Math 461, Sec. 0501 Cumulative File of Quizzes & Solutions

Quiz 1 , Mon. 9/9

(1). Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + 6 &= 0 \\ 3x_1 + 6x_2 + 12x_3 - x_4 &= -5 \\ 3x_2 + 6x_3 - 3x_4 &= 9 \end{aligned}$$

(a) (*4 points*) Write the augmented matrix for this system. Reduce the resulting matrix to row-echelon form by a series of elementary row operations.

Solution: the matrix and successive row-reduced intermediate matrices are:

$$\begin{pmatrix} 1 & 2 & 4 & 0 & -6 \\ 3 & 6 & 12 & -1 & -5 \\ 0 & 3 & 6 & -3 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & -1 & 13 \\ 0 & 3 & 6 & -3 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 4 & 0 & -6 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & -13 \end{pmatrix}$$

(b) (*4 points*) Use the row-echelon form you found in (a) to conclude whether the original system has a solution (x_1, x_2, x_3, x_4). If so, tell how many solutions there are. Justify your answer.

Solution. Since all 3 equations have pivots, there is at least one solution. Since the number of pivots is less than the number of variables, at least one variable mustt be free, and there are infinitely many solutions. In fact (although this was not required on the quiz), back-solving from the row-echelon form shows

$$x_4 = -13, \quad x_1 = -2x_4 - 12 = 14, \quad x_2 = -2x_3 - 10, \quad x_3 \textit{ free}$$

(2). (*2 points*) Suppose that the augmented matrix of a linear system of equations in the four variables x_1, x_2, x_3, x_4 has been reduced by elementary row operations to the row-echelon form

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Give the general solution of the system.

Solution: backsolving gives $x_4 = 3,\ x_3 = -5,\ x_1 = -2x_2 - 2,\ x_2$ *free.*

Quiz 2 , Fri. 9/13

(1). (a) (*4 points*) Give a complete description ('parametric' or in terms of specified 'free' variables) of the set of all vectors $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4$ such that

$$x_1 \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \\ 4 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution. Row reducing the coefficient matrix gives

$$\begin{pmatrix} 0 & 1 & -2 & -1 \\ 2 & -2 & -3 & 1 \\ 0 & 1 & 4 & 1 \\ 4 & 5 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 2 & -2 & -3 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & 4 & 1 \\ 0 & 9 & 6 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 2 & -2 & -3 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 24 & 8 \end{pmatrix}$$

and one further step (replace the last row by itself minus 3 times the 3rd row) yields a matrix with final row all 0's. So x_4 has no pivot and is free, and the other variables all have pivots, leading to the solution $x_3 = (-1/3)x_4$, $x_2 = (1/3)x_4$, $x_1 = x_2 + (3/2)x_3 - (1/2)x_4 = (-2/3)x_4$, or solution-set

$$\Big\{ \mathbf{x} \in \mathbf{R}^4 \;:\; x_4 \in \mathbf{R},\ x_1 = \frac{-2}{3}x_4,\ x_2 = \frac{1}{3}x_4\ x_3 = \frac{-1}{3}x_4 \Big\} = span\Big\{ \begin{pmatrix} -2 \\ 1 \\ -1 \\ 3 \end{pmatrix} \Big\}$$

(b) (*3 points*) Is there a solution vector $\mathbf{y} \in \mathbf{R}^4$ of the equation $A\mathbf{y} = \mathbf{b}$ for every $\mathbf{b} \in \mathbf{R}^4$, where

$$A = \begin{pmatrix} 0 & 1 & -2 & -1 \\ 2 & -2 & -3 & 1 \\ 0 & 1 & 4 & 1 \\ 4 & 5 & 0 & 1 \end{pmatrix} \qquad ?$$

Solution. The row-echelon reduction given in (a) shows that there are only 3 pivots, but 4 variables, so Theorem 4 of Sec. 1.4 says that there **cannot** *be a solution for every* $\mathbf{b}$*, even though we saw in (a) that there are infinitely many solutions for* $\mathbf{b} = \mathbf{0}$.

