Stat 400, Summer I June 24, 1996

Handouts on Transformation of Random Variables & Simulation

I. TRANSFORMATION OF CONTINUOUS R.V.'S

The main idea of this topic is that a *known* function $Y = g(X)$ of a continuously distributed random variable X with *known* density function $f_X(t)$ is itself a continuous random variable with a distribution function F_Y and density function $f_Y(y)$ which we can figure out explicitly. We use this in two different ways: (1) to generate interesting new probability models cheaply from old ones, in a way which leads to many natural applications, and (2) to show that any continuous random variable with known *cdf* can be expressed as a known function of a $Unif[0,1]$ random variable, and so can be simulated on a computer using as building-block an algorithm for generating $Unif[0,1]$ pseudorandom variables on the computer.

We restrict attention to functions g which are differentiable and *strictly monotonic* (which means either always increasing, with positive derivative, or always decreasing, with negative derivative) on the range (a,b) on which the density f_X is positive. (That interval, which may be infinite in one or both directions, is the range of 'possible values' for the random variable X. **First, suppose g is strictly increasing.** That means, for any two real numbers $z, x \in (a,b)$

$$z \le x \qquad \textit{if and only if} \qquad g(z) \le g(x)$$

To every value $y \in (g(a), g(b))$, there is one and only value $z \in (a,b)$ for which $y = g(z)$, and we denote this value by $z = g^{-1}(y)$. *(You should review these ideas under the heading of 'Inverse Functions' in your calculus book if this is not familiar.)* Therefore, even if we replace y by a random variable value Y and then put $z = X = g^{-1}(Y)$, $x = g^{-1}(y)$ in the previous displayed equation, we have

$$Y \le y \qquad \textit{if and only if} \qquad X \le x = g^{-1}(y)$$

from which it follows that

$$F_Y(y) = P(Y \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y)) \tag{1}$$

Equation (1) shows how to find the *distribution function* of $Y = g(X)$ in terms of an *increasing* function g and the distribution function of X. The correct distribution function formula in the case of a random variable $Y = g(X)$ with *decreasing* function g is:

$$F_Y(y) = P(Y \le y) = P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y)) \tag{1'}$$

Here are some examples (of the increasing-g case):

Example A. Lognormal distribution Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$, and $Y = g(X) = \exp(X)$. Then Y is called *lognormal* with parameters μ, σ^2. Sovling $y = g(x) = e^x$

for y gives $x = g^{-1}(y) = \log(y)$. Equation (1) shows that $F_Y(y) = F_X(\log(y)) = \Phi\big((\log(y) - \mu)/\sigma\big)$. Differentiating this formula by means of the **chain rule** gives, for positive values y,

$$f_Y(y) = F_Y'(y) = \Phi'\big(\frac{\log(y) - \mu}{\sigma}\big) \cdot \frac{1}{\sigma y} \;=\; \frac{1}{\sqrt{2\pi}\,\sigma y} \cdot \exp\big(-\frac{(\log(y) - \mu)^2}{2\sigma^2}\big)$$

Example B. Weibull distribution Suppose that $X \sim Expon(\lambda)$ and that for a fixed constant $\alpha > 0$, $Y = g(X) = X^{1/\alpha}$, so that $y = g(x) = x^{1/\alpha}$ is solved to give $g^{-1}(y) = y^{\alpha}$. Then, since $F_X(x) = 1 - e^{-\lambda x}$, Equation (1) and the chain rule show for positive values y that

$$f_Y(y) = F_Y'(y) = \frac{d}{dy}\big[1 - \exp\big(-\lambda y^{\alpha}\big)\big] = \lambda\alpha x^{\alpha-1}\exp\big(-\lambda y^{\alpha}\big)$$

Note that the parameters here agreee with those in the book only if we take $\lambda = \beta^{-\alpha}$.

