Review Problems for Test 3

(1). Suppose that a 5×5 matrix A has eigenvalues $1, 1, -2, 0, 4$ and corresponding non-zero eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$, with v_2 not a multiple of $\mathbf{v}_1$.

(a) Explain how you know that A can be diagonalized, and howto do it. You can use words, symbols, or MATLAB commands, but be as precise as you can.

(b) Find the eigenvalues and eigenvectors of the 5×5 matrix $B = A^2 - A$.

(c) Suppose that $\mathbf{x}$ is fixed. How rapidly does the length of $A^k \mathbf{x}$ grow as a function of k ? What feature of the vector $\mathbf{x}$ does this rate of growth depend on ?

(2). Find the eigenvectors and eigenvalues of the matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ -1 & 1 & 4 \\ 0 & -1 & 2 \end{pmatrix}$$

(3). Find the eigenvectors and eigenvalues of the matrix

$$\begin{pmatrix} 4 & 0 & 0 \\ -1 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix}$$

(4). Find an orthonormal basis for the orthogonal complement in $\mathbf{R}^4$ of the subspace spanned by the vectors $\mathbf{v}_1 = (1, -1, 2, 2)^T$ and $\mathbf{v}_2 = (2, 1, 2, -1)^T$. *Hint: find two vectors spanning the null space of the matrix* $(\mathbf{v}_1 \,|\, \mathbf{v}_2)^T$, *and orthonormalize them.*

(5). Explain why the set of vectors in $\mathbf{R}^4$ orthogonal to $(4, 5, 1, 2)^T$ is a (3-dimensional) hyperplane, and find its equation.

(6). Find the QR decomposition of the matrix

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and use it to find the least-squares solution $\mathbf{x}$ of the equation $G\mathbf{x} = (1, 2, -1, -2)^T$.

(7). Find a vector $\mathbf{w}$ in $\mathbf{R}^5$ orthogonal to the columns of

$$C = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -3 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

such that $(1, 2, 3, 4, 5)^T \in \text{span}(col(C), \mathbf{w})$.

(8). Find the solution of the differential equation

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}(t) \quad , \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the approximate *direction* of $\mathbf{x}(t)$ when t gets large ?