Math 461, HW 1 Solutions

p. 11, #19. The row-reduction to echelon form of this augmented matrix is

$$\begin{pmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{pmatrix} \mapsto \begin{pmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{pmatrix}$$

The system is consistent if and only if the coefficient part of the last row is not all zeroes, i.e., $h \neq 2$.

p. 11, #25. Again we row-reduce to echelon form:

$$\begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{pmatrix} \mapsto \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{pmatrix} \mapsto \begin{pmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{pmatrix}$$

Since the last row of the coefficient part of the augmented matrix is all zeroes (and the other rows have a pivot), consistency is equivalent to the lower-right element being 0, i.e., to validity of the equation $k+2g+h=0$.

p. 26, #23. If a 3×5 coefficient matrix has three pivot columns, i.e., one in each row, then one can back-solve, first for the variable corresponding to the last pivot column, then for the variable corresponding to the previous pivot column, and finally for the variable corresponding to the first pivot column. Along the way, 2 of the variables will turn out to be free, which means that the system actually is consistent with infinitely many solutions.

p. 38, #11. The augmented matrix with columns $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{b}$ has echelon reduction:

$$\begin{pmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this, we conclude that there are infinitely many sets of weights x_1, x_2, x_3 with x_3 free, for which $x_1\mathbf{a}_1 + x_2\mathbf{a_2} + x_3\mathbf{a_3} = \mathbf{b}$.

p. 39, #28. (a). $27.6x_1 + 30.2x_2$ million Btu.

(*b*). $$x_1 \begin{pmatrix} 27.6 \\ 3100 \\ 250 \end{pmatrix} + x_2 \begin{pmatrix} 30.2 \\ 6400 \\ 360 \end{pmatrix}$$

```
 (c). MATLAB sol'n:
>> A = [  27.6 30.2 162 ; 3100 6400 23610; 250 360 1623];
>> A(2,:) = A(2,:) - (A(2,1)/A(1,1))*A(1,:);
>> A(3,:) = A(3,:) - (A(3,1)/A(1,1))*A(1,:);
>> A(3,:) = A(3,:) - (A(3,2)/A(2,2))*A(2,:);
>> A(1,:) = A(1,:) - (A(1,2)/A(2,2))*A(2,:);
>> x = [ A(1,3)/A(1,1) A(2,3)/A(2,2)]
x =
    3.9000    1.8000     %% respective amounts of A and B
```

p. 48, #17. Row operations lead to

$$\begin{pmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So exactly 3 rows have pivot positions. Since this is less than the number of variables, the equation $A\mathbf{x} = \mathbf{b}$ does **not** have a solution for each $\mathbf{b} \in \mathbf{R}^4$.

p. 48, #21. In general, three vectors cannot span all of $\mathbf{R}^4$. In this particular case, since each of the three vectors has components summing to 0, any vector $x_1\mathbf{v}_1 + x_2\mathbf{v_2} + x_3\mathbf{v}_3 \in \mathbf{R}^4$ must have the same property, but the general vector in $\mathbf{R}^4$ certainly does not have this property.

```
p.49, #39.  Matlab solution:
>> A = [12 -7 11 -9 5; -9 4 -8 7 -3; -6 11 -7 3 -9; 4 -6 10 -5 12];
>> for i=2:4 A(i,:) = A(i,:)- (A(i,1)/A(1,1))*A(1,:); end;
>> for i=[1 3 4] A(i,:) = A(i,:)- (A(i,2)/A(2,2))*A(2,:); end;
>> for i=1:3 A(i,:) = A(i,:)- (A(i,3)/A(4,3))*A(4,:); end;
>> A(3:4,:) = A([4 3],:)
A =
   12.0000         0         0   -5.7143  -13.1429
         0   -1.2500         0    0.3720    0.3869
         0         0    5.6000   -2.7333    8.1333
         0         0         0         0   -2.0000
```

Since there is a pivot in every row, the columns of the matrix do span $\mathbf{R}^4$.

p. 55, #13. This is the parametric vector form of a set defined in terms of free variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5+4x_3 \\ -2-7x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}$$

p. 55, #16. We reduce the augmented matrix to find solutions:

$$\begin{pmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution has the form x_3 free, $x_2 = 3x_3 + 3,\ x_1 = -4x_3 - 5.$ In parametric vector form, this is

$$\begin{pmatrix} -4x_3-5 \\ 3x_3+3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$$

Of these last two displayed vectors, the first is a particular solution of the (inhomogeneous) equation-system of this problem, while the span of the second is the general solution of the homogeneous equation-system (i.e., of problem 6, p. 55).

