In-Class Test 1 Solutions

(1). Is the vector $\mathbf{v_3}$ in span $\{\mathbf{v_1}, \mathbf{v_2}\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix} \qquad ?$$

Solution. Combine the three columns into a single matrix A*: then we do the row-reductions to look for a nontrivial solution to* $A\mathbf{x} = \mathbf{0}$*:*

$$\begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \\ 4 & -1 & 2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & 7 \\ 0 & 7 & 14 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This shows that there are as many pivots as rows, therefore only the unique trivial solution: so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *are linearly independent, and* $\mathbf{v}_3 \notin \mathit{span}\{\mathbf{v}_1, \mathbf{v}_2\}$*.*

(2). Explain how you know that the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

is invertible, and find A^{-1}.

Solution. Since A^T *has a pivot in every row, it is invertible, and therefore so is* A*. It is easy to check that the elementary row operations bringing* A *into reduced row-echelon form (as the Identity matrix) are expressed by multiplication on the left by* $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ -1/3 & 0 & 1/3 \end{pmatrix}$.

(3). Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ which maps

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix}$$

Find the standard matrix representation of T.

Solution. Clearly

$$\mathbf{e}_1 \mapsto \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}, \; \mathbf{e}_2 \mapsto \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \; \mathbf{e}_3 \mapsto \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

so the answer is: $\begin{pmatrix} -1 & 1 & 1 \\ 4 & -2 & 3 \\ 5 & -3 & 6 \end{pmatrix}$.

(4). Is the linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ with standard matrix $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 3 \\ 3 & 5 & 10 \end{pmatrix}$ onto ? Is it one-to-one ? Justify your answers.

Solution. Simple row-reductions show that this matrix has a pivot in every row. Therefore it is invertible, which for square matrix means the associated transformation is both one-to-one and onto.

(5). Suppose that a company makes three different products A, B, and C out of three raw materials U, V, and W, and that a single unit of A, B, or C can be produced from amounts of U, V, and W given in the respective columns of the following matrix.

One unit of	A	B	C	
requires	3	6	2	units of U
and	6	4	8	units of V
and	9	2	4	units of W

(a) How many units of U, V, and W would it take to make 3 units of A plus 4 units of B plus 5 units of C ?

(b) How many units of A, B, and C altogether could the company make with exactly 20 units of U, 20 units of V, and 20 units of W (with no raw material left over) ?

Solution. (a) Denote the matrix by A. *The answer here is* $A \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} =$

$\begin{pmatrix} 43 \\ 74 \\ 55 \end{pmatrix}$. *(b). Now we are solving the system* $A\mathbf{x} = (20, 20, 20)^T$*: the unique solution is* $x_1 = 5/3$, $x_2 = 5/2$, $x_3 = 0$.

(6). (a) In a set of 4 vectors in $\mathbf{R}^3$, at least one can be expressed as a linear combination of the others. **Always true.** *Must be linearly dependent.*

(b) A rectangular 10×6 matrix can be brought into reduced row-echelon form by a series of elementary row operations. **Always true.**

(c) A square matrix has exactly as many linearly independent columns as pivots. **Always true.** *This assertion is like the MATLAB problem we did on HW2: when you select a matrix from a subset of columns containing pivots, you have linearly independent vectors, because the homogeneous system defined by the selected matrix has only the unique (trivial) solution.*

(d) For $n \times k$ matrix A and fixed vector $\mathbf{b} \in \mathbf{R}^n$, the linear system $A\mathbf{x} = \mathbf{b}$ has the same number of solutions $\mathbf{x}$ $(0, 1, \text{ or } \infty)$ as the homogeneous equation $A\mathbf{x} = \mathbf{0}$. **Sometimes true.** *If* A *is onto, the number of solutions is the same, because the general solution is the general solution vector of the homogeneous equation plus any particular solution vector for the inhomogeneous equation.*

In-Class Test 2 Solutions

(1). If $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & -4 & 0 & -1 \end{pmatrix}$, then find the (3,2) element of the inverse of A by means of determinants.

Solution. The formula from Cramer's rule is $(-1)^{3+2}$ *times the determinant of the* 3×3 *matrix obtained by removing the 2nd row and 3rd column, divided by* $\det(A)$*, or*

$$-\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -4 & -1 \end{pmatrix} \Big/ \left\{ 1\,(-1)\,\det\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -3 \end{pmatrix} \right\} = \frac{-2}{-3(1) + 2(-2)} = \frac{2}{7}$$

(2). Suppose that B is a 4×7 matrix such that

$$\mathbf{row-echelon\ form\ of\ \ B} \;=\; \begin{pmatrix} 1 & 3 & 0 & -1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determine each of the following explicitly if that is possible: otherwise explain why the given information is insufficient.

