Persistence of regularity for solutions of the Boussinesq equations in Sobolev spaces

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Wednesday 20th December, 2017

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Abstract

We address the global regularity of solutions to the Boussinesq equations with zero diffusivity in two spatial dimensions. Previously, the persistence in the space $H^{1+s}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ for all $s \geq 0$ has been obtained. In this paper we address the persistence in general Sobolev spaces, establishing it on a time interval which is almost independent of the size of the initial data. Namely, we prove that if $(u_0, \rho_0) \in W^{1+s,q}(\mathbb{R}^2) \times W^s,q(\mathbb{R}^2)$ for $s \in (0,1)$ and $q \in [2,\infty)$, then the solution $(u(t), \rho(t))$ of the Boussinesq system stays in $W^{1+s,q}(\mathbb{R}^2) \times W^s,q(\mathbb{R}^2)$ for $t \in [0,T^*]$, where $T^*$ depends logarithmically on the size of initial data. If we furthermore assume that $s q > 2$, then we get the global persistence in the space $W^{1+s,q}(\mathbb{R}^2) \times W^s,q(\mathbb{R}^2)$ for the initial data with compact support, as well as for data in $W^{1+s,q}(\mathbb{T}^2) \times W^s,q(\mathbb{T}^2)$, without any restriction on $s \in (0,1)$ and $q \in [2,\infty)$.

Mathematics Subject Classification: 35K55, 35M33, 76B03, 76D05
Keywords: Boussinesq equations, commutator estimate, Kato-Ponce type inequalities, global well-posedness

1 Introduction

In this paper, we address the persistence of regularity for the 2D Boussinesq equations with zero diffusivity

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \Delta u + \nabla p = \rho e_2$$

$$\text{div } u = 0$$

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0$$

in general Sobolev spaces. Here, $u$ is the velocity solving the 2D Navier-Stokes equation ([CF, DG, FMT, R, T2, T3]) driven by $\rho$, which represents the density or temperature of the fluid, depending on the physical context, and $e_2 = (0,1)^T$. This system arises in several different physical scenarios. One situation is the limiting case of small diffusion Rayleigh-Bénard problem, where $\rho$ represents the temperature. Also, the Boussinesq system serves as a simplified model for the 3D Navier-Stokes equations since it incorporates a vortex stretching effect.
Due to the smoothing effect, the Boussinesq system with viscosity and non-zero diffusion

\[ \frac{\partial p}{\partial t} + u \cdot \nabla \rho = \kappa \Delta \rho \]  

replacing (1.3) is easier to treat than the same system without the diffusion. The global well-posedness of the system (1.1)–(1.3) with (1.3) replaced by (1.4), where \( \kappa > 0 \), was obtained in [CD]. In 2006, Chae [C] proved the global existence and uniqueness of solutions to the system (1.1)–(1.3), proving that if \( (u_0, \rho_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \), then the solution \((u, \rho)\) stays in \( H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \) for all \( t > 0 \), where \( m \) is an integer greater than or equal to 3. The main idea in the proof was to show that \( \int_0^T \| \nabla \rho \|_{L^\infty} \, dt < \infty \). At the same time, Hou and Li [HL] proved the persistence of regularity for the solutions in \( H^m(\mathbb{R}^2) \times H^{m-1}(\mathbb{R}^2) \) also for integer \( m \) greater than or equal to 3, with the key ingredient in their proof being an upper bound for \( \| \nabla u \|_{L^\infty} \). The persistence results in the class \( H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2) \) for a larger range of \( s \) were obtained more recently. For \( s = 1 \), one may refer to [DP, LLT], in which the global existence and uniqueness results were proven for the two-dimensional non-diffusive Boussinesq system with viscosity only in the horizontal direction. Recently, the case \( s \in (1,3) \) was resolved in [HKZ1, HKZ2]. The key step in the first paper was an estimate for \( \nabla B(u,v) \) while a borderline commutator estimate for \( \|[\Lambda^s \partial_j, v] \rho \|_{L^2(\mathbb{R}^2)} \) was obtained in the second paper. For further results on the well-posedness and persistence, we refer the reader to [ACW, BS, BrS, CG, CLR, CN, CW, ES, HK1, HK2, HKR, KTW, LPZ, T1, W]. Note that the global existence for another extreme case when (1.1) is replaced by

\[ \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \rho e_2 \]  

is still an open problem.

In this paper, we consider the persistence of the system (1.1)–(1.3) in the Sobolev space \( W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2) \) where \( s \in (0,1) \) and \( q \in [2, \infty) \). In comparison with the \( L^2 \) based result, we are faced with the following difficulties. First, the velocity does not need to belong to \( L^2(\mathbb{R}^2) \) and thus we cannot use the energy inequality. In order to avoid this problem, we couple the \( L^p \) estimate for the velocity and that of the vorticity. Second, the available commutator estimates and the Kato-Ponce inequality are inadequate for our purpose. What we need here is the estimate for \( \|[\Lambda^s \partial_j, g] f \|_{L^2(\mathbb{R}^2)} \). As in [KP], we write this commutator as a Coifman-Meyer operator in the frequency regions \( \xi \gg \eta \) and \( \xi \ll \eta \) and use the complex interpolation in the remaining region. Also, the Brezis-Gallouet and related Beale-Kato-Majda inequalities ([BKM]) do not apply in our situation. Instead we prove

\[ \| A u \|_{L^\infty} \leq C \left( 1 + \log(1 + \| \Lambda^s \omega \|_{L^r}) \right)^{1+1/s} (1 + \| \omega \|_{L^q} + \| \nabla (|\omega|^{q/2}) \|_{L^2}^{q/2}) \]  

(cf. Lemma 4.1 below). Since the power of the logarithm is too high, we may only obtain a local persistence result, with the existence time depending on the initial data logarithmically. As we may see, a huge difficulty to get a global result is the estimate of \( \| \omega \|_{L^\infty} \). If we furthermore assume that \( sq > 2 \), then we may prove

\[ \| \omega(t) \|_{L^\infty} \leq C \sqrt{1 + t} \]  

(cf. Lemma 6.2), allowing us to prove the global existence in \( W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2) \).

In Theorem 6.3 below we also obtain the global persistence in the intersection space \( (W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)) \cap (H^{1+s}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)) \) where \( s \in (0,1) \) and \( q \in [2, \infty) \). As a corollary of this result, we obtain
the global persistence in the space $W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$ for the initial data with compact support. The global existence in $W^{1+s,q}(\mathbb{T}^2) \times W^{s,q}(\mathbb{T}^2)$ is also established in Theorem 7.1 below.

2 On almost persistence

In this section we prove that if the initial data $(u_0, \rho_0)$ belongs to $W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$ where $s \in (0,1)$ and $q \in (2,\infty)$, then $(u(t), \rho(t)) \in W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)$ for $t \in [0,T^*)$, where $T^*$ depends logarithmically on the size of the initial data. Further below we prove the global persistence in the intersection space $(W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)) \cap (H^{1+s}(\mathbb{R}^2) \times H^s(\mathbb{R}^2))$.

Before stating the first main theorem, we assert the global persistence in $W^{1,q}(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$.