(c) (*3 points*) Find the general solution $\mathbf{z} \in \mathbf{R}^4$ (either in set notation with simplified constraints, or in terms of 'free variables') of the equation

$$A\,\mathbf{z} \;=\; \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution. First note that $z_1 = z_2 = z_3 = 0,\, z_4 = 1$ *gives a particular solution of this equation. Then the general solution is the sum of this vector and any element of the general solution in (a) of the homogeneous equation. So the desired general solution, in terms of the free variable* x_4, *is*

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} -2/3 \\ 1/3 \\ -1/3 \\ 1 \end{pmatrix}$$

Quiz 3 , Wed. 9/18

(1). *(3 points) Are the following three vectors linearly independent ?*

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} , \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Solution. First we use a row reduction of the matrix consisting of these three columns, which we think of as the coefficient matrix A *for which the assertion of* **linear dependence** *would be that the equation* $A\mathbf{x} = \mathbf{0}$ *has a nontrivial solution.*

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the echelon form has only 2 pivots but three variables, the equation $A\mathbf{x} = \mathbf{0}$ *does have nontrivial solutions in terms of the free variable* x_3, *namely* $x_2 = (-1/2)x_3$, $x_1 = -x_2 - x_3 = (-1/2)x_3$. *The existence of* *nontrivial solutions (which is all we needed to establish for this problem, says precisely that the displayed vectors are linearly dependent.*

(2). (*3 points*) Suppose that A is a 4×4 matrix whose row-echelon reduced form has a pivot in every row. What can you say about

$$\text{range}(A) \;=\; \left\{ A\mathbf{x} : \quad \mathbf{x} \in \mathbf{R}^4 \right\} \qquad ?$$

Solution. We have a Theorem which says that there is a solution to the equation $A\mathbf{x} = \mathbf{b}$ *for each choice of* $\mathbf{b} \in \mathbf{R}^4$ *if and only if the row-echelon form of* A *has a pivot in each row. Thus* $range(A) = \mathbf{R}^4$.

(3). (*4 points*) Describe the general set of vectors $\mathbf{y} \in \mathbf{R}^3$ such that

$$\begin{pmatrix} 1 & -5 & -7 \\ -3 & 5 & 7 \end{pmatrix} \mathbf{y} \;=\; \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

Give either the formulas for all components of $\mathbf{y}$ as functions of the free variable(s) or the *parametric vector* form for the solution set.

Solution. Again the mechanics comes down to row-reductions as part of solving equations. First comes the single row operation on the augmented matrix, leading to the echelon form:

$$\begin{pmatrix} 1 & -5 & -7 & -2 \\ -3 & 5 & 7 & -2 \end{pmatrix} \mathbf{y} \;=\; \begin{pmatrix} 1 & -5 & -7 & -2 \\ 0 & -10 & -14 & -8 \end{pmatrix} \mathbf{y}$$

Back-solving, we find that y_3 *is free, and* $y_2 = (-1/10)(14y_3 - 8) = -1.4y_3 + .8$, $y_1 = 5y_2 + 7y_3 - 2 = 5(-1.4y_3 + .8) + 7y_3 - 2 = 2$. *The parametric form of the general solution set is*

$$\begin{pmatrix} 2 \\ .8 \\ 0 \end{pmatrix} \;+\; y_3 \begin{pmatrix} 0 \\ -1.4 \\ 1 \end{pmatrix}$$

Note the correspondence between the parametric form of the solution set and the Theorem which says that the solutions are sums of a particular solution — which is the first of the two vectors in the last displayed line — and the general solution of the homogeneous equation, which is the span of the second vector in the last displayed line.

Quiz 4 , Math 461

(1). Consider the transformation $T: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ which maps

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ to } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ to } \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ to } \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

(a) (*5 points*) Write the standard matrix representation for the transformation T, and find $T\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$. Then give the standard matrix representation of the transformation $S: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ defined by the equation, for all $\mathbf{x} \in \mathbf{R}^3$, that $S\mathbf{x} = T\mathbf{x} - \mathbf{x}$.

Solution. The matrices A for T, B for T and mapped vector are

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 3 \\ 1 & 0 & 3 \end{pmatrix}, \quad T\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

(b) (*5 points*) Completely describe (in parametric vector form if possible) the set of weight-vectors $\mathbf{x} \in \mathbf{R}^3$ such that $T\mathbf{x} = \mathbf{0}$, where T is as defined in part (a), and *also* completely describe *range*(T) as a set of vectors in $\mathbf{R}^3$.