(a) A basis for $col(B)$. *Insufficient information: we would need to see the columns of* B *themselves. The point is that the elementary row operations change the columns (and the space they span) completely.*

(b) A basis for $col(B^T)$. *Sol'n. The first three rows of the row-echelon form give a basis for the row-space: elementary row operations do not change the space spanned by the rows.*

(c) A basis for $Nul(B)$. *Sol'n. This is a subspace of* $\mathbf{R}^7$, *obtained as the set of* $(x_1, \ldots, x_7)^T$ *solving* $\mathbf{rref}(B)\,\mathbf{x} = \mathbf{0}$. *Note that the given row-echelon form is not the* **reduced** *row-echelon form, which we must first calculate (displayed below). The non-pivot variables* x_3, x_4, x_6, x_7 *are free, and a basis of* $Nul(B)$ *is given by the columns of the displayed matrix* N.

$$\mathbf{rref}(B) \;=\; \begin{pmatrix} 1 & 0 & -6 & 8 & 0 & 3 & -4 \\ 0 & 1 & 2 & -3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \;, \quad N \;=\; \begin{pmatrix} 6 & -8 & -3 & 4 \\ -2 & 3 & 1 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) $rank(B)$. *Sol'n. This is* 3, *the number of pivots, equal to the row-space and the column-space dimensions.*

(3). A vector space V has a basis $\{\mathbf{v}_1, \mathbf{v}_2, , \ldots, \mathbf{v}_7\}$. In addition, it contains vectors $\mathbf{w}_j$, $j = 1, \ldots, 5$. If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_5$ all lie in $\mathbf{span}(\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\})$, then explain clearly, with specific reference to results from the course, why $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5, \mathbf{v}_6, \mathbf{v}_7\}$ is also a basis of V.

Solution. The given conditions show that any vector $\mathbf{x}$ *of* V *has the form* $\sum_{j=1}^{7} c_j\,\mathbf{v}_j$, *since the* $\mathbf{v}_j$ *form a basis, and each* $\mathbf{v}_j$ *for* $j \le 5$ *has the* *form* $\sum_{k=1}^{5} a_{jk}\,\mathbf{w}_k$. *Thus* $\mathbf{x} = \sum_{j=1}^{5} \sum_{k=1}^{5} c_j\, a_{jk}\, \mathbf{w}_k + c_6\, \mathbf{v}_6 + c_7\, \mathbf{v}_7$, *so* *span*$(\{\mathbf{w}_j,\ 1 \le j \le 5,\ \mathbf{v}_j,\ j = 6,7\}) = V$. *But since* V *has a basis with* 7 *elements, any set of 7 spanning vectors must also be linearly independent, i.e. a basis, since there could not be a smaller spanning set.*

(4). Consider the basis $\mathcal{A}$ of $\mathbf{R}^3$ given by

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix}$$

(a) What vector $\mathbf{w} \in \mathbf{R}^3$ has coordinates $[\mathbf{w}]_{\mathcal{A}} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ with respect to the basis $\mathcal{A}$?

Solution. $(-1)\mathbf{a}_1 + (1)\mathbf{a}_2 + (-1)\mathbf{a}_3 = (-7,-1,1)^T$.

(b) Express the vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

in terms of coordinates with respect to $\mathcal{A}$.

Solution. Let B *be the matrix with columns* $\mathbf{b}_j$ *and* A *be the matrix with columns* $\mathbf{a}_j$. *The vectors* $\mathbf{b}_j$ *are already expressed in coordinates with respect to the standard basis* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. *The answer is the set of three columns of the matrix* M *solving* $AM = B$, *or*

$$M = A^{-1} B = \begin{pmatrix} .2 & .4 & .8 \\ -.1 & .3 & .6 \\ .1 & .2 & -.1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} .6 & 1 & 1.2 \\ .2 & .5 & .9 \\ .3 & 0 & .1 \end{pmatrix}$$

(5). A Markov chain with 3 states A, B, and C has transition probabilities in unit time given as follows:

from state **A**, prob's of going to **A**, **B**, **C** are: 0.5, 0.4, 0.1
from state **B**, prob's of going to **A**, **B**, **C** are: 0.3, 0.3, 0.4
from state **C**, prob's of going to **A**, **B**, **C** are: 0, 0.1, 0.9

(a) Starting in state **B**, what is the probability of being in state **C** exactly 3 time-units later ?