**Proposition 2.1.** Assume that, for some $q \geq 2$, we have $(u_0, \rho_0) \in W^{1,q}(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$ with $\text{div} \, u_0 = 0$. Then there exists a unique solution $(u, \rho)$ to the equations (1.1)–(1.3) such that $u \in C([0,\infty), W^{1,q}(\mathbb{R}^2))$ and $\rho \in C([0,\infty), L^q(\mathbb{R}^2))$. Moreover, with $\omega = \nabla \times u$, there exists $C$ depending on $\|u_0\|_{W^{1,q}}$ and $\|\rho_0\|_{L^q}$ such that

$$\|\nabla u(t)\|_{L^q} \leq C\sqrt{t} + 1, \quad t \in [0, \infty)$$

(2.1)

and

$$\int_0^t \|\nabla (|\omega|^{q/2})\|_{L^2}^2 \, ds \leq C(1 + t)^{q/2}$$

(2.2)

with

$$\|\rho(t)\|_{L^q} = \|\rho_0\|_{L^q}, \quad t \in [0, \infty)$$

(2.3)

hold.

The following is our first main statement.

**Theorem 2.2.** Let $s \in (0,1)$ and $q \in (2,\infty)$. Assume that $\|u_0\|_{W^{1+s,q}} \leq M_0$ with $\text{div} \, u_0 = 0$ and $\|\rho_0\|_{W^{s,q}} \leq M_1$, where $M_0, M_1 \geq 1$ are constants. Then there exists a constant $C$ depending on $\|u_0\|_{W^{1,q}}$ and $\|\rho_0\|_{L^q}$ such that for

$$T^* = \min \left\{ \left( 1 + \frac{1}{C(1 + \log^{1/q}(1 + M_0 + M_1))} \right)^{2/3} - 1, \frac{1}{C(1 + \log^{1/q(1 + M_0 + M_1))^{2q/(q-2)}} \right\}$$

(2.4)

there exists a unique solution $(u, \rho)$ to the equations (1.1)–(1.3) such that $u \in C([0,T^*), W^{1+s,q}(\mathbb{R}^2))$ and $\rho \in C([0,T^*), W^{s,q}(\mathbb{R}^2))$.

It is clear from the proof below that we can take $C$ to be proportional to $\|u_0\|_{W^{1,q}} + \|\rho_0\|_{L^q}$.

**Proof of Proposition 2.1.** Since the case $q = 2$ is covered in [LLT], we assume $q > 2$. We start by multiplying the equation (1.1) with $|u|^{q-2}u$ and integrating it with respect to $x$ obtaining

$$\frac{1}{q} \frac{d}{dt} \|u\|_{L^q}^q + \int (u \cdot \nabla u) \cdot u |u|^{q-2} \, dx - \int \Delta u \cdot u |u|^{q-2} \, dx + \int \nabla p \cdot u |u|^{q-2} \, dx$$

$$= \int \rho e_2 \cdot u |u|^{q-2} \, dx.$$  

(2.5)
Due to the divergence-free condition for \( u \), we have \( \int (u \cdot \nabla u) \cdot u |u|^{q-2} \, dx = 0 \) while integrating by parts, we obtain
\[
- \int \Delta u \cdot u |u|^{q-2} \, dx = \int \partial_j u_k \partial_j u_k |u|^{q-2} \, dx + (q-2) \int (u_k \partial_j u_k)(u_l \partial_j u_l) |u|^{q-4} \, dx, 
\]
(2.6)

where the summation convention on repeated indices is used throughout. Denoting the right side of the above equality by \( D_0 \), we get from (2.5)
\[
\frac{1}{q} \frac{d}{dt} \|u\|_{L^q}^q + D_0 = - \int \nabla p \cdot u |u|^{q-2} \, dx + \int \rho e_2 \cdot u |u|^{q-2} \, dx 
\leq \|\nabla p\|_{L^q} \|u\|_{L^q}^{q-1} + \|\rho\|_{L^q} \|u\|_{L^q}^{q-1}. 
\]
(2.7)

Noting that \( D_0 \geq 0 \), we only need to estimate the pressure term. We take the divergence of the equation (1.1) and obtain
\[
- \Delta p = \text{div} \, (u \cdot \nabla u) - \partial_2 \rho. 
\]
(2.8)

Writing
\[
\nabla p = \nabla (-\Delta)^{-1} \{ \text{div} \, (u \cdot \nabla u) - \partial_2 \rho \},
\]
(2.9)

we may apply the Calderón-Zygmund theorem in order to get
\[
\|\nabla p\|_{L^q} \leq C(\|u \cdot \nabla u\|_{L^q} + \|\rho\|_{L^q}) \leq C(\|u\|_{L^\infty} \|\nabla u\|_{L^q} + \|\rho\|_{L^q}) 
\leq C(\|u\|_{W^{1,\infty}} \|\nabla u\|_{L^q} + \|\rho\|_{L^q}) \leq C(\|u\|_{L^q} + \|\nabla u\|_{L^q}) \|\nabla u\|_{L^q} + C\|\rho\|_{L^q} 
\leq C(\|u\|_{L^q} + \|\omega\|_{L^q}) \|\omega\|_{L^q} + C\|\rho\|_{L^q}, 
\]
(2.10)

where \( \omega = \nabla \times u \) is the vorticity and where the constant depends on \( q \) (considered fixed). In order to get the bound for \( \|\rho\|_{L^q} \), we multiply the equation (1.3) by \( |\rho|^{q-2} \rho \) and integrate with respect to \( x \) to get
\[
\frac{1}{q} \frac{d}{dt} \|\rho\|_{L^q}^q = 0, 
\]
(2.11)

where we used
\[
\int u \cdot \nabla \rho |\rho|^{q-2} \rho \, dx = 0.
\]
(2.12)

Therefore, the \( L^q \) norm of \( \rho \) is preserved by this system, i.e.,
\[
\|\rho(t)\|_{L^q} = \|\rho_0\|_{L^q}, \quad t > 0. 
\]
(2.13)

Combining (2.7) with (2.10) and (2.13) leads to
\[
\frac{1}{q} \frac{d}{dt} \|u\|_{L^q}^q + D_0 \leq C(\|u\|_{L^q} + \|\omega\|_{L^q}^2 + 1) \|u\|_{L^q}^{q-1}. 
\]
(2.14)

Taking the curl of the equation (1.1), we get
\[
\omega_1 + u \cdot \nabla \omega - \Delta \omega = \nabla \times (\rho e_2) = \partial_1 \rho. 
\]
(2.15)

We multiply both sides of the equation (2.15) by \( |\omega|^{q-2} \omega \) and integrate it to get an \( L^q \) estimate of \( \omega \),
\[
\int \omega_1 |\omega|^{q-2} \omega \, dx + \int u \cdot \nabla \omega |\omega|^{q-2} \omega \, dx - \int \Delta \omega |\omega|^{q-2} \omega \, dx = \int \partial_1 \rho |\omega|^{q-2} \omega \, dx. 
\]
(2.16)
Due to the divergence-free condition, we have $\int u \cdot \nabla |\omega|^{q-2} \omega \, dx = 0$. Integrating by parts the third term on the left side of (2.16), we arrive at

$$\frac{1}{q} \frac{d}{dt} \|\omega\|_{L^q}^q + (q - 1) \left( \frac{2}{q} \right)^2 \|\nabla(|\omega|^{q/2})\|_{L^2}^2 = -(q - 1) \int \rho |\omega|^{q-2} \partial_t \omega \, dx. \tag{2.17}$$