Solution. Row-reductions on the augmented matrix $(A \,|\, \mathbf{b})$ give

$$\begin{pmatrix} 1 & -1 & 2 & b_1 \\ 0 & 3 & 3 & b_2 \\ 1 & 0 & 3 & b_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 2 & b_1 \\ 0 & 1 & 1 & b_2/3 \\ 0 & 1 & 1 & b_3 - b_1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 2 & b_1 \\ 0 & 1 & 1 & b_2/3 \\ 0 & 0 & 0 & b_3 - b_1 - b_2/3 \end{pmatrix}$$

The fact that not all rows of the coefficient matrix have pivots means that range(T) is not all of $\mathbf{R}^3$, and in fact consists only of $\{\mathbf{b} \in \mathbf{R}^3 : b_1 = b_3 - b_2/3\}$. The set of $\mathbf{x}$ solving $A\mathbf{x} = \mathbf{0}$ is obtained as the vectors with x_3 free, $x_2 = -x_3$, $x_1 = x_2 - 2x_3 = -3x_3$, which in parametric vector form is the line

$$x_3 \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

Quiz 5 , Math 461

(1). Let A denote the 3×3 matrix

$$A = \begin{pmatrix} 1 & 3 & 9 \\ 2 & 6 & 10 \\ 1 & 4 & 4 \end{pmatrix}$$

(a) Evaluate $\det(A)$ by any method. Explain why your method shows that A is invertible.

Solution. The first method is to expand by minors. For example, if you expand across the first row, then you get $1(24-40)-3(8-10)+9(8-6) = 8$. *A second method is to row-reduce using elementary row operations and keep track of the effect on the determinant, e.g.*

$$\begin{pmatrix} 1 & 3 & 9 \\ 2 & 6 & 10 \\ 1 & 4 & 4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & 9 \\ 0 & 0 & -8 \\ 0 & 1 & -5 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & -5 \\ 0 & 0 & -8 \end{pmatrix}$$

where the first row-operations leave the determinant unchanged and the second (swapping the second and third rows) multiplies it by -1. *Therefore the final determinant is* -1 *times the product of the diagonal elements of the last upper-diagonal matrix, or* 8.

(b) Find the entry in the 1^{st} row and 2^{nd} column of the inverse of A.

Solution. The $(1,2)$ *element of* A^{-1} *by Cramer's rule is* $(-1)^{1+2} A_{21}/\det(A) = -\det \begin{pmatrix} 3 & 9 \\ 4 & 4 \end{pmatrix} /8 = 3.$

(c) Use the identity $A\,A^{-1} = I$ to find $\det(A^{-1})$.

Solution. $1 = \det(I) = \det(A\,A^{-1}) = \det(A)\,\det(A^{-1})$, *which shows that* $\det(A^{-1}) = 1/\det(A) = 1/8.$

Quiz 6 , Math 461

(1). The 4×6 matrix A and its reduced row-echelon form are given by

$$A = \begin{pmatrix} 0 & 1 & 7 & 9 & 2 & -1 \\ 0 & -2 & 5 & 10 & 0 & 2 \\ 1 & 1 & 4 & 0 & -3 & 6 \\ 0 & -4 & 3 & 7 & 1 & 1 \end{pmatrix} , \quad \mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & -8.59 & 12.12 \\ 0 & 1 & 0 & 0 & -.69 & -.47 \\ 0 & 0 & 1 & 0 & 1.57 & -1.65 \\ 0 & 0 & 0 & 1 & 0.92 & 1.12 \end{pmatrix}$$

(a) Give at least two different bases for the vector space $\mathbf{col}(A)$, i.e., the space spanned by the columns of A.

Solution. The test for whether a subset of columns spans the same space as the whole matrix is whether there is a pivot in every column **for the matrix consisting of the restricted set of columns**. *This is so because if* $\mathbf{b} \in range(A)$, *then there is a solution* $\mathbf{x}$ *of* $A\mathbf{x} = \mathbf{b}$, *and after row-reduction, for all choices of free variables in the general solution the variables in pivot columns are uniquely determined, and choosing the free variables equal to* 0 *means that solution* $\mathbf{x}$ *exists with non-zero entries only in the pivot columns. So two different bases are given by* $\mathcal{B}_1$ *equal to the first through 4th columns of* A, *and by* $\mathcal{B}_2$ *consisting of columns 1,2,3,5 of* A.