Math 461, HW 2 Solutions

p.71, #11. Row-reductions for the coefficient matrix give

$$\begin{pmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{pmatrix} \mapsto \begin{pmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & -5 & h+4 \end{pmatrix} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & h-6 \end{pmatrix}$$

which means that the columns are linearly *dependent* iff the homogeneous equation has a consistent nontrivial solution, which happens iff $h = 6$.

#42. (MATLAB) This problem uses the MATLAB command **rref**. Then note that we are trying to choose the largest possible subset of columns such that we have a pivot in every column.

```
>> rref(C)
     1     0     2     0     2     0
     0     1    -3     0    -2     0
     0     0     0     1    -1     0
     0     0     0     0     0     1
     0     0     0     0     0     0
```

From this it is easy to conclude that columns 1,2,4,6 of the original matrix form a largest possible set of linearly independent columns. (The first four rows have pivots.)

p.79, #38. (MATLAB) Use **rref**. Then specify free variables, and back-solve by inspection, to get:

```
>> rref(B)
    1.0000         0         0    0.7500
         0    1.0000         0    1.2500
         0         0    1.0000   -1.7500
         0         0         0         0
```

giving the general solution x_4 free, $x_3 = 1.75x_4$, $x_2 = -1.25x_4$, $x_1 = -0.75x_4$.

p.90, #12. In problem 8, the matrix is given as a product

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \quad \text{for} \quad \vartheta = \pi/2$$

#37. (MATLAB) The transformation is 1-to-1 iff there is a pivot in every row, which we check using **rref**.

p.99, #7. The Kirchhoff's Law equations are:

$$A \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{pmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{pmatrix} = \begin{pmatrix} 40 \\ 30 \\ 20 \\ -10 \end{pmatrix}$$

which is a little messy to row-reduce by hand, but either hand-calculator arithmetic or MATLAB using **rref** yields the solution $I_1 = 11.434$, $I_2 = 10.550$, $I_3 = 8.036$, $I_4 = 5.840$.

#12. The problem is vaguely stated, but if we understand the given matrix A as the proportions of cars picked up at various places which are returned to the different possible places *within a single day*, then we are to calculate respectively $A\mathbf{b} = (307.08, 48.1\,94.82)^T$ and $AA\mathbf{b} = (309.75, 48.03, 92.21)^T$, where $\mathbf{b} = (304, 48, 98)^T$ to give the row vectors showing numbers of cars at different locations for Tuesday and Wednesday.

p.116, #17. By looking at the matrix sizes for A, B, we conclude that B must be 2×3. The linear equations for its first two columns, and the augmented-matrix row-reductions used to provide the solutions, are as follows:

$$\begin{pmatrix} 1 & -2 & -1 \\ -2 & 5 & 6 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -9 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -5 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \end{pmatrix}$$

from which we read of the solutions $b_{11} = 7,\ b_{21} = 4,\ b_{12} = -8,\ b_{22} = -5.$

p.126, #13. $AB = AC$. If A is invertible, then multiplying this equation on the left by A^{-1} gives $B = A^{-1}(AB) = A^{-1}(AC) = C$. However, if A is not invertible, in the most extreme case if $A = \mathbf{0}$, then the equation $AB = AC$ does not give enough information to conclude that $B = C$.

p.132, #41. (MATLAB) MATLAB command $\mathbf{A} \setminus \mathbf{b}$ can be used to get the exact solution and solution to the rounded system.

```
>> A = [4.5 3.1 19.249 19.25; 1.6 1.1 6.843 6.84];
>> B = [ A(:,1:2)\ A(:,3)  A(:,1:2)\ A(:,4)]
    3.9400    2.9000   %% 1st column is ans to (a),
    0.4900    2.0000   %%      2nd to (b)
>> 100*abs((B(:,2)-B(:,1))./B(:,1))
  26.3959
308.1633    %% Percent errors
>> cond(A(:,1:2))
    3.3630e+03
```

The very large relative (percentage) errors in solving the rounded system arise because the matrix is nearly singular, as can be seen by checking that the determinant of the coefficient matrix is small (-0.01), or that the *condition number* $= 3.363e3$.