By Hölder’s inequality and using

$$\|\omega\|^{(q-2)/2} \partial_t \omega \|_{L^2} = \frac{2}{q} \|\partial_t (|\omega|^{q/2})\|_{L^2}, \tag{2.18}$$

we have

$$\left| \int \rho |\omega|^{q-2} \partial_t \omega \, dx \right| \leq \frac{C}{q} \|\rho\|_{L^s} \|\nabla(|\omega|^{q/2})\|_{L^2} \|\omega\|_{L^2}^{(q-2)/2}. \tag{2.19}$$

We apply the inequality (2.19) to the equation (2.17) and use Young’s inequality to get

$$\frac{1}{q} \frac{d}{dt} \|\omega\|_{L^q}^q + (q - 1) \left( \frac{2}{q} \right)^2 \|\nabla(|\omega|^{q/2})\|_{L^2}^2 \leq \frac{1}{2}(q - 1) \left( \frac{2}{q} \right)^2 \|\nabla(|\omega|^{q/2})\|_{L^2}^2 + C \|\rho\|_{L^2}^2 \|\omega\|_{L^q}^{q-2}, \tag{2.20}$$

where, recall, the constant $C$ depends on $q$. Absorbing the first term on the right side of the above inequality, we get

$$\frac{d}{dt} \|\omega\|_{L^q}^q + \frac{1}{C} \|\nabla(|\omega|^{q/2})\|_{L^2}^2 \leq C \|\rho\|_{L^2}^2 \|\omega\|_{L^q}^{q-2}. \tag{2.21}$$

The inequality (2.21), combined with (2.13), implies

$$\|\omega(t)\|_{L^q} \leq C \sqrt{1 + t}, \quad t \in [0, \infty), \tag{2.22}$$

as well as

$$\int_0^t \|\nabla(|\omega|^{q/2})\|_{L^2}^2 \, ds \leq C(1 + t)^{q/2}, \tag{2.23}$$

where $C$ depends on the initial data $\|u_0\|_{W^{1,q}}$ and $\|\rho_0\|_{L^q}$. Using (2.22) in (2.14) and applying the Gronwall’s inequality then concludes the proof.

\[\square\]

### 3 A commutator lemma

The main step in the proof of Theorem 2.2 is a commutator estimate, which is stated next. Let $\Lambda = (-\Delta)^{1/2}$.

**Lemma 3.1.** Let $s \in (0, 1)$ and $f, g \in \mathcal{S}(\mathbb{R}^2)$. For $1 < q < \infty$ and $j \in \{1, 2\}$,

$$\| [\Lambda^s \partial_j, g] f \|_{L^q(\mathbb{R}^2)} \leq C \|f\|_{L^{r_1}} \|\Lambda^{1+s} g\|_{L^{r_1}} + C \|\Lambda^s f\|_{L^{r_2}} \|g\|_{L^{r_2}} \tag{3.1}$$

holds, where $r_1, r_1, r_2 \in [q, \infty]$ and $r_2 \in [q, \infty)$ satisfy $1/q = 1/r_1 + 1/r_1 = 1/r_2 + 1/r_2$, and where $C$ is a constant depending on $r_1, r_1, r_2, s$, and $q$. 

\[\square\]
Recall that
\[ [\Lambda^s \partial_j, g]f = \Lambda^s \partial_j (gf) - g\Lambda^s \partial_j f. \] (3.2)

This lemma can be seen as an extension of the Kato-Ponce inequality since we allow \( \tilde r_1 = \infty \). The proof uses the ideas from [KP].

**Proof.** Taking the Fourier transform of \([\Lambda^s \partial_j, g]f\), we get
\[
([\Lambda^s \partial_j, g]f)(\xi) = i \int_{\mathbb{R}^2} (|\xi|^s \xi_j - |\xi - \eta|^s (\xi - \eta)_j) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta.
\] (3.3)

Therefore,
\[
[\Lambda^s \partial_j, g]f = c_0 \int \int e^{ix \cdot \xi} (|\xi|^s \xi_j - |\xi - \eta|^s (\xi - \eta)_j) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta \, d\xi
\] (3.4)

where \( c_0 = i/4\pi^2 \), and where \( \Phi_k : \mathbb{R} \to [0, 1] \) are \( C^\infty \) cut-off functions such that
\[
\sum_{k=1}^{3} \Phi_k = 1 \text{ on } [0, \infty)
\] (3.5)

with
\[
\text{supp } \Phi_1 \subseteq [-1/2, 1/2], \quad \text{supp } \Phi_2 \subseteq [1/4, 3], \quad \text{supp } \Phi_3 \subseteq [2, \infty).
\] (3.6)

Denote
\[
A_k(\xi, \eta) = (|\xi + \eta|^s(\xi + \eta)_j - |\xi|^s \xi_j) \hat{f}(\xi) \hat{g}(\eta) \Phi_k(|\xi|/|\eta|), \quad k = 1, 2, 3.
\] (3.7)

The commutator (3.2) may be rewritten as
\[
[\Lambda^s \partial_j, g]f = c_0 \sum_{k=1}^{3} \int \int e^{ix \cdot (\xi + \eta)} A_k(\xi, \eta) \, d\eta \, d\xi.
\] (3.8)

First, we write \( A_1 \) as
\[
A_1(\xi, \eta) = \frac{|\xi + \eta|^s(\xi + \eta)_j - |\xi|^s \xi_j}{|\eta|^{1+s}} \Phi_1 \left( \frac{|\xi|}{|\eta|} \right) \hat{f}(\xi)(\Lambda^{1+s}g)^*(-\eta)
\]
\[
= \sigma_1(\xi, \eta) \hat{f}(\xi)(\Lambda^{1+s}g)^*(-\eta).
\] (3.9)

It is easy to check that
\[
|\sigma_1| \leq C, \quad (3.10)
\]

where \( C \) is a constant and where we used \( 2|\xi| \leq |\eta| \) on \( \text{supp } \Phi_1 \). Taking the first derivative with respect to \( \xi_l \), where \( l = 1, 2 \), we obtain
\[
\left| \frac{\partial \sigma_1}{\partial \xi_l} \right| \leq C \frac{C}{|\eta|} \leq \frac{C}{|\xi| + |\eta|} \quad (3.11)
\]
while differentiating with respect to $\eta_l$ for $l = 1, 2$ leads to
\[
\left| \frac{\partial \sigma_1}{\partial \eta} \right| \leq C \frac{C}{|\xi| + |\eta|}.
\]
Continuing by induction, we get
\[
|\frac{\partial^2 \sigma_1}{\partial \eta^2} | \leq C(|\alpha|, |\beta|)(|\xi| + |\eta|)^{-|(|\alpha| + |\beta|)}, \quad \alpha, \beta \in \mathbb{N}_0^2.
\]