Another interpretation and valid solution of the problem: since $\mathbf{rref}(A)$ *has a pivot in every row,* $\mathbf{col}(A) = \mathbf{R}^4$, *and any two bases of* $\mathbf{R}^4$ *would work, including sets of 4 columns of* $\mathbf{rref}(A)$, *which would otherwise not automatically give bases of* $\mathbf{col}(A)$.

(b) Is *every* set of 4 out of the first 5 columns of A a basis for $\mathbf{col}(A)$? Justify your answer.

Solution. The answer is **yes**, *according to the criterion above. Each* 4×4 *matrix constructed of 4 columns of the first 5 columns of matrix* A *is seen with very easy row-reductions to have a pivot in every column, or equivalently to have a pivot in every row, or equivalently to be invertible.*

(c) Give a basis for the vector space $\mathbf{null}(A)$.

Solution. Recall that we are looking for a minimal set of vectors spanning the set of general solutions $\mathbf{x}$ *of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$, *which is exactly the same as the set of solutions of* $\mathbf{rref}(A)\mathbf{x} = \mathbf{0}$. *Recognizing that* *the latter general solution has variables* x_5, x_6 *free and the other variables unieuqly solved in terms of them using* $\mathbf{rref}(A)$, *we have the parametric vector form of the null space*

$$x_5 \begin{pmatrix} 8.59 \\ .69 \\ -1.57 \\ -.92 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -12.12 \\ .47 \\ 1.65 \\ -1.12 \\ 0 \\ 1 \end{pmatrix}$$

The null space therefore has the last two displayed vectors as a basis, since these two vectors are obviously linearly independent (neither being a multiple of the other).

Quiz 7 , Math 461

(1). The 7×6 matrix A and its row-echelon form B are given by

$$A = \begin{pmatrix} 1 & 0 & -1 & 4 & 1 & -3 \\ -1 & 2 & 4 & -6 & -1 & 7 \\ 1 & 2 & 2 & 8 & -4 & 0 \\ 1 & -2 & -4 & 6 & 1 & 0 \\ -3 & 2 & 6 & -14 & -3 & 13 \\ 1 & 4 & 5 & 0 & 1 & 5 \\ 1 & -2 & -4 & 6 & 1 & -7 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 & 4 & 1 & -3 \\ 0 & 2 & 3 & -2 & 0 & 4 \\ 0 & 0 & 0 & 6 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Explain in detail how you know that the 5^{th} column of A is a vector of $\mathbf{R}^7$ which lies in the span of the first 4 columns of A.

Solution 1. Consider the system of equations $(\mathbf{a}_1 \,|\, \mathbf{a}_2 \,|\, \mathbf{a}_3 \,|\, \mathbf{a}_4)\,\mathbf{x} = \mathbf{a}_5$, *where* $\mathbf{a}_j$ *denote the columns of A. Row-reduction of the augmented matrix for this system gives echelon form equal to the first five columns of* B, *which means that a consistent solution* $\mathbf{x} \in \mathbf{R}^4$ **does** *exist. In fact, since the variable* x_3 *is free in this system, there are infinitely many solutions, or in other words, by choosing the free variable* $x_3 = 0$, *there is a solution showing that* $\mathbf{a}_5$ *is actually a linear combination of* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$.

Solution 2. We know that in the matrix consisting of the first 5 columns of A, the pivot columns span the column space, which implies $\mathbf{a}_5 \in span(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4)$.

(b) How many linearly independent *rows* does A have ? Explain how you know.

Solution. We know that $\dim(col(A)) = \#$ *pivot columns* $= 4$*, which is the same as the rank of* A*, and also is the dimension of the span of the rows, and the latter is the number of linearly indpendent rows. A simpler way to see the same answer is to notice that the row-operations which led to* B *do not change the number of linearly independent rows, and that the number of linearly independent rows is evidently the same as the number of pivots in* B*, which is 4.*

(2). Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \in \mathbf{R}^{10}$ are linearly independent vectors, and that

$$\mathbf{w} \in \mathbf{R}^{10} \quad , \quad \mathbf{w} \notin \text{span}\left(\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\right\}\right)$$

Explain in detail why $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{w}\}$ is a linearly independent set of vectors.