Example C. General function of Uniform. Now take $X \sim Uniform[0,1]$, so that $F_X(x) = x$ for $0 \le x \le 1$, and put $g(x) = G^{-1}(x)$, where G is the specified distribution function for which we want to produce an associated random variable. Put $Y = g(X) = G^{-1}(X)$. Note that $g^{-1}(y) = G(y)$, since the inverse of the inverse of a function G is the function G itself. Then we apply Equation (1) again to find that this random variable Y has distribution function given by

$$F_Y(y) \;=\; F_X(G(y)) \;=\; G(y) \tag{2}$$

The interpretation of formula (2) is that if we have a $Uniform[0,1]$ random variable X produced or 'simulated' for us on the computer, and if we have implemented a *FUNCTION* subroutine to calculate $g(\cdot) = G^{-1}(\cdot)$, then $g(X)$ is a random variable with the desired cdf G.

We give two more simple examples to illustrate the general idea that for many interesting random variables whose probability distributions depend upon parameters, there is a simple function of the random variable whose distribution no longer depends upon those parameters.

Example D. Standardizing Normal RV's. We saw in class the operation of 'standardizing' a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Here is the same idea from the perspective of equation (1). Now we define the linear 'standardizing' function $g(x) = (x-\mu)/\sigma$. Clearly $y = g(x)$ is solved in terms of y to give $x = g^{-1}(y) = \mu + \sigma \cdot y$, and formula (1) says that the standardized *rv* $Y = (X - \mu)/\sigma$ has *cdf* given by

$$F_Y(y) = F_X(\mu + \sigma \cdot y) = \Phi\big(\frac{(\mu + \sigma y) - \mu}{\sigma}\big) = \Phi(y)$$

So Y is a *standard normal rv.*

Example E. Rescaling Exponential(λ). Let $X \sim \mathit{Expon}(\lambda)$, $g(x) = \lambda \cdot x$, $Y = \lambda \cdot X$. Then $F_X(x) = 1 - e^{-\lambda x}$, $g^{-1}(y) = y/\lambda$, and for positive y, by Equation (1)

$$F_Y(y) = F_X(y/\lambda) = 1 - e^{-(y/\lambda)\cdot\lambda} = 1 - e^{-y}$$

which means that the rescaled variable Y is $\mathit{Expon}(1)$ distributed. Thus $X = Y/\lambda$: to get an $\mathit{Expon}(\lambda)$ *rv*, just take an $\mathit{Expon}(1)$ *rv* and divide by λ.

We give one more example which is of interest later on. First, suppose that X is a standard $\mathcal{N}(0,1)$ random variable. Recall that X is *symmetric*, in the sense that it is just as likely to take positive values in an interval $[x, x+\delta]$ as to take negative values in the mirror-image interval $[-x-\delta, -x]$. Therefore, if we want to find the distribution of the positive-valued random variable $Y = X^2$, we calculate for positive y (using the same idea as in deriving equation, but slightly different details because x^2 is not an increasing function on the whole — positive and negative — axis)

$$F_Y(y) = P(X^2 \le y) = P(|X| \le \sqrt{y}) = P(-\sqrt{y} \le X \le \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

From this last equation, it follows (using $\Phi'(x) = \exp(-x^2/2)/\sqrt{2\pi}$) that

$$f_Y(y) = 2 \cdot \frac{1}{2} \cdot y^{-1/2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-(\sqrt{y})^2/2} = \Gamma(\frac{1}{2}) \cdot 2^{-1/2} \cdot y^{1-\frac{1}{2}} \cdot e^{-y/2}$$

This last density is a Gamma density with parameters $\frac{1}{2}$, $\frac{1}{2}$, also called χ^2_1 or *chi-squared with one degree of freedom.*

PROBLEMS ON TRANSFORMATION OF RANDOM VARIABLES

TRAN.1. Show that a $Uniform[12,17]$ random variable can be obtained as a simple function of a $Uniform[0,1]$ random variable, and find the function.

TRAN.2. Find the cumulative distribution function and the probability density function of the random variable $3 \cdot V^2 + 1$, where $V \sim \mathit{Expon}(1)$.