Solution. The answer is the (3,2) element of P^3 *or*

$$(.1\ .4\ .9)\begin{pmatrix} .5 & .3 & 0 \\ .4 & .3 & .1 \\ .1 & .4 & .9 \end{pmatrix}\begin{pmatrix} .3 \\ .3 \\ .4 \end{pmatrix} = (.3\ .51\ .85)\begin{pmatrix} .3 \\ .3 \\ .4 \end{pmatrix} = .09{+}.153{+}.34 = .583$$

(b) What is the steady-state probability of being in state **B** ?
Solution. The answer is the second entry of the steady-state probability vector $\mathbf{q}$ *which solves* $(P-I)\mathbf{q} = \mathbf{0}$, $q_1+q_2+q_3 = 1$. *We solve by row-reduction, first finding a vector* $\mathbf{p}$ *satisfying* $(P-I)\mathbf{p} = 0$, *then taking* $\mathbf{q} = c\mathbf{p}$ *for* $c = 1/(p_1+p_2+p_3)$.

$$\begin{pmatrix} -.5 & .3 & 0 \\ .4 & -.7 & .1 \\ .1 & .4 & -.1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 4 & -1 \\ 4 & -7 & 1 \\ -5 & 3 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 4 & -1 \\ 0 & -23 & 5 \\ 0 & 23 & -5 \end{pmatrix}$$

So p_3 *is free, and* $p_2 = 5p_3/23$, $p_1 = 3p_3/23$, *so that, taking* $c = 23/(31p_3)$ *or equivalently* $p_3 = 23/31$, *we find steady-state probability vector* $\mathbf{q} = (3,5,23)/31$. *The long-term probability of being in state B is* $5/23$.

In-Class Test 3 Solutions

(1). Suppose that the 5×5 matrix B has eigenvalues $2, -1, -1, -3, 4$, and corresponding non-zero eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$, with $\mathbf{v}_3$ not a multiple of $\mathbf{v}_2$.

(a) Explain how you know that B can be diagonalized, what it means, and how to do it. You can use words, symbols, or MATLAB commands, but be as precise as possible.

(b) If $\mathbf{I}$ denotes the 5×5 identity matrix, then what are the eigenvalues and eigenvectors of the 5×5 matrix $A = B^3 - \mathbf{I}$?

(c) Suppose that $\mathbf{x}$ is fixed. How rapidly does the length of $B^k\mathbf{x}$ grow as a function of k when k is large ? What feature(s) of the vector $\mathbf{x}$ does this rate of growth depend on ?

Solution. (a) If we define $P = (\mathbf{v}_1\,|\,\mathbf{v}_2\,|\,\cdots\,|\,\mathbf{v}_5)$, *then*

$$AP = (2\mathbf{v}_1\,|\,-\mathbf{v}_2\,|\,-\mathbf{v}_3\,|\,-3\mathbf{v}_4\,|\,4\mathbf{v}_5) = P\,D \quad , \quad D = diag(2,-1,-1,-3,4)$$

then we have $A = PDP^{-1}$, *which is what is meant by diagonalization. Note that* P *is invertible even though the eigenvectors of* A *are not distinct, because we have a linearly independent set of eigenvectors. (b) The eigenvectors are as before,* $\mathbf{v}_i$, $i = 1, \ldots, 5$, *and in terms of the eigenvalues* λ_i *for* A, *the eigenvalues for* B *are* $\lambda_i^3 - 1$, *or* $7, -2, -2, -28, 63$. *(c) Note that* $A^k\mathbf{x} = \sum_{i=1}^5 (P^{-1}\mathbf{x})_i \, \lambda_i^k \, \mathbf{v}_i$. *So the rate of growth is* $|\lambda_i|^k$, *where* $|\lambda_i|$ *is the largest eigenvalue magnitude for which the corresponding coordinate* $(P^{-1}\mathbf{x})_i$ *for the eigenvector basis is non-zero, so is* $|4|^k$ *if* $(P^{-1}\mathbf{x})_5 \neq 0$, 3^k *if* $(P^{-1}\mathbf{x})_5 = 0 \neq (P^{-1}\mathbf{x})_4$, *etc.*

(2). Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ denote the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) Give a basis for the space of all vectors in $\mathbf{R}^4$ orthogonal to all 3 vectors $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$.

