Review Problems for Test 2

(**1**). Let $A = \begin{pmatrix} 3 & 0 & 1 & 4 \\ -5 & 4 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(a) Calculate $\det(A)$. (b) What does the value in (a) tell you about the four-dimensional volume spanned by $\{\Sigma_{i=1}^{4} t_i(A\mathbf{b}_i) \ : \ 0 \le t_i \le 1, \ i = 1, \ldots, 4\}$, where the vectors $\mathbf{b}_i$ are the columns of B ?
(c) Calculate the (4,1) element of A^{-1} by determinants.

(2). Suppose that A is a 5×6 matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_6 \in \mathbf{R}^5$, and denote the columns of A^T by $\mathbf{b}_1, \ldots, \mathbf{b}_5$, where

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 6 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \text{rref}(A^T) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let S denote the linear transformation from $\mathbf{R}^6$ to $\mathbf{R}^5$ with standard matrix representation A. Give bases for the ker(S) = null-space of S, for range(S), for Nul(A^T), and for Col(A^T).

(3). Suppose that $S : \mathbf{R}^{10} \to \mathbf{R}^4$ and $T : \mathbf{R}^4 \to \mathbf{R}^6$ are linear transformations.

(a). What is the size of the standard matrix representation of the transformation $T \circ S$ which sends $\mathbf{x} \in \mathbf{R}^{10}$ to $T(S(\mathbf{x}))$?

(b). If the matrix representing S has exactly 4 linearly independent rows, then what is range(S) ?

(c). If the condition of (b) holds, and also the only vector which T sends to $\mathbf{0} \in \mathbf{R}^6$ is the $\mathbf{0}$-vector in $\mathbf{R}^4$, then what is $\dim(range(T))$?

(4). Consider the two bases $\mathcal{B}_1 = \{1, t, t^2, t^3\}$ and $\mathcal{B}_2 = \{1, 2t, t^2 - 1, 3t^3 - t^2 + t\}$ of $\mathcal{P}_3$. (a) Give the matrix which transforms each vector $[\mathbf{v}]_{\mathcal{B}_1}$ of coordinates for a polynomial $\mathbf{v} \in \mathcal{P}_3$ with respect to $\mathcal{B}_2$ into the vector of coordinates $[\mathbf{v}]_{\mathcal{B}_2}$ of coordinates with respect to $\mathcal{B}_2$. (b) Give the matrix which transforms $[\mathbf{v}]_{\mathcal{B}_2}$ to $[\mathbf{v}]_{\mathcal{B}_1}$ and use it to write down the general expression for the standard basis representation of the polynomial $a(1) + b(2t) + c(t^2 - 1) + d(3t^3 - t^2 + t)$.

(5). A Markov chain with three states has one-step transition probabilities defined as follows:

from state **1**, prob's of going to **1**, **2**, **3** are: 0.6, 0.1, 0.3
from state **2**, prob's of going to **1**, **2**, **3** are: 0.1, 0.4, 0.5
from state **3**, prob's of going to **1**, **2**, **3** are: 0.3, 0.5, 0.2

(a) Starting in state **3**, what is the probability of being in state **1** exactly 2 steps later ?

(b) What is the steady-state probability of being in state **2** ?