Math 461, HW 3 Solutions

p.190, #9. First choose the co-factor for the (3,1) element, then the one for the (1,3) element of the remaining 3×3 matrix, to obtain: $\det = (-1)^4 \cdot 2 \cdot (-1)^4 \cdot 5 \cdot \det \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix} = 10.$

#35. $\det(EA) = \det \begin{pmatrix} a+ck & b+kd \\ c & d \end{pmatrix} = (a+ck)d - (b+kd)c = ad - bc = (1) \cdot (ad - bc)$, which is as it should be since $\det(E) = 1$.

#43. (MATLAB) Generate random numbers in 5×5 matrix Y by the command **Y=rand(5,5)**.

```
>> for i=1:3,
    X = rand(5,5); Y = rand(5,5);
    [det(X+Y) det(X)+det(Y)]
   end;
...
   -0.2673    0.1220
    0.1679    0.0486
   -0.1569   -0.1219
```

Thus $\det(A+B) \neq \det(A) + \det(B)$. Other examples are easy to produce.

p.209, #5. Check first by minors that if A is the coefficient matrix, then $\det(A) = 2(-1) - 1(-6) = 4$. Then the Cramer's rule solution is

$$x_1 = \frac{1}{4} \det \begin{pmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{pmatrix} = \frac{3}{2}\,, \quad x_2 = \frac{1}{4} \det \begin{pmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{pmatrix} = 4, \quad x_3 = \frac{-7}{2}$$

p.223, #7. Not a subspace, because scalar multiplication of a plynomial with integer coefficients by an irrational number, like $\sqrt{2}$, gives a plynomial of the same degree but with *non*-integer coefficients.

#17. The displayed vector with free variables has the form

$$a\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}+b\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}+c\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$$

So the vectors appearing in this combination span the space W.

#35. (MATLAB) The problem is simply to show that the equation $(\mathbf{v}_1\,|\,\mathbf{v}_2\,|\,\mathbf{v}_3)\,\mathbf{x} = \mathbf{w}$ has a solution, which is easily done in MATLAB by

```
>> [7 -4 -9; -4 5 4; -2 -1 4; 9 -7 -7] \ [-9 7 4 8]'
   7.5000
   3.0000
   5.5000
```

p.234, #3. The echelon form of A has pivots in both rows, and the non-pivot columns correspond to free variables x_3, x_4. The reduced row-echelon form of the matrix is $\begin{pmatrix}1&0&-7&6\\0&1&4&-2\end{pmatrix}$. So, after backsolving for the pivot-variables x_1, x_2 in terms of the free variables, we find that the null space is spanned by vectors $(7,-4,1,0)^{tr}$, $(-6,2,0,1)^{tr}$.

#31. (a) For $\mathbf{p}(t) = a+bt+ct^2$, $\mathbf{q}(t) = e+ft+gt^2$ and scalar λ, we have $T(\mathbf{p}+\mathbf{q}) = \begin{pmatrix}a+e\\a+b+c+e+f+g\end{pmatrix} = T(\mathbf{p})+T(\mathbf{q})$, and $T(\lambda\,\mathbf{p}) = \begin{pmatrix}\lambda a\\\lambda(a+b+c)\end{pmatrix}$. (b) For the kernel, we are solving $a=0,\ a+b+c=0$, giving $a=0, c=-b$, and $ker(T) = \text{span}(t-t^2)$. The range of T is all of $\mathbf{R}^2$, since $T(x_1+(x_2-x_1)t) = \begin{pmatrix}x_1\\x_2\end{pmatrix}$.

#39. (MATLAB) The discussion in the text says that bases of the column space can be found as the columns with pivots of the row-echelon form.

```
>> rref(A)
   1.0000         0    0.3333         0    3.3333
        0    1.0000    0.3333         0   -8.6667
        0         0         0    1.0000   -4.0000
        0         0         0         0         0
```

(a) The matrix B consisting of the columns (1,2,4) with pivots therefore has the same column space as the original matrix A. That is, the columns (3 and 5) without pivots are in the span of the columns with pivots.
(b) Using the reduced echelon form above, we conclude that the null space of A (general solution of the homogeneous equation) is: x_3, x_5 free, $x_4 = 4x_5$, $x_2 = -x_3/3+26x_5/3$, $x_1 = -x_3/3-10x_5/3)$, so the null space in spanned by the

$$\begin{pmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \quad \begin{pmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{pmatrix} \quad \in \mathbf{R}^5$$

(c) Not one-to-one because the null-space is nontrivial. Not onto because the range, which is the same as the column-space of B and thus is spanned by three vectors, cannot be all of $\mathbf{R}^4$.