(b) Give an orthogonal basis for the space $col(A)$ of vectors in $\mathbf{R}^4$.

(c) Do the two sets of vectors in (a), (b) together form a basis of $\mathbf{R}^4$? Why or why not ?

Solution. (a) To find a basis for the null-space of A^T, *row-reduce:*

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies for homogeneous equation solution $\mathbf{x}$ *that* x_3, x_4 *are free, and null-space basis is* $(-4, 1, 1, 0)^T$, $(-1, 0, 0, 1)^T$. *(b) Since* $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$, *and* $\mathbf{a}_1 \perp \mathbf{a}_2$, *the desired orthogonal basis is* $\{\mathbf{a}_1, \mathbf{a}_2\}$. *(c) In (a) we have a basis for* $(col(A))^\perp$, *and in (b) a basis for* $col(A)$. *The union of these bases is linearly independent, because the first basis is orthogonal to the second one; and the union spans* $\mathbf{R}^4$ *because all* $\mathbf{x} \in \mathbf{R}^4$ *can be written* $w + z$ *with* $w \in col(A)$, $z \perp col(A)$.

(3). Solve the differential equation in $\mathbf{R}^3$ for all $t \geq 0$:

$$\dot{\mathbf{x}}\,(t) = A\,\mathbf{x}(t) = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \mathbf{x}(t) \quad , \qquad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

Note: the eigenvalues of the matrix A *are* 0, 2, 3.

Solution. We solve for three eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$*:*

$$\text{for e.v. } 2, \quad \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad \text{since} \quad \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & -2 \\ 0 & -1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \implies \begin{array}{c} x_2 = 0 \\ x_1 = -2x_3 \end{array}$$

$$\text{for e.v. } 0, \quad \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \text{since} \quad \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 2 \end{pmatrix} \mathbf{x} = \mathbf{0} \implies \begin{array}{c} x_1 = 0 \\ x_2 = 2x_3 \end{array}$$

$$\text{for e.v. } 3, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{since} \quad \begin{pmatrix} -1 & 0 & 0 \\ -1 & -2 & -2 \\ 0 & -1 & -1 \end{pmatrix} \mathbf{x} = \mathbf{0} \implies \begin{array}{c} x_1 = 0 \\ x_3 = -x_2 \end{array}$$

It follows that $\mathbf{x}(0) = \mathbf{v}_2 + \mathbf{v}_3$ *and* $\mathbf{x}(t) = \mathbf{v}_2 + e^{3t}\mathbf{v}_3 = (0, 2+e^{3t}, 1-e^{3t})^T$.

(4). A given 5×3 matrix C has the QR factorization:

$$C = QR = \begin{pmatrix} .5 & 0 & 0 \\ -.5 & .5 & .5 \\ .5 & .5 & .5 \\ 0 & .5 & -.5 \\ 0 & .5 & -.5 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) What is the relationship between the column-spaces *col*(C), *col*(Q) ? Justify your answer.

(b) Find the element $\hat{\mathbf{b}} \in$ *col*(C) which makes the length of the vector $\mathbf{b} - \hat{\mathbf{b}}$ as small as possible.

(c) Find the vector $\mathbf{x} \in \mathbf{R}^3$ such that $\mathbf{b} - C\mathbf{x}$ has length as small as possible.

Solution. (a) *col*$(Q) =$ *col*(C) *since* $C = QR$ *implies* *col*$(C) \subset$ *col*(Q), *while* $Q = R^{-1}C$ *implies* *col*$(Q) \subset$ *col*(C). *(b) Using (a), the unique solution is the projection of* $\mathbf{b}$ *onto* *col*(Q), *or* $\hat{\mathbf{b}} = QQ^T\mathbf{b}$. *(c) We want* $\mathbf{x}$ *such that* $QR\mathbf{x} = C\mathbf{x} = QQ^T\mathbf{b}$, *or equivalently* $R\mathbf{x} = Q^T\mathbf{b}$ *or* $\mathbf{x} = R^{-1}Q^T\mathbf{b}$. *The unique vector* $\mathbf{x} \in \mathbf{R}^3$ *minimizing* $\|\mathbf{b} - C\mathbf{x}\|$ *is*

$$\frac{1}{2} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} Q^T = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix} \mathbf{b}$$