion Process

General form of Jump-Diffusion Process

The one-dimensional Jump-Diffusion process can be defined as:

$$dX_t = d_t(\theta; X_t)dt + \gamma_t(\theta; X_t)dW_t + \delta_t(\theta; X_t, \Delta Z_t^\theta)dZ_t^\theta, \quad t > 0 \quad (1)$$

For some $\theta \in \Theta \subset \mathbb{R}^p$, let $d_t(\theta; \cdot)$, $\gamma_t(\theta; \cdot)$ and $\delta_t(\theta; \cdot)$ be predictable functionals. The process $\{W_t\}$ is the Wiener process and $\{Z_t^\theta\}$ is a Compound Poison process

$$Z_t^\theta = \sum_{i=1}^{N_t} Y_i$$

where $\{N_t\}$ is a counting process with intensity $\{\alpha_t(\theta)\}$, and $\{Y_i\}$ are i.i.d random variables with distribution F_θ independent of $\{N_t\}$.

Examples of Jump-Diffusion Processes

1. *A Neurophysiological Model*

$$dV_t = (-\rho V_t + \lambda)dt + dM_t$$

(e.g. Kallianpur(1983)), where M_t is a discontinuous martingale with a (centered) generalized Poisson distribution and $V(t)$ is the membrane potential.

2. *A CIR Interest Rate Model*

$$dX_t = \alpha(\beta - X_t)dt + \sigma\sqrt{X_t}dW_t$$

where $X_0 > 0$, $\alpha > 0$, $\beta > 0$ and $\sigma > 0$.

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Examples of Jump-Diffusion Processes

3 *A Counting Process Model*

$$dX_t = \theta J_t dt + dM_t$$

with multiplicative intensity $\Lambda_t = \theta J_t$, $J_t > 0$ a.s. being predictable and M_t a square integrable martingale.

The Likelihood Approach in Parameters Estimation

Formal Steps in Deriving Likelihood and MLE:

- Find out the Drift, Diffusion and Jump characteristics of X_t.
- Verify there exist a $\theta_0 \in \Theta$ s.t. the measure P_θ induced by X_t will be dominated by P_{θ_0}.
- Verify certain integrability conditions under the measure P_{θ_0}.
- Derive the Radon-Nikodym Derivative, so the Likelihood function.
- Derive the MLE.

Formal Steps Applied on a Simplified Jump-Diffusion Process

We consider a sub-model of SDE (1):

$$dX_t = d_t(\theta; X_t)dt + \gamma_t(X_{t-})dW_t + \delta_t(X_{t-}, \Delta Z_t)dZ_t, \quad t > 0, \quad X_0 = x_0$$

deriving from SDE (1) by assuming the follow:

- $d_t(\theta; X_t) = d_t(X_t) + \gamma_t(X_t)\pi_t\theta$ where π_t is a known nonrandom quantity.
- The jump intensity λ_t and jump size distribution $F(y)$ do NOT depend on θ.

Formal Steps Applied on a Simplified Jump-Diffusion Process

Then it can be checked that the Log-Likelihood function of θ given the *continuously* observed data X_t for any fixed θ_0 is given by:

$$\ell_t(\theta) = \theta \int_0^t \pi_s \gamma_t (X_{t-})^{-1} dX_s^{(c)}(\theta_0) - \frac{1}{2}\theta^2 \int_0^t \pi_s^2 ds$$

with

$$\int_0^t \pi_s \gamma_t (X_{t-})^{-1} dX_s^{(c)}(\theta_0)$$
$$= \int_0^t \pi_s \gamma_t (X_{t-})^{-1} dX_s(\theta_0) - \int_0^t \pi_s \gamma_t (X_{t-})^{-1} d_s(\theta, X_s) ds - \sum_{s \leq t} \pi_s \gamma_t (X_{t-})^{-1} \Delta X_s$$

where

$$X_t^{(c)}(\theta) = X_t - X_0 - \sum_{s \leq t} \Delta X_s - \int_0^t d_s(\theta, X_s) ds$$

Formal Steps Applied on a Simplified Jump-Diffusion Process

Thus the MLE is given by:

- Under P_{θ_0}

$$\hat{\theta}_t = \left[\int_0^t \pi_s^2 ds\right]^{-1} \left\{\int_0^t \pi_s \gamma_s (X_{s-})^{-1} dX_s^{(c)}(\theta_0)\right\}$$

- Or under P_θ

$$\begin{aligned}\hat{\theta}_t &= \left[\int_0^t \pi_s^2 ds\right]^{-1} \left\{\int_0^t \pi_s \gamma_s (X_{s-})^{-1} dX_s^{(c)}(\theta)\right. \\ &\quad \left. + \theta \int_0^t \pi_s \gamma_s (X_{s-})^{-1} \gamma_s (X_{s-}) \pi_s ds\right\} \\ &= \left[\int_0^t \pi_s^2 ds\right]^{-1} \int_0^t \pi_s dW_s + \theta\end{aligned}$$

Formal Steps Applied on a Simplified Jump-Diffusion Process

So one sees, under P_θ,

$$\widehat{\theta}_t \simeq N\left(\theta, \left[\int_0^t \pi_s^2 ds\right]^{-1}\right)$$

Examples

1. $dX_t = \theta dt + dW_t + dN_t, \quad t \geq 0, \quad X_0 = x_0$

   W_t : standard Wiener process

   N_t : Poisson process with intensity λ

   Then the MLE's of (θ, λ) are given by:

$$\hat{\theta}_t = \frac{X_t^{(c)}}{t}, \quad \hat{\lambda}_t = \frac{N_t}{t}$$

2. $dX_t = \theta X_t dt + \sigma dW_t + dN_t, \quad t \geq 0, \quad X_0 = x_0$

   Then the MLE's of (θ, λ) are given by:

$$\hat{\theta}_t = \frac{\int_0^t X_{s-} dX_s^{(c)}}{\int_0^t X_s^2 ds} \quad \text{and} \quad \hat{\lambda}_t = \frac{N_t}{t}$$

Examples

3 Model for Security Prices:

$$dX_t = \theta X_t dt + X_{t-} dW_t + X_{t-} dZ_t, \quad Z_t = \sum_{i=1}^{N_t} \varepsilon_i$$

N_t is a Poisson process with parameter λ

ε_i are i.i.d. rv's bounded below by -1.

The MLE for θ is given by

$$\hat{\theta}_t = t^{-1} \log\left(\frac{X_t}{x_0}\right) + \frac{1}{2} - t^{-1} \sum_{s \le t} \left(\frac{X_s}{X_{s-}}\right)$$

and

$$\hat{\theta}_t \simeq N(\theta, \frac{1}{t})$$