By the Coifman-Meyer theorem, we get
\[
\left\| \int \int e^{ix \cdot (\xi + \eta)} A_1(\xi, \eta) \, d\eta \, d\xi \right\|_{L^q} \leq C\|f\|_{L^{r_1}} \|\Lambda^{1+q}g\|_{L^{r_1}}
\]
where $1/q = 1/r_1 + 1/r_2$. Postponing the treatment of $A_2$, we deal with $A_3$ next. We rewrite this symbol as
\[
A_3(\xi, \eta) = |\xi + \eta|^{-s} \Phi_3 \left( \frac{|\xi|}{|\eta|} \right) \hat{f}(\xi) \hat{g}(\eta) + (|\xi| - |\eta|) \xi_j \Phi_3 \left( \frac{|\xi|}{|\eta|} \right) \hat{f}(\xi) \hat{g}(\eta)
\]
\[
= i \frac{|\xi + \eta|^s}{|\xi|^s} \frac{d}{d\eta} (\Phi_3 \left( \frac{|\xi|}{|\eta|} \right)) \Lambda^s f \nabla g (\eta) + (|\xi| - |\eta|) \xi_j \Phi_3 \left( \frac{|\xi|}{|\eta|} \right) \Lambda^s f \nabla g (\eta)
\]
\[
= \sigma_{31}(\xi, \eta) \Lambda^s f \nabla g (\eta) + \sigma_{32}(\xi, \eta) \Lambda^s f \nabla g (\eta).
\]
Note that in the region $\Phi_3 > 0$, we have $|\xi| > 2|\eta|$. For $\sigma_{31}$ it is easy to check that
\[
|\frac{\partial^2 \sigma_3}{\partial \eta^2} | \leq C(|\xi| + |\eta|)^{-|(|\alpha| + |\beta|)}, \quad \alpha, \beta \in (\mathbb{Z}^+)^2.
\]
In order to estimate the second term on the far right side of (3.15), we write
\[
\xi_j (|\xi + \eta|^s - |\xi|^s) = \xi_j \int_0^1 \frac{d}{dr} (|\xi + r\eta|^s) \, dr = \xi_j \int_0^1 s|\xi + r\eta|^s - 2(\xi + r\eta) \cdot \eta \, dr,
\]
which implies that the second term may be rewritten as
\[
\sigma_{32}(\xi, \eta) \Lambda^s f \nabla g (\eta) = \int \frac{d}{d\eta} (\Phi_3 \left( \frac{|\xi|}{|\eta|} \right)) \Lambda^s f \nabla g (\eta) \int_0^1 \frac{s|\xi + r\eta|^s - 2(\xi + r\eta) \cdot \eta}{|\xi|^s} \, dr.
\]
A direct computation shows that
\[
|\frac{\partial^2 \sigma_3}{\partial \eta^2} | \leq C(|\xi| + |\eta|)^{-|(|\alpha| + |\beta|)}, \quad \alpha, \beta \in \mathbb{N}_0^2.
\]
Therefore, we obtain that
\[
\left\| \int \int e^{ix \cdot (\xi + \eta)} A_3 \, d\eta \, d\xi \right\|_{L^q} \leq C\|\Lambda^s f\|_{L^{r_2}} \|\nabla g\|_{L^{r_2}},
\]
where \(1/q = 1/r_2 + 1/\tilde{r}_2\). Finally we treat the second term of the sum on the right side of (3.8), in which case \(|\xi|\) and \(|\eta|\) are comparable. Note that

\[
A_2(\xi, \eta) = |\xi + \eta|^s(\xi + \eta)jF_2 \left( \frac{|\xi|}{|\eta|} \right) \hat{f}(\xi)\hat{g}(\eta) - |\xi|^s\xi_jF_2 \left( \frac{|\xi|}{|\eta|} \right) \hat{f}(\xi)\hat{g}(\eta) = A_{21} - A_{22}. \tag{3.21}
\]

First we deal with the simpler term \(A_{22}\), which may be written as

\[
A_{22}(\xi, \eta) = \frac{|\xi|^s\xi_j}{|\xi|^s|\eta|} F_2 \left( \frac{|\xi|}{|\eta|} \right) \left( \Lambda^s f \right)^{\ast}(\xi)(\Lambda g)^{\ast}(\eta). \tag{3.22}
\]

Applying the Coifman-Meyer theorem we get

\[
\left\| \int \int e^{ix\cdot(\xi + \eta)} A_{22}(\xi, \eta) \, d\eta \, d\xi \right\|_{L_q} \leq C \|\Lambda^s f\|_{L^{r_2}} \|\Lambda g\|_{L^{\tilde{r}_2}}. \tag{3.23}
\]

In order to conclude the lemma we only need to get a similar estimate for \(A_{21}\), to which the above method does not apply. The main reason is that when we take the derivative, the factor of \(|\xi + \eta|^s\) appears in the denominator. Thus, as in [KP], we use the complex interpolation to avoid this difficulty. First, we have

\[
\left\| \int \int e^{ix\cdot(\xi + \eta)} A_{21}(\xi, \eta) \, d\eta \, d\xi \right\|_{L_q} \leq C(b) \|\Lambda^s f\|_{L^{r_2}} \|\Lambda g\|_{L^{\tilde{r}_2}}, \tag{3.24}
\]

where \(s = a + ib\), with \(a\) a sufficiently large positive number and \(b \in \mathbb{R}\), and where \(C(b)\) is a polynomial. Next we consider the case \(s = ib\) with \(b \in \mathbb{R}\). In order to get an estimate for \(\left\| \int \int e^{ix\cdot(\xi + \eta)} A_{21} \, d\eta \, d\xi \right\|_{L_q}\), we rewrite

\[
\int \int e^{ix\cdot(\xi + \eta)} A_{21}(\xi, \eta) \, d\eta \, d\xi = \int \int e^{ix\cdot(\xi + \eta)} |\xi + \eta|^s(\xi + \eta)j \hat{f}(\xi)\hat{g}(\eta) \Phi_2 \, d\eta \, d\xi
\]

\[
= \int \int e^{ix\cdot(\xi + \eta)} |\xi + \eta|^s \left( \frac{\xi + \eta}{|\eta|} \right) \hat{f}(\xi)(\Lambda g)^{\ast}(\eta) \Phi_2 \, d\eta \, d\xi
\]

\[
= \int \int e^{ix\cdot|\xi|^s} \left( \frac{\xi}{|\eta|} \right) \hat{f}(\xi - \eta)(\Lambda g)^{\ast}(\eta) \Phi_2 \, d\eta \, d\xi. \tag{3.25}
\]

Using the Hörmander-Mikhlin multiplier theorem for the symbol \(|\xi|^s = |\xi|^ib\), we have

\[
\left\| \int \int e^{ix\cdot(\xi + \eta)} A_{21}(\xi, \eta) \, d\eta \, d\xi \right\|_{L_q} \leq C(b) \left\| \int e^{ix\cdot\xi} \int \frac{\xi_j}{|\eta|} \hat{f}(\xi - \eta)(\Lambda g)^{\ast}(\eta) \Phi_2 \, d\eta \, d\xi \right\|_{L_q}
\]

\[
= C(b) \left\| \int \int e^{ix\cdot(\xi + \eta)} \left( \frac{\xi + \eta}{|\eta|} \right) \Phi_2 \hat{f}(\xi)(\Lambda g)^{\ast}(\eta) \, d\eta \, d\xi \right\|_{L_q}. \tag{3.26}
\]