Math 461, HW 4 Solutions

p.243, #15. The method is to choose a subset of the vectors $\mathbf{v}_j$ corresponding to the pivot columns of the matrix $(\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \cdots \mathbf{v}_6)$: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis because

$$\text{rref} \begin{pmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

#36. Same idea using sets of pivot columns as bases: $\{\mathbf{u}_1, \mathbf{u}_2\}$ is basis for span(H); $\{\mathbf{v}_1, \mathbf{v}_2\}$ is basis for span(K); but for span(H+K) (which means the span of the set of 6 vectors obtained from H and K together, if U denotes the 4×3 matrix with columns $\mathbf{u}_j$, and V the corresponding matrix with columns $\mathbf{v}_j$, the MATLAB command **rref**$([UV])$ shows that the 4×6 matrix has pivot columns (therefore column-space basis) $\mathbf{u_1}, \mathbf{u_2}, \mathbf{v_1}$.

p.253, #17. The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is equal to $5\mathbf{v}_1 - 2\mathbf{v}_2 = -5\mathbf{v}_1 - 2\mathbf{v}_3$.

#36. $\mathcal{B}$ is a basis for H, i.e., is linearly independent, because **rref**$(B) =$ **rref**$(\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3)$ has pivots in rows 1, 2, and 3. The $\mathcal{B}$ coordinate vector of

$\mathbf{x}$ is obtained by solving the equation $B\,[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$, using the first 3 entries of the last column of $\mathbf{rref}(\mathbf{v}_1\,|\,\mathbf{v}_2\,|\,\mathbf{v}_3\,|\,\mathbf{x})$ which is $(3,5,2)^T$.

p.260, #21. It is enough to show that each of the standard basis elements 1, t, t^2, t^3 of $\mathcal{P}_3$ can be obtained as a linear combination of the first four Hermite polynomials, i.e.: $1 = 1$, $t = (1/2)(2t)$, $t^2 = (1/4)(-2+4t^2) + (1/2)1$, $t^3 = (1/8)(-12t+8t^3) + (3/4)(2t)$.

#34. (MATLAB) The 7×7 matrix whose columns are the $\mathcal{B}$ coordinate vectors of the vectors in $\mathcal{C}$ can be read off from the given trig-identity expansions, as

$$P \;=\; \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 18 \\ 0 & 0 & 0 & 4 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{pmatrix}$$

The columns of P are already exhibited in row-echelon form with pivot in every row, so are linearly independent. They span the same set H as the standard basis $\mathcal{B}$, and by their linear independence are also a basis.

p.269, #9. The number (6) of columns is the sum of the null-space dimension (4) and the column-space dimension, which is therefore 2.

#35. (a) For the given matrix A,

$$\text{rref}(A) \;=\; \begin{pmatrix} 1 & 0 & 6.5 & 0 & 5 & 0 & -3 \\ 0 & 1 & 5.5 & 0 & .5 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5.5 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies that the desired matrices C (matrix whose columns are basis for col(A)) and N (matrix whose columns are basis for null(A)) are respectively

$$C = \begin{pmatrix} 7 & -9 & 5 & -3 \\ -4 & 6 & -2 & -5 \\ 5 & -7 & 5 & 2 \\ -3 & 5 & -1 & -4 \\ 6 & -8 & 4 & 9 \end{pmatrix}, \; N \; = \begin{pmatrix} -6.5 & -5 & 3 \\ -5.5 & -.5 & -2 \\ 1 & 0 & 0 \\ 0 & 5.5 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Here the columns of C are just the columns of A containing pivots, and the columns of N are obtained by taking the negatives of the elements of the nonzero rows in the non-pivot columns of rref(A) as the entries in the pivot-column numbered rows of N, and extending this matrix by an identity matrix in the non-pivot-column-numbered rows of N, as described in Example 8. Note that we cannot instantly see from rref(A) which rows of A form a basis of row(A), because they could have been re-arranged by elementary row operations in arriving at the echelon form. But we can find out which rows to use by selecting the ones with numbers the same as the pivot columns of rref(A^T) shown below, i.e., rows 1 through 4. (Alternatively, just use first 4 rows of rref(A).)