Solution 1 (formal). If $\sum_{j=1}^{6} c_j \mathbf{v}_j + a\mathbf{w} = \mathbf{0}$ *for some scalars* c_j, a, *then we first exclude the possibility* $a \neq 0$ *because that would imply* $\mathbf{w}$ *could be expressed as a linear combination of* $\mathbf{v}_1, \ldots, \mathbf{v}_5$*; and then we conclude all* $c_j = 0$ *by linear independence of the* v_j*'s. This shows the constants* $a.$, c_j *are* $\mathbf{0}$*, and proves linear independence of the six vectors.*

Solution 2. We are told $W_0 = span(\mathbf{v}_1, \ldots, \mathbf{v}_5) \subset span(\mathbf{v}_1, \ldots, \mathbf{v}_5, \mathbf{w}) = W_1$*. The first space has dimension* 5 *(matrix* $(\mathbf{v}_1 \,|\, \cdots \,|\, \mathbf{v}_5)$ *has 5 pivots), but the second space is strictly larger, so must have dimension at least 6, and therefore* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{w}\}$ *must be a basis of it. Also: (a third way): the matrix* $(\mathbf{v}_1 \,|\, \cdots \,|\, \mathbf{v}_5 \,|\, \mathbf{w})$ *must also have a pivot in the 6th column because* $\mathbf{w} \notin span(\mathbf{v}_1, \ldots, \mathbf{v}_5)$*. Hence all 6 columns are linearly independent.*

Quiz 8 , Math 461

(1). Let $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$. You may take as given that A has one eigenvalue equal to 1, with corresponding eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(a) (*4 pts*) Find the other eigenvalue of A and its corresponding eigenvector.

Solution. Since $\det(A-\lambda I) = (5-\lambda)(2-\lambda-4 = \lambda^2-7\lambda+6 = (\lambda-1)(\lambda-6)$, *the eigenvalues are* 1, 6, *and an eigenvector* $\mathbf{x}$ *for eigenvalue 6 must satisfy* $5x_1 + 2x_2 = 6x_1$, *or* $x_1 = 2x_2$, *and* $(2,1)^T$ *is such an eigenvector.*

(b) (*3 pts*) Show how to diagonalize A, i.e., how to represent it in the form $A = P\,D\,P^{-1}$, where D is a diagonal 2×2 matrix. Give P and D explicitly.

Solution. $P = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & \\ 0 & 6 \end{pmatrix}$.

(c) (*3 pts*) If a matrix $B = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}^{-1}$, then find a scalar (not matrix) numerical expression for the (1,1) element of B^6 .

Solution. Since $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = -\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$, *we have*

$$B^6 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^6 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$(B^6)_{11} = (3 \;\; 2) \begin{pmatrix} 1 & 0 \\ 0 & 2^6 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 4 \cdot 2^6 - 3 = 253$$

Quiz 9 , Math 461

(1). (a) Suppose that the matrix A is given by

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

Find the solution $\mathbf{x}(t) \in \mathbf{R}^2$ for all $t > 0$ to the system of linear constant-coefficient differential equations

$$\dot{\mathbf{x}}\,(t) = A\,\mathbf{x}(t) \quad \textit{with initial condition} \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(b) Find a non-zero initial vector $\mathbf{v} \in \mathbf{R}^2$ for which the solution of $\dot{\mathbf{x}}\,(t) = A\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) = \mathbf{v}$ tends to 0 as $t \to \infty$.

Solution. Since $\binom{2}{-1}$, $\binom{1}{1}$ *are the eigenvectors of* A *for respective e.v.'s* $3, -2$, *the general solution is* $c_1 e^{3t} \binom{2}{-1} + c_2 e^{-2t} \binom{1}{1}$. *For (a), we choose* c_1, c_2 *so that*

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \textit{or} \qquad \mathbf{c} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 4/3 \end{pmatrix}$$

For (b), take $c_1 = 0,\ c_2 \neq 0$.

(2). Suppose that the 2×2 real matrix B is given by

$$B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

Find the eigenvalues and eigenvectors of B.