Since

\[
\left| \partial_\xi^\alpha \partial_\eta^\beta \left( \frac{\xi + \eta}{|\eta|} \right) \Phi_2 \left( \frac{|\xi|}{|\eta|} \right) \right| \leq C(|\xi| + |\eta|)^{-|\alpha|+|\beta|}, \quad \alpha, \beta \in \mathbb{N}^2_0, \tag{3.27}
\]

using the Coifman-Meyer theorem, we have

\[
\left\| \int \int e^{ix\cdot(\xi + \eta)} A_{21}(\xi, \eta) \, d\eta \, d\xi \right\|_{L_q} \leq C(b) \|f\|_{L^{r_2}} \|\Lambda g\|_{L^{\tilde{r}_2}}. \tag{3.28}
\]
Thus, using the complex interpolation inequality, we get
\[ \left\| \int e^{i\xi \cdot (\xi + \eta)} A_{21}(\xi, \eta) \, d\eta \right\|_{L^p} \leq C(b) \| \Lambda^s f \|_{L^r} \| \Lambda g \|_{L^{r/2}} \] (3.29)
for any \( s \) such that \( \Re(s) \in (0, 1) \). Taking \( s \in (0, 1) \), since \( \Im(s) = 0 \), we have
\[ \left\| \int e^{i\xi \cdot (\xi + \eta)} A_{21}(\xi, \eta) \, d\eta \right\|_{L^{r/2}} \leq C \| \Lambda^s f \|_{L^r} \| \Lambda g \|_{L^{r/2}} \] (3.30)
and the proof is completed. \( \square \)

4 An \( L^\infty \) bound

The following statement provides \( L^\infty \) bounds on the velocity and the vorticity in terms of \( L^q \) norms of the derivatives of the vorticity.

**Lemma 4.1.** Assume that \( u \in S(\mathbb{R}^2)^2 \) is a vector field satisfying \( \text{div} \, u = 0 \). Let \( 0 < s < \infty \) and \( 1 \leq r \leq \infty \) be such that \( rs > 2 \). For \( 2 \leq q < \infty \), with \( \omega = \nabla \times u \), the inequalities
\[ \| \Lambda u \|_{L^\infty} \leq C \left( 1 + \log \left( 1 + \| \Lambda^s \omega \|_{L^r} \right) \right) \left( 1 + \| \omega \|_{L^r} + \| \nabla (|\omega|^{q/2}) \|_{L^2}^{2/q} \right) \] (4.1)
and
\[ \| \omega \|_{L^\infty} \leq C \left( 1 + \log \left( 1 + \| \Lambda^s \omega \|_{L^r} \right) \right) \left( 1 + \| \omega \|_{L^r} + \| \nabla (|\omega|^{q/2}) \|_{L^2}^{2/q} \right) \] (4.2)
hold, where \( C = C(r, s, q) \).

**Proof.** First we prove that
\[ \| f \|_{L^\infty} \leq C (\| f \|_{L^b} + \| \Lambda^s f \|_{L^r}) , \quad b \geq 1 \quad \text{if} \quad f \in S(\mathbb{R}^2) \] (4.3)
where the constant \( C \) depends only on \( r \) and \( s \), i.e., it is independent of \( b \). Without loss of generality, we may assume \( \| f \|_{L^b} + \| \Lambda^s f \|_{L^r} = 1 \). Using the standard Littlewood-Paley notation, we have
\[ \| f \|_{L^\infty} \leq \| S_0 f \|_{L^\infty} + \sum_{j=0}^{\infty} \| \Delta_j f \|_{L^\infty} \] (4.4)
By Bernstein’s inequality, we bound the first term on the right side as
\[ \| S_0 f \|_{L^\infty} \leq C \| S_0 f \|_{L^b} \leq C . \] (4.5)
In order to estimate the second term on the right side of (4.4), we write
\[ \| \Delta_j f \|_{L^\infty} \leq C 2^{2j/r} \| \Delta_j f \|_{L^r} \leq C 2^{2j/r - j} \| \Lambda^s \Delta_j f \|_{L^r} \leq C 2^{j(2 - s)/2} , \] (4.6)
where we used Bernstein’s inequality in the first step. Since \( rs > 2 \), we may sum up the terms and obtain \( \| f \|_{L^\infty} \leq C \), thus leading to (4.3). By rescaling (4.3), we obtain the Gagliardo-Nirenberg inequality
\[ \| f \|_{L^\infty} \leq C \| f \|_{L^b}^{1 - a} \| \Lambda^s f \|_{L^r}^a \] (4.7)
where
\[ a = \frac{2}{2 + b(s - 2/r)}. \] (4.8)

Applying the above inequality to \( Au \) with \( b \geq q \) to be determined, we get
\[ \|Au\|_{L^\infty} \leq C\|Au\|_{L^p} - a\|A^{1+\delta}u\|_{L^r} \leq Cb\|\omega\|_{L^q} - a\|A^\omega\|_{L^r} \leq Cb(1 + \|\omega\|_{L^r})\|A^\omega\|_{L^r}^q, \] (4.9)
where the parameters \( r, s, q \) are fixed, while we need to track the dependence of the constants on \( b \).

Also, we have
\[ \|u\|_{L^r} = \|\omega\|_{L^q}^{2/q} \leq C\left( \frac{b}{q} \right)^{2/q} \|\omega\|_{L^q}^{2/q} \|(\omega\|_{L^2}^q)_{1-q/b} \right)^{2/q} \leq C\left( \frac{b}{q} \right)^{2/q} \|\nabla(\omega|q/2)\|_{L^2}^{2/q} \leq Cb\|\omega\|_{L^r} + \|\nabla(\omega|q/2)\|_{L^2}^{2/q}. \] (4.10)

Using (4.10) in (4.9), we get
\[ \|Au\|_{L^\infty} \leq Cb^{1+1/q}(1 + \|\omega\|_{L^r} + \|\nabla(\omega|q/2)\|_{L^2}^{2/q})\|A^\omega\|_{L^r}^q. \] (4.11)

Therefore, by setting \( b = q + \log(1 + \|A^\omega\|_{L^r}) \), we obtain
\[ \|Au\|_{L^\infty} \leq C \left( 1 + \log(1 + \|A^\omega\|_{L^r}) \right)^{1+1/q}(1 + \|\omega\|_{L^r} + \|\nabla(\omega|q/2)\|_{L^2}^{2/q}). \] (4.12)
Indeed, note that \( \|A^\omega\|_{L^r}^q = \exp(a \log\|A^\omega\|_{L^r}) \), which is bounded by a constant by our choice of \( b \) and (4.8). Thus (4.1) is established. Similarly to (4.9), we get
\[ \|\omega\|_{L^\infty} \leq C\|\omega\|_{L^p} - a\|A^\omega\|_{L^r}. \] (4.13)

Repeating the above procedure, we get
\[ \|\omega\|_{L^\infty} \leq C \left( 1 + \log(1 + \|A^\omega\|_{L^r}) \right)^{1/q}(1 + \|\omega\|_{L^r} + \|\nabla(\omega|q/2)\|_{L^2}^{2/q}), \] (4.14)
which gives (4.2).

\section{The proof of Theorem 2.2}

We are now ready to prove Theorem 2.2. Since the persistence in \( W^{1,q}(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \) has been addressed in Proposition 2.1, we focus on the estimate of the highest derivatives of \( u \rho \), namely on \( \|A^\omega\|_{L^r} \) and \( \|A^\rho\|_{L^r} \). Since the cancellation property is not available, we obtain an extra term involving \( \|Au\|_{L^\infty} \). Using results in Section 4, we may bound it by \( \log^{1+1/q}(1 + \|A^\omega\|_{L^r})(1 + \|\omega\|_{L^r} + \|\nabla(\omega|q/2)\|_{L^2}^{2/q}) \), leading to logarithmic dependence on the norms. Note that since \( 1 + 1/q > 1 \), we do not obtain the global existence (note, however, the global results in Sections 6 and 7).