(b). To produce a matrix M whose columns form a basis of $Nul(A^T)$, we operate on the non-pivot columns of rref(A^T) in the same way we did before in finding a basis for $Nul(A)$ from rref(A), giving:

$$\text{rref}(A^T) = \begin{pmatrix} 1 & 0 & 0 & 0 & -0.1818 \\ 0 & 1 & 0 & 0 & -3.7273 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2.5455 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0.1818 \\ 3.7273 \\ 0 \\ -2.5455 \\ 1 \end{pmatrix}$$

The number of rows of S is equal to the number of columns of A (because that is how many entries each vector in nul(A) has). The number of columns of S is the number of rows of R (equal to the dimension of row(A), which is also the dimension of col(A) or the rank of A) plus the dimension of Nul(A): thus S is square because the number of columns of A is $rank(A) + dim(Nul(A))$. Similarly, the fact that T is square reflects the analogous fact for A^T. Finally, $\det(S) = \mathbf{det}([R^T\ N]) = -11046$, and $\det(T) = \mathbf{det}([C\ M]) = -941.818$. So both S, T are invertible.

#19, p. 276 (MATLAB) (a). If the matrix P were the change-of-coordinates matrix for $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \rightarrow \mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then its columns would be the the $\mathcal{C}$ coordinate vectors for the $\mathcal{B}$ basis elements, i.e.,

$$P \;=\; \begin{pmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{pmatrix} \;=\; \Big([\mathbf{u}_1]_{\mathcal{C}}\,\Big|\,[\mathbf{u}_2]_{\mathcal{C}}\,\Big|\,[\mathbf{u}_3]_{\mathcal{C}}\Big)$$

So the matrix $U = (\mathbf{u}_1, |\, \mathbf{u}_2 \,|\, \mathbf{u}_3)$ can be given in terms of the corresponding matrix V with columns $\mathbf{v}_j$ by $U \;=\; V\,P$ which after MATLAB calculation of the product gives

$$U \;=\; (\mathbf{u}_1, |\, \mathbf{u}_2 \,|\, \mathbf{u}_3) \;=\; \begin{pmatrix} -6 & -6 & -5 \\ -5 & -9 & 0 \\ 21 & 32 & 3 \end{pmatrix}$$

(b) In the same way, we are now being asked to find the matrix $W = (\mathbf{w}_1 \,|\, \mathbf{w}_2 \,|\, \mathbf{w}_3)$ such that $V = WP$, or $W = V\,P^{-1}$. Via MATLAB, the answer is

$$V * inv(P) \;=\; \begin{pmatrix} 28 & 38 & 21 \\ -9 & -13 & -7 \\ -3 & 2 & 3 \end{pmatrix}$$

p.296, #11. The matrix P and steady-state probability vector q are given by

$$P \;=\; \begin{pmatrix} .7 & .6 \\ .3 & .4 \end{pmatrix} \quad , \qquad \mathbf{q} \;=\; \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

The vector $\mathbf{q}$ could be obtained either as the first column of P^{100} or as the solution of the equation $(P - I)\mathbf{q} = \mathbf{0}$. The fraction listening to the news after a long time is $q_1 = 2/3$.

Math 461, HW 5 Solutions

p.308, #37. (MATLAB) Let the given matrix be denoted A. The first step is to find the eigenvalues by the comand eig(A). They are 13, 3, 13. Next, the command rref(A - 13*eye(3)) provides a reduced matrix whose null-space consists of eigenvectors $\mathbf{x}$ for e.v. 13, with x_2, x_3 free and $x_1 = -2x_2 - x_3$. This space has basis $(-2, 1, 0)^T$, $(-1, 0, 1)^T$. The eigenvector with e.v. 3 is any nonzero vector in the null-space of rref(A-3*eye(3)), and so is given by $(5/9, -2/9, 1)^T$.

p.317, #12. Expand the determinant of $A - \lambda I$ by the (3,3) minor to obtain characteristic polynomial $(2 - \lambda)(4 - \lambda)(-1 - \lambda)$.

p.325, #11. We are given that the eignevalues are 1, 2, 3. Solve successively for eigenvectors:

for e.v. 1 : $-x_1 + 4x_2 - 2x_3 = x_1,\ -3x_1 + 4x_2 = x_2 \implies x_2 = x_1 = x_3$

for e.v. 2 : $-x_1 + 4x_2 - 2x_3 = 2x_1,\ -3x_1 + 4x_2 = 2x_2 \implies x_2 = 1.5x_1 = x_3$

for e.v. 3 : $-x_1 + 4x_2 - 2x_3 = 3x_1,\ -3x_1 + 4x_2 = 3x_2 \implies x_2 = 3x_1,\ x_3 = 4x_1$

