Law of Large Numbers for Increasing Subsequences of Random Permutations

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Let the random variable $Z_{n,k}$ denote the number of increasing subsequences of length k in a random permutation from S_n, the symmetric group of permutations of $\{1, ..., n\}$. We show that the weak law of large numbers holds for Z_{n,k_n} if $k_n = o(n^{\frac{2}{5}})$; that is,

$$\lim_{n\to\infty} \frac{Z_{n,k_n}}{EZ_{n,k_n}} = 1 \text{ in probability.}$$

The proof uses the second moment method and demonstrates that this method cannot work if the condition $k_n = o(n^{\frac{2}{5}})$ does not hold. It follows from results concerning the longest increasing subsequence of a random permutation that the law of large numbers cannot hold for Z_{n,k_n} if $k_n \geq cn^{\frac{1}{2}}$, with $c > 2$. Presumably there is a critical exponent l_0 such that the law of large numbers holds if $k_n = O(n^l)$, with $l < l_0$, and does not hold if $\limsup_{n\to\infty} \frac{k_n}{n^l} > 0$, for some $l > l_0$. Several phase transitions concerning increasing subsequences occur at $l = \frac{1}{2}$, and these would suggest that $l_0 = \frac{1}{2}$. However, we show that the law of large numbers fails for Z_{n,k_n} if $\limsup_{n\to\infty} \frac{k_n}{n^{\frac{4}{9}}} = \infty$. Thus the critical exponent, if it exists, must satisfy $l_0 \in [\frac{2}{5}, \frac{4}{9}]$. To show that the the law of large numbers fails, we use a celebrated result by Baik, Deift and Johansson (1999) concerning the length of the longest increasing subsequence in a random permutation, as well as the following result, which is of independent interest: Place n cards numbered from 1 to n on a line in increasing order from left to right. Randomly designate k_n of the cards. Pick up the $n-k_n$ other cards and randomly reinsert them between the k_n designated cards that remained on the line. Denote the resulting measure on S_n by $\mu_{n;k_n}$, and denote the uniform measure on S_n by U_n. Then the law of large numbers holds for $Z_{n;k_n}$ if and only if the distance between U_n and μ_{n,k_n} converges to 0 in the total variation norm as $n \to \infty$.