\textbf{Proof of Theorem 2.2}. Applying the operator \( A^\omega \) to the equation (2.15), multiplying it with \( |A^\omega|^{q-2}A^\omega \), and integrating in \( x \), we get
\[ \frac{1}{q} \frac{d}{dt} \|A^\omega\|_{L^q}^q - \int A^\omega |\Delta \omega| |A^\omega|^{q-2}A^\omega \ dx + \int A^\omega (u \cdot \nabla \omega) |A^\omega|^{q-2}A^\omega \ dx 
\]
\[ = \int A^\omega \partial_t |\Delta \omega|^{q-2}A^\omega \ dx. \] (5.1)
Next, we integrate by parts in the second term on the left side of the above equation and move the third
term to the right side in order to get

\[
\frac{1}{q} \frac{d}{dt} \| \Lambda^s \omega \|_{L^q}^q + (q - 1) \left( \frac{2}{q} \right)^2 \| \nabla (|\Lambda^s \omega|^{q/2}) \|_{L^2}^2 \\
= \int \Lambda^s \partial_t \rho |\Lambda^s \omega|^{q-2} \Lambda^s \omega \, dx - \int \Lambda^s (u \cdot \nabla \omega) |\Lambda^s \omega|^{q-2} \Lambda^s \omega \, dx \\
= I_1 + I_2, \tag{5.2}
\]

where we also used

\[
\int |\Lambda^s \nabla \omega|^{2} |\Lambda^s \omega|^{q-2} \, dx = \left( \frac{2}{q} \right)^2 \| \nabla (|\Lambda^s \omega|^{q/2}) \|_{L^2}^2. \tag{5.3}
\]

For the term $I_1$, we integrate by parts and use Hölder’s inequality to obtain

\[
I_1 = - (q - 1) \int \Lambda^s (\rho \psi_2) |\Lambda^s \omega|^{q-2} \Lambda^s \partial_t \omega \, dx \leq C \| \Lambda^s \rho \|_{L^q} \| \Lambda^s \nabla \omega |\Lambda^s \omega|^{(q-2)/2} \|_{L^2} \| |\Lambda^s \omega|^{(q-2)/2} \|_{L^{2q/(q-2)}} \\
= C \| \Lambda^s \rho \|_{L^q} \| \nabla (|\Lambda^s \omega|^{q/2}) \|_{L^2} \| \Lambda^s \omega \|_{L^{2q/(q-2)}}. \tag{5.4}
\]

Since

\[
I_2 = - \int (\Lambda^s (u \cdot \nabla \omega) - u \cdot \Lambda^s \nabla \omega) |\Lambda^s \omega|^{q-2} \Lambda^s \omega \, dx, \tag{5.5}
\]

we have by Hölder’s inequality and Lemma 3.1

\[
I_2 \leq \| \Lambda^s (u \cdot \nabla \omega) - u \cdot \Lambda^s \nabla \omega \|_{L^q} \| \Lambda^s \omega \|_{L^{q-1}}^{q-1} \\
\leq C (\| \Lambda^s \omega \|_{L^q} \| \Lambda u \|_{L^\infty} + \| \omega \|_{L^\infty} \| \Lambda^{1+\kappa} u \|_{L^q}) \| \Lambda^s \omega \|_{L^q}^{q-1} \\
\leq C \| \Lambda^s \omega \|_{L^q}^q (\| \Lambda u \|_{L^\infty} + \| \omega \|_{L^\infty}). \tag{5.6}
\]

Using (5.4) and (5.6) in (5.2), we get

\[
\frac{1}{q} \frac{d}{dt} \| \Lambda^s \omega \|_{L^q}^q + (q - 1) \left( \frac{2}{q} \right)^2 \| \nabla (|\Lambda^s \omega|^{q/2}) \|_{L^2}^2 \\
\leq C \| \Lambda^s \rho \|_{L^q} \| \nabla (|\Lambda^s \omega|^{q/2}) \|_{L^2} \| |\Lambda^s \omega|^{(q-2)/2} \|_{L^2} + C \| \Lambda^s \omega \|_{L^q} (\| \Lambda u \|_{L^\infty} + \| \omega \|_{L^\infty}) \\
\leq C \| \Lambda^s \rho \|_{L^q}^q + 1 \left( \frac{q-1}{q^2} \right) \| \nabla (|\Lambda^s \omega|^{q/2}) \|_{L^2}^2 + C \| \Lambda^s \omega \|_{L^q}^q \\
+ C \| \Lambda^s \omega \|_{L^q}^q (\| \Lambda u \|_{L^\infty} + \| \omega \|_{L^\infty}), \tag{5.7}
\]

where we used Young’s inequality in the last step. Absorbing the second term on the right side into the
left side of the above inequality, we obtain

\[
\frac{d}{dt} \| \Lambda^s \omega \|_{L^q}^q + \| \nabla (|\Lambda^s \omega|^{q/2}) \|_{L^2}^2 \\
\leq C \| \Lambda^s \rho \|_{L^q}^q + C \| \Lambda^s \omega \|_{L^q}^q + C \| \Lambda^s \omega \|_{L^q}^q (\| \Lambda u \|_{L^\infty} + \| \omega \|_{L^\infty}). \tag{5.8}
\]

Adding the inequalities (2.14) and (5.8) and omitting the $D_0$ term, we arrive at

\[
\frac{d}{dt} (\| u \|_{L^q}^q + \| \Lambda^s \omega \|_{L^q}^q) + \| \nabla (|\Lambda^s \omega|^{q/2}) \|_{L^2}^2 \\
\leq C (\| u \|_{L^q}^q + \| \omega \|_{L^q}^q + 1) \| u \|_{L^q}^{q-1} + C \| \Lambda^s \rho \|_{L^q}^q + C \| \Lambda^s \omega \|_{L^q}^q \\
+ C \| \Lambda^s \omega \|_{L^q}^q (\| \omega \|_{L^\infty} + \| \Lambda u \|_{L^\infty}). \tag{5.9}
\]
Next, we need an appropriate estimate for \(\|\Lambda^s \rho\|_{L^q}\) and \(\|\omega\|_{L^\infty} + \|Au\|_{L^\infty}\). We apply the operator \(\Lambda^s\) to the equation (1.3), multiply it with \(|\Lambda^s \rho|^{q-2}\Lambda^s \rho\), and integrate it with respect to \(x\) to get

\[
\frac{1}{q} \frac{d}{dt} \|\Lambda^s \rho\|_{L^q}^q = - \int \Lambda^s (u \cdot \nabla \rho) |\Lambda^s \rho|^{q-2} \Lambda^s \rho \, dx = - \int \Lambda^s \text{div} (u \rho) |\Lambda^s \rho|^{q-2} \Lambda^s \rho \, dx, \tag{5.10}
\]

where we used the divergence-free condition for \(u\). Since

\[
- \int \Lambda^s \text{div} (u \rho) |\Lambda^s \rho|^{q-2} \Lambda^s \rho \, dx = \int (\Lambda^s \text{div} (u \rho) - u \cdot \nabla \Lambda^s \rho) |\Lambda^s \rho|^{q-2} \Lambda^s \rho \, dx \tag{5.11}
\]

we obtain by Lemma 3.1

\[
- \int \Lambda^s \text{div} (u \rho) |\Lambda^s \rho|^{q-2} \Lambda^s \rho \, dx \leq \|\Lambda^s \text{div} (u \rho) - u \cdot \nabla \Lambda^s \rho\|_{L^q} \|\Lambda^s \rho\|_{L^q}^{q-1} \leq C \|\Lambda^s \rho\|_{L^q} \|\Lambda u\|_{L^\infty} \|\Lambda^s \rho\|_{L^q}^{q-1} + C \|\rho\|_{L^r} \|\Lambda^{1+s} u\|_{L^r} \|\Lambda^s \rho\|_{L^q}^{q-1}, \tag{5.12}
\]

where \(r_1 = 2q/(2 - s)\) and \(s_1 = 2q/s\). Denoting

\[
I = \begin{cases} [q, 2q/(2 - qs)], & \text{if } qs < 2 \\ [q, \infty), & \text{otherwise}, \end{cases} \tag{5.13}
\]

we observe that \(\rho_0 \in W^{s,q}\) implies \(\rho_0 \in L^r \) for \(r \in I\), and thus

\[
\|\rho(t)\|_{L^r} = C(M_1, r_1), \quad t > 0. \tag{5.14}
\]

For \(\|\Lambda^{1+s} u\|_{L^{r_1}}\), by the Sobolev embedding theorem and the Gagliardo-Nirenberg inequality, we have

\[
\|\Lambda^{1+s} u\|_{L^{r_1}} \leq C \|\Lambda^s \omega\|_{L^{r_1}} \leq C \|\Lambda^s \omega\|^{q/2}_{L^{2q/s}} \|\nabla|\Lambda^s \omega|^{q/2}\|_{L^2}^{2s/q} \leq C \|\Lambda^s \omega\|^{q/2}_{L^{2q/s}} \|\nabla|\Lambda^s \omega|^{q/2}\|_{L^2}^{2s/q} \leq C \|\Lambda^s \omega\|_{L^{q/2/s}} \|\nabla|\Lambda^s \omega|^{q/2}\|_{L^2}^{2s/q}, \tag{5.15}
\]

where \(\alpha = 1 - s/2\). Therefore, combining (5.10) and (5.12) with (5.15) leads to

\[
\frac{1}{q} \frac{d}{dt} \|\Lambda^s \rho\|_{L^q}^q \leq C \|\Lambda u\|_{L^\infty} \|\rho\|_{W^{s,q}}^q + C \|\rho\|_{L^r} \|\omega\|_{W^{s,q}} + C \|\omega\|_{W^{s,q}} \|\Lambda^s \omega\|_{L^q} \|\Lambda^s \rho\|_{L^q}^{q-1} + \frac{q - 1}{q^2} \|\nabla|\Lambda^s \omega|^{q/2}\|_{L^2}^2, \tag{5.16}
\]

where we used Young’s inequality and (5.14). Adding the above estimate to (5.9) and setting

\[
F(t) = \|u\|_{L^q}^q + \|\Lambda^s \omega\|_{L^q}^q + \|\rho\|_{L^r}^q + \|\Lambda^s \rho\|_{L^q}^q, \tag{5.17}
\]

we get

\[
\frac{1}{q} F'(t) + \frac{q - 1}{q} \frac{2}{q^2} \|\nabla|\Lambda^s \omega|^{q/2}\|_{L^2}^2 \leq C(1 + F(t)) + CF(t) \|\omega\|_{L^q} + CF(t)(\|\Lambda u\|_{L^\infty} + \|\omega\|_{L^\infty}).
\]
Finally, we handle the terms \( \|\Lambda u\|_{L^\infty} \) and \( \|\omega\|_{L^\infty} \). By Lemma 4.1, we get from (5.17)

\[
\frac{1}{q} F'(t) + (q-1) \frac{2}{q^2} \|\nabla(|\Lambda^s\omega|^{q/2})\|_{L^2}^2 \\
\leq C(1 + F(t)) + CF(t)\|\omega\|_{L^r} \\
+ CF(t)(1 + \|\nabla(|\omega|^{q/2})\|_{L^2}^{2/q}) (1 + \log^{1+1/q}(1 + \|\Lambda^s\omega\|_{L^r}))
\]

\[= C(1 + F(t)) + CF(t)\sqrt{1 + t} + CF(t)X(t)(1 + \log^{1+1/q}(1 + \|\Lambda^s\omega\|_{L^r})), \quad (5.18)\]

where we chose \( r \geq \max(2/s, q) \) and denoted \( X(t) = 1 + \|\nabla(|\omega|^{q/2})\|_{L^2}^{2/q} \). By (2.23)

\[
\int_0^t X^q(s) \, ds \leq C(1 + t)^{q/2} \quad (5.19)
\]

holds. As for the term \( \|\Lambda^s\omega\|_{L^r} \), we have by the Gagliardo-Nirenberg and Young’s inequalities,

\[
\|\Lambda^s\omega\|_{L^r} = \|\Lambda^s\omega|^{q/2}\|_{L^{2/q}}^{2/q} \leq (\|\Lambda^s\omega|^{q/2}\|_{L^2}^{2/q})^{2/q} \]

\[\leq \|\Lambda^s\omega\|_{L^r} + \|\nabla(|\Lambda^s\omega|^{q/2})\|_{L^2}^{2/q} \leq \|\Lambda^s\omega\|_{L^r} + C + \frac{q-1}{q^2} \|\nabla(|\Lambda^s\omega|^{q/2})\|_{L^2}^2. \quad (5.20)\]

Now we conclude the proof by applying the following Gronwall type lemma to (5.18). \( \square \)

**Lemma 5.1.** Assume that \( F(t) \) is a continuously differentiable function in a neighborhood of 0 and \( F(0) \leq M \). Let \( X(t), Y(t) \) and \( D(t) : [0, \infty) \to [0, \infty) \) be such that

\[
\int_0^t X(s) \, ds \leq C(1 + t)^{q/2} \quad (5.21)
\]

for some \( q \geq 2 \) and \( Y(t) \leq D(t)/2 + F(t) + \overline{M} \), where \( M \) and \( \overline{M} \) are constants. Furthermore, assume that \( F(t) \) satisfies

\[
F'(t) + D(t) \leq C(1 + F(t)) + CF(t)\sqrt{1 + t} + CF(t)X(t)(1 + \log(1 + Y(t)))^\alpha \quad (5.22)
\]

for some \( \alpha > 0 \). With \( T^* = \infty \) if \( 0 < \alpha \leq 1 \) and

\[
T^* = \min \left\{ \left( \frac{1}{C(1 + \log^{\alpha-1}(M + 1))} \right)^{2/3} - 1, \frac{1}{C(1 + \log^{\alpha-1}(M + 1))^{2q/(q-2)}} \right\} \quad (5.23)
\]

for some constant \( C \) if \( \alpha > 1 \), we then have

\[
F(t) < \infty, \quad t \in [0, T^*) \quad (5.24)
\]

where \( C \) is a constant depending only on \( M, \overline{M}, \) and \( T^* \).