Solution. Now $\binom{2}{-1}$, $\binom{1}{1}$ *are respectively the real and imaginary parts of an eigenvector, as we can check directly by noting*

$$B \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} (1-i)$$

Thus $(2+i, -1+i)^T$ *is the eigenvector for eigenvalue* $1-i$, *and by complex conjugation* $(2-i, -1-i)^T$ *is the eigenvector for eigenvalue* $1+i$.

Quiz 10 , Math 461

(1). Find the unique linear combination $\mathbf{v}$ of the two vectors

$$\mathbf{v_1} = \begin{pmatrix} .1 \\ .3 \\ .3 \\ .9 \end{pmatrix} \quad , \quad \mathbf{v_2} = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

in $\mathbf{R}^4$ which is as close as possible to the vector $\mathbf{w} = (1, 2, 3, 4)^T$. Your answer should be the single vector $\mathbf{v}$, not the coefficients used to obtain it.

Hint: orthogonalize the two vectors $\mathbf{v}_1, \mathbf{v}_2$ *! Note that* $\mathbf{v}_1$ *has unit length.*

Solution.

```
>> A = [[.1 .3 .3 .9]' [5 1 1 1]'];  %% dot-product = 2
>> v3 = (A(:,2)-2*A(:,1))'
  4.80   0.40   0.40  -0.80
>> B = [A(:,1), v3]
    0.1000    4.8000
    0.3000    0.4000
    0.3000    0.4000
    0.9000   -0.8000
>> [1 2 3 4] * B
    5.2000    3.6000
>> v3' * v3
  24
>> 5.2*B(:,1)+(3.6/24)*B(:,2)  %%% This is the v vector
    1.2400
    1.6200
    1.6200
    4.5600
%%% So the combination is
       5.2 * v1 + (3.6/24) * v3
     = 5.2 * v1 + .15 * (v2- 2*v1)
     = 4.9 * v1 + .15 * v2
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Quiz 11 , Math 461

(1). In the vector space $V = \mathbf{R}^3$ with the inner product defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = 2v_1 w_1 + v_2 w_2 + v_3 w_3 \tag{1}$$

you are given the three vectors

$$\mathbf{y} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} , \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} , \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

(a) *With respect to the inner product defined in (1),* find an orthonormal basis of the vector space $W = span(\mathbf{v}_1, \mathbf{v}_2)$ in V.

Solution. Using this new inner-product, we find $\langle \mathbf{v}_2, \mathbf{v}_1 \rangle = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 4$. *Therefore we perform Gram-Schmidt orthonormalization to get* $\mathbf{v}_2 - \frac{4}{4}\mathbf{v}_1 = (0,1,1)^T \perp \mathbf{v}_1$, *anmd the o.n. basis is* $\mathbf{u}_1 = \mathbf{v}_1/\sqrt{4} = (.5, .5, -.5)^T$, $\mathbf{u}_2 = (0,1,1)^T/\sqrt{2}$.

(b) *With respect to the inner product defined in (1), and the associated length* $\|\mathbf{w}\| = (\langle \mathbf{w}, \mathbf{w} \rangle)^{1/2}$, find the vector $\hat{\mathbf{y}} \in W = span(\mathbf{v}_1, \mathbf{v}_2)$ which minimimizes $\|\mathbf{y} - \hat{\mathbf{y}}\|^2$.

Solution. The projection is $\hat{\mathbf{y}} = \langle \mathbf{y}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{y}, \mathbf{u}_2 \rangle \mathbf{u}_2 = -\mathbf{u}_1 - \sqrt{2}\mathbf{u}_2 = (-0.5, -1.5, -0.5)^T$.

(c) *With respect to the inner product and length* $\| \cdot \|$ *defined above,* find a unit vector in $V = \mathbf{R}^3$ orthogonal to $W = span(\mathbf{v}_1, \mathbf{v}_2)$.

Solution. By the projection principle, $\mathbf{y} - \hat{\mathbf{y}} \perp W$, *so we make a unit vector by* $(\mathbf{y} - \hat{\mathbf{y}})/\|\mathbf{y} - \hat{\mathbf{y}}\| = 1.5(1,-1,1)^T/(1.5\sqrt{4}) = (0.5, -0.5, 0.5)^T$.