Accordingly, we get a basis of eigenvectors $(1, 1, 1)^T$, $(2, 3, 3)^T$, $(1, 3, 4)^T$, and the diagonalization is:

$$\begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}^{-1}$$

#35. (MATLAB) After entering the 5×5 matrix as B, we do the follwing steps: eig(B) gives the list of eigenvalues with multiplicity: 3, 5, 5, 1, 1; next rref(B-3*eye(5)) has null-vector (therefore eigenvector of B for e.v. 3) of $(.5, -.25, -1, -.25, 1)^T$; similarly rref(B-1*eye(5)) has two-dimensional null-space, the eigenvectors for e.v. 1, spanned by the two basis vectors $(.8, -.6, -.4, 1, 0)^T$, $(.6, -.2, -.8, 0, 1)^T$; finally, a basis of 2 eigenvectors for e.v. 5 is given as a basis for the null-space of

```
>>rref(B-5*eye(5))
    1.0  0    0   -2.0      -1.0
    0    1.0  0    0.3333   0.3333
```

```
0    0    1.0  1.0        1.0
0    0    0    0          0
0    0    0    0          0
```

so the vectors $(2,-1/3,-1,1,0)^T$, $(1,-1/3,-1,0,1)^T$ will serve. Since in all, we have a basis of 5 eigenvectors, the matrix B is diagonalized with

$$P=\begin{pmatrix} .5 & .8 & .6 & 2 & 1\\ -.25 & -.6 & -.2 & -1/3 & -1/3\\ -1 & -.4 & -.8 & -1 & -1\\ -.25 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 1\end{pmatrix},\quad D=\begin{pmatrix} 3&0&0&0&0\\ 0&5&0&0&0\\ 0&0&5&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\end{pmatrix}$$

p.333, #6. $T(\mathbf{p}(t))=(1+t^2)\mathbf{p}(t)$, so the image in (a) is $(1+t^2)(2-t+t^2)=2-t+3t^2-t^3+t^4$, and we check linearity in (b) by observing $(1+t^2)\lambda\mathbf{p}(t)=\lambda(1+t^2)\mathbf{p}(t)$ and $(1+t^2)(\mathbf{p}(t)+\mathbf{q}(t))=(1+t^2)\mathbf{p}(t)+(1+t^2)\mathbf{q}(t)$. Finally, the matrix representation is obtained by reading off the coordinates of $T(1)=1+t^2$, $T(t)=t+t^3$, $T(t^2)=t^2+t^4$ respectively as columns, giving

$$\begin{pmatrix} 1&0&0\\ 0&1&0\\ 1&0&1\\ 0&1&0\\ 0&0&1\end{pmatrix}$$

#32. (Matlab) Any basis of eigenvectors will do. First, eig(C) gives the list of e.v.'s 5, 4, 2, 4, and rref(C-5*eye(4)) has null-vector $(2.75,-.75,1,1)^T$ equal to eigenvector for A. Similarly, rref(C-4*eye(4)) gives basis of eigenvectors with e.v. 4 as $(-10,-7/3,1,0)^T$, $(13,5/3,0,1)^T$, and null-vector for rref(C-2*eye(4)) is eigenvector $(0,-1.5,1.5,1)^T$ for e.v. 2.

p.341, #16. First we solve the characteristic polynomial $(5-\lambda)(3-\lambda)+2=\lambda^2-8\lambda+17$ at $\lambda=4\pm i$. Then, by Theorem 9, the desired similarity is achieved with $C=\begin{pmatrix}4&-1\\1&4\end{pmatrix}$ and P with columns the real and imaginary parts of the complex eigenvector for e.v. $4-i$. Since that complex eigenvector solves $5x_1-2x_2=(4-i)x_1$, or $(1+i)x_1=2x_2$, we can take as eigenvector $(2, 1+i)^T$, and $P = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. Alternatively, the eignevector for $4-i$ can also be taken as $0.5(1-i)(2,\, 1+i)^T = (1-i, 1)^T$.