**Proof.** For the case \( \alpha \in [0, 1] \), as in [HKZ1], we have a basic inequality

\[
FX(1 + \log(1 + Y))^\alpha \leq FX(1 + \log(1 + Y)) \leq FX + \frac{Y}{C} + FX \log(1 + CFX). \quad (5.25)
\]
Hence, we have

\[
F'(t) + D(t) \leq C(1 + F(t)) + Y(t) + CF(t)X(t) + CF(t)\sqrt{1 + t} \\
+ CF(t)X(t)\log(1 + CF(t)X(t)) \\
\leq C + CF(t) + \frac{D(t)}{2} + CF(t)X(t) + CF(t)\sqrt{1 + t} \\
+ CF(t)X(t)\log(1 + F(t)) + CF(t)X(t)\log(1 + CX(t)).
\]

(5.26)

The desired result is then obtained by the usual Gronwall lemma.

If on the other hand \( \alpha > 1 \), we get

\[
FX(1 + \log^\alpha(1 + Y)) \leq FX + \frac{Y}{C} + CFX\log^\alpha(1 + CFX).
\]

(5.27)

Similarly to (5.26), we arrive at

\[
F'(t) + D(t) \leq C + CF(t) + \frac{D(t)}{2} + CF(t)X(t) + CF(t)\sqrt{1 + t} \\
+ CF(t)X(t)\log^\alpha(1 + F(t)) + CF(t)X(t)\log^\alpha(1 + CX(t))
\]

(5.28)

and the lemma is established by the classical Gronwall’s inequality. \( \square \)

6 Persistence for \( qs > 2 \) and for the intersection spaces

Given initial datum \((u_0, \rho_0) \in W^{1+s,q} \times W^{s,q}\) we do not know in general whether the solution \((u, \rho)\) stays in the space \(W^{1+s,q} \times W^{s,q}\) for all time. However, we can prove that this is so if \( sq > 2 \) or if the initial data \((u_0, \rho_0)\) belongs to the intersection space \( X = (H^{1+s} \times H^s) \cap (W^{1+s,q} \times W^{s,q})\). We consider the case \( sq > 2 \) in the next theorem, while the intersection space is addressed in Theorem 6.3 below.

**Theorem 6.1.** Let \( s \in (0, 1) \) and \( q \in [2, \infty) \) be such that \( sq > 2 \). Assume that \( \|u_0\|_{W^{1+s,q}} \leq M_0 \) with \( \text{div} u_0 = 0 \) and \( \|\rho_0\|_{W^{s,q}} \leq M_1 \), where \( M_0, M_1 \geq 1 \) are constants. Then there exists a unique solution \((u, \rho)\) of the equations (1.1)–(1.3) such that \( u \in C([0, \infty), W^{1+s,q}(\mathbb{R}^2)) \) and \( \rho \in C([0, \infty), W^{s,q}(\mathbb{R}^2))\).

The key ingredient making the global persistence work is the following uniform bound on the vorticity.

**Lemma 6.2.** Assume that \( q > 2 \) and \( \omega_0 \in W^{1,q}(\mathbb{R}^2) \), and let \( \rho \in L^\infty(0, \infty; L^\infty(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \). Then the unique solution \( \omega \) of the equation (2.15) with the initial data \( \omega_0 \) satisfies \( \omega \in C([0, \infty); L^\infty(\mathbb{R}^2)) \). Moreover for any \( T > 0 \) the inequality

\[
\|\omega(t)\|_{L^\infty} \leq C(\|\rho\|_{L^\infty(0, \infty; L^\infty)}; \|\omega_0\|_{W^{1,q}})\sqrt{1 + t}, \quad t \in [0, T]
\]

(6.1)

holds.

**Proof.** Fix \( T > 0 \). By (2.22) we have

\[
\|\omega(t)\|_{L^q} \leq C(\|\rho\|_{L^\infty(0, \infty; L^q)})\sqrt{1 + T}, \quad t \in [0, T].
\]

(6.2)
For \( p \geq q \), denote \( \phi_p = \int |\omega|^p \). Using the estimate

\[
\left| \int \rho|\omega|^{2p-2} \partial_1 \omega \, dx \right| \leq \frac{C}{p} \|\rho\|_{L^\infty} \|\nabla(|\omega|^p)\|_{L^2} \|\omega\|_{L^{2p-2}},
\]  

we obtain from (2.17) that

\[
\frac{1}{2p} \phi_{2p}' + \frac{2p-1}{2p^2} \|\nabla(|\omega|^p)\|_{L^2}^2 \leq C_p \|\rho\|_{L^\infty}^2 \int \omega^{2p-2}
\leq C_p \|\rho\|_{L^\infty}^2 (\phi_p)^{2/p} (\phi_{2p})^{(p-2)/p}.
\]  

(6.4)

Applying the Gagliardo-Nirenberg inequality \( \|f\|_{L^2} \leq C \|f\|_{L^2}^{1/2} \|\nabla f\|_{L^2}^{1/2} \) to \( f = |\omega|^p \) leads to

\[
\phi_{2p} \leq C \phi_p \|\nabla(|\omega|^p)\|_{L^2},
\]  

(6.5)

which combined with (6.4) implies

\[
\frac{1}{2p} \phi_{2p}' + \frac{2p-1}{2p^2} \left( \frac{\phi_{2p}}{\phi_p} \right)^2 \leq C_p \|\rho\|_{L^\infty}^2 (\phi_p)^{2/p} (\phi_{2p})^{(p-2)/p}.
\]  

(6.6)

Note that if

\[
\phi_{2p} \geq C^{p/(p+2)} p^{2p/(p+2)} \rho_{L^\infty}^{2p/(p+2)} \phi_p^{(2p+2)/(p+2)},
\]  

(6.7)

for a sufficiently large constant \( C \), the second term on the left side dominates the term on the right and thus

\[
\phi_{2p} \leq 0.
\]  

(6.8)

Denoting

\[
K_p = \|\omega\|_{L^\infty(0,T;L^p(\mathbb{R}^2))},
\]  

(6.9)

\( \phi_{2p} \) may be estimated by

\[
\phi_{2p}(t) \leq \max \{ \phi_{2p}(0), C^{1/2p} p^{1/p} \rho_{L^\infty}^{1/p} K_p^{(p+1)/(p+2)} \}, \quad t \in [0, T].
\]  

(6.10)

Taking the supremum in \( t \), we obtain

\[
K_{2p} \leq \max \{ C \|\omega_0\|_{W^{s,q}}, C^{1/2p} p^{1/p} \rho_{L^\infty}^{1/p} K_p^{(p+1)/(p+2)} \}.
\]  

(6.11)

By induction and (6.2), we arrive at

\[
K_{2n+1,q} \leq \max \{ C \|\omega_0\|_{W^{s,q}}, K_q \} C \sum_{i=1}^{\infty} 1/2^i q \prod_{i=0}^{\infty} (2^i q)^{1/2^i q} (1 + \|\rho\|_{L^\infty(0,\infty;L^\infty)}) \sum_{i=0}^{\infty} 1/2^i q
\leq C \|\rho\|_{L^\infty(0,\infty;L^\infty)} \|\omega_0\|_{W^{s,q}} \sqrt{1 + T}, \quad t \in [0, T].
\]  

(6.12)

Since the constant \( C \) does not depend on \( n \), we may pass to the limit \( