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Review Problems for Final Math 461 Exam

(1). (*Determinants, minors, etc.*) Let

$$A = (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3) = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

(a) Find the volume for the solid geometric body (tetrahedron) $\{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 \,:\, 0 \le x_1,\, x_2,\, x_3 \le 1\}$ with vertices $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

(b) Find A^{-1} via Cramer's rule.

(c) Find $A^{-1}\,(1,2,3)^T$ via row-reduction.

(2). (*MATLAB, Invertibility*) Suppose $B = (\mathbf{w}_1 \,|\, \mathbf{w}_2 \,|\, \cdots \,|\, \mathbf{w}_6)$ is a 6×6 matrix.

(a) Give 4 different MATLAB commands involving B, each producing an output from which you can read off *by inspection* whether the columns of B are linearly independent. Explain what each output is and why it gives enough information to distinguish linear independence.

(b) Now assume B is invertible. Give a MATLAB command (or short sequence of commands) to produce an orthonormal basis of $W = \mathit{span}(\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3)$ and an orthonormal basis of $W^\perp$.

(c) Explain in symbols what the elements of the 6×6 matrix

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>> diag(diag(B'*B).^ (-1/2))  * B'*B * diag(diag(B'*B).^ (-1/2))
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are. Can you interpret these elements ?

(3). (*ODE's, complex eigenvalues*) Solve the differential equation for all $t > 0$:

$$\dot{\mathbf{x}}\,(t) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x}(t) \quad , \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Hint: solve for $(x_2(t),\, x_3(t))$ *first, using first principles or formulas from the book. Then (harder part) solve the equation as an inhomogeneous equation for* x_1, *plugging in the functions already found for* $x_2(t),\, x_3(t)$.

(4). (*Null & Col Spaces*) By hand, find bases for $Nul(A)$, $col(A)$, and $(row(A))^{\perp}$, where

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 1 \\ 1 & 2 & -1 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

(5). (*Symmetric matrices*) Suppose that

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Find the values of

$$\max_{\mathbf{x}:\, \|\mathbf{x}\|=1} \mathbf{x}^T B \mathbf{x} \quad , \qquad \min_{\mathbf{y}:\, \|\mathbf{y}\|=1} \mathbf{y}^T B \mathbf{y}$$

and give the vectors $\mathbf{x}$, $\mathbf{y}$ for which the *max* and *min* are achieved.

(6). (*Least-squares*) A vector of observations $\mathbf{y} = (y_1, \ldots, y_{50})^T$ is to be expressed as accurately as possible in mean-squared error sense according to the model $y_i \approx a + bx_i + cx_i^2$. Suppose that

$$\sum_{i=1}^{50} x_i = 10 \ , \ \sum_{i=1}^{50} x_i^2 = 20 \ , \ \sum_{i=1}^{50} x_i^3 = 40 \ , \ \sum_{i=1}^{50} x_i^4 = 90$$

$$\sum_{i=1}^{50} y_i = 20 \ , \ \sum_{i=1}^{50} x_i y_i = 40 \ , \ \sum_{i=1}^{50} x_i^2 y_i = 60$$

Find the least-squares values of a, b, c.

(7). The 3×3 matrix A is given in the form $A = P\,D\,P^{-1}$, where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \quad , \qquad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Find the approximate values of $\|A^k\,\mathbf{w}\|^{1/k}$ and $A^k\mathbf{w}/\|A^k\,\mathbf{w}\|$ when k is large, where $\mathbf{w} = (1, 1, 3)^T$.

(8). (*Change of Basis*) Suppose that

$$C = (\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, \mathbf{c}_3) = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ -1 & 0 & 1 \end{pmatrix} \quad , \quad B = (\mathbf{b}_1 \,|\, \mathbf{b}_2 \,|\, \mathbf{b}_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

(a) Find the 3×3 matrix A whose j'th column is the coordinate vector of $\mathbf{c}_j$ with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

(b) What is the relationship between the eigenvectors and eigenvalues of the matrices A (defined in part (a)) and C ?

(9). (*Orthogonal polynomials*) Let $\mathcal{P}_2$ denote the vector space of polynomials $a_0 + a_1 t + a_2 t^2$ of degree 2 or less, and define an inner product on such polynomials by $\langle p(t), q(t) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

(a) Find an orthonormal basis for $\mathcal{P}_2$ with respect to this inner product.

(b) Explain why this inner-product *cannot* also define an inner product on the vector space $\mathcal{C}[-1, 1]$ of continuous functions on the interval $[-1, 1]$.

(10). (*Transformations*) For each of the following transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$, explain why the transformation is linear, give the standard matrix representation, the null-space and range space, and the eigenvalues:

(a) S which multiplies the z coordinate by -3 and rotates points in the (x, y) plane through 90 degrees (counterclockwise);

(b) T which maps $\mathbf{x}$ to $2((1, 0, -1)^T \mathbf{x})\,(1, 0, -1)^T + 3((1, 1, 1)^T \mathbf{x})\,(1, 1, 1)^T$.