#28. (Matlab) Let A be the matrix. Then eig(A) gives the eigenvalues as $-0.4-i$, $-0.4+i$, $-0.2-0.5i$, $-0.2+0.5i$. Now the series of commands

```
>> [V G] = eig(A);
>> c = V(:,1)/real(V(4,1));
>> d = V(:,3)/real(V(2,3));
>> P = [real(c), imag(c), real(d), imag(d)];
```

gives

$$P = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 1 & -1 \\ 2 & 0 & 1 & 1 \\ 1 & -1 & -2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -0.4 & 1 & 0 & 0 \\ -1 & -0.4 & 0 & 0 \\ 0 & 0 & -0.2 & 0.5 \\ 0 & 0 & -0.5 & -0.2 \end{pmatrix}$$

Note that for some reason, MATLAB reverses the order of the eigenvectors associated with $-.4-i$ and $-.4+i$ among the columns of V , so that the latter comes first, and similarly reverses the order of the eigenvectors associated with $-.2-.5i$ and $-.2+.5i$ among the columns of V , so that the latter is the third column.

p.361, #2. We express the initial value in coordinates with respect to $\mathbf{v}_1$, $\mathbf{v}_2$, as $\mathbf{x}(0) = .5\mathbf{v}_1 + 2.5\mathbf{v}_2$. Then, letting $\mathbf{x}(t) = c_1(t)\mathbf{v}_1 + c_2(t)\mathbf{v}_2$, with $\mathbf{c}(0) = (.5, 2.5)^T$, we find from the matrix form of the differential equation that

$$c_1'(t) = -3c_1(t),\;\; c_2'(t) = -c_2(t) \qquad \Longrightarrow \qquad c_1(t) = .5e^{-3t},\;\; c_2(t) = 2.5e^{-t}$$

Hence $\mathbf{x}(t) = .5e^{-3t}\,\mathbf{v}_1 + 2.5e^{-t}\,\mathbf{v}_2$.

#15. (Matlab) Here there turn out to be only real eigenvalues: eig(A) gives 1, -1, -2, and the same commands rref(A-eye(3)), etc. as before, lead to null vectors (eigenvectors of A) forming the basis $\mathbf{b}_1 = (-4, 1, 4)^T$, $\mathbf{b}_2 = (-6, 1, 5)^T$, $\mathbf{b}_3 = (-1, 0, 1)^T$. So the general differential-equation solution is given by $\mathbf{x}(t) = c_1e^t\,\mathbf{b}_1 + c_2e^{-t}\,\mathbf{b}_2 + c_3e^{-2t}\,\mathbf{b}_3$. The origin is a *saddle point*: trajectories are attracted exponentially in the $\mathbf{b}_2$, $\mathbf{b}_3$ directions and grow exponentially in the $\mathbf{b}_1$ direction. For large values of t, the trajectories of $\mathbf{x}(t)$ are asymptotic to the direction $\mathbf{b}_1$.

Math 461, HW 6 Solutions

p.392, #12. Projection is

$$\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} \Big/ \begin{pmatrix} -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right\} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} .4 \\ -1.2 \end{pmatrix}$$

#36. (MATLAB)

```
>> A' * A
   100     0     0     0
     0   100     0     0
     0     0   100     0
     0     0     0   100
>> U = A / 10;

%% U'*U is 4x4 identity, U'*U is 8x8 with lots of non-zero entries
%% but  rank(U*U')  is still 4.

(b)
>> y = rand(8,1); p = U*U'*y; z=y-p;
>> z'*p
    2.4980e-16

%% p is in col(A)=col(U) because = U b   with b =
>> (U'*y)'
   0.3654    0.5019    1.1225   -0.2190
(c) >> U'*z   %% all entries mult. 1.0e-15
(d) >> z in (col A)perp  by projection principle.
```

p.400, #12. Since $\mathbf{v}_1 \perp \mathbf{v}_2$,

$$\mathbf{z} = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{30}{10}\mathbf{v}_1 + \frac{26}{26}\mathbf{v}_2 = \begin{pmatrix} -1 \\ -5 \\ -3 \\ 9 \end{pmatrix}$$

#25. (MATLAB) $U^T\mathbf{y}$ gives 4 coefficients, so the answer is: $U * U' * \mathbf{y}$.

p.407, #10. $\mathbf{u}_1 = (-1,3,1,1)^T$, $\mathbf{u}_2 = (6,-8,-2,-4)^T - \frac{-36}{12}\mathbf{u}_1 = (3,1,1,-1)^T$, $\mathbf{u}_3 = (6,3,6,-3)^T - \frac{6}{12}\mathbf{u}_1 - \frac{30}{12}\mathbf{u}_2 = (-1,-1,3,-1)^T$.

#25. (Matlab)

```
>> A = [-10 13 7 -11; 2 1 -5 3; -6 3 13 -3;
         16 -16 -2 5; 2 1 -5 -7];
>> Q=A; Q(:,1)=Q(:,1)/(Q(:,1)' * Q(:,1))^.5;
   Q(:,2) = Q(:,2)-(Q(:,2)'*Q(:,1))*Q(:,1);
   Q(:,2)=Q(:,2)/(Q(:,2)'*Q(:,2))^.5;
   Q(:,3)=Q(:,3)- Q(:,1:2) * Q(:,1:2)' * Q(:,3);
   Q(:,3)=Q(:,3)/(Q(:,3)'*Q(:,3))^.5;
   Q(:,4)=Q(:,4)- Q(:,1:3) * Q(:,1:3)' * Q(:,4);
   Q(:,4)=Q(:,4)/(Q(:,4)'*Q(:,4))^.5; R = Q'*A;
>> [Q' R]
-0.500  0.100 -0.300  0.8000  0.100    20 -20 -10    10
 0.500  0.500 -0.500  0       0.500     0   6  -8    -6
 0.577  0.000  0.577  0.577   0.000     0   0 10.392 -5.196
 0      0.707 -0.000 -0.0000 -0.707     0   0   0     7.071

>> [Q2 R2] = qr(A);
>> Q2(:,1:4)'    %%% this is same as Q'  above, R2(1:4,:) = R
```

p.416, #6. By matrix multiplication and then augmented-matrix row-reduction,

$$A^T A \;=\; \begin{pmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \quad , \qquad A^T \mathbf{y} \;=\; \begin{pmatrix} 27 \\ 12 \\ 15 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 6 & 3 & 3 & 27 \\ 0 & 3/2 & -3/2 & -3/2 \\ 0 & -3/2 & 3/2 & 3/2 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 6 & 0 & 6 & 30 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So the general solution has x_3 free, $x_2 = x_3 - 1$, $x_1 = -x_3 + 5$, and has the form $(5, -1, 0)^T + x_3(-1, 1, 1)^T$.

#16. From the formula, the least-squares solution is $(A^T A)^{-1} A^T \mathbf{b} = R^{-1}(R^T)^{-1} R^T Q^T \mathbf{b} =$

$$R^{-1} Q^T \mathbf{b} \;=\; \frac{1}{10} \begin{pmatrix} 5 & -3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} .5 & .5 & .5 & .5 \\ -.5 & .5 & -.5 & .5 \end{pmatrix} \begin{pmatrix} -1 \\ 6 \\ 5 \\ 7 \end{pmatrix} \;=\; \begin{pmatrix} 2.9 \\ .9 \end{pmatrix}$$

p.425, #8. (a) The design matrix and least-squares solution are

$$A = \begin{pmatrix} x_1 & \cdots & x_n \\ x_1^2 & \cdots & x_n^2 \\ x_1^3 & \cdots & x_n^3 \end{pmatrix}^T$$

```
(b). >> x= (4:2:18)'; y= [1.58 2.08 2.5 2.8 3.1 3.4 3.8 4.32]';
   A = [ x x.*x x.^3 ];
>> beta = (A'*A) \ A' * y ; yhat = A * beta;
>> beta'
   0.5132   -0.0335    0.0010
>> yhat'
1.5822  2.0935  2.4833  2.8003  3.0933  3.4111  3.8024  4.3160
>> y'
1.5800  2.0800  2.5000  2.8000  3.1000  3.4000  3.8000  4.3200
```

#10 (MATLAB) (a) The linear model has design matrix $A = \begin{pmatrix} e^{-.02t_1} & \cdots & e^{-.02t_k} \\ e^{-.07t_1} & \cdots & e^{-.07t_k} \end{pmatrix}^T$, and $\beta = (M_A, M_B)^T$.

```
>> t = [10 11 12 14 15]'; y = [21.34 20.68 20.05 18.87 18.30]';
>> A = [exp(-.02*t) exp(-.07*t)];
>> M = (A'*A) \ A'*y
   19.9411
   10.1015
>> [y' ; (A*M)']
   21.3400   20.6800   20.0500   18.8700   18.3000
   21.3427   20.6802   20.0472   18.8624   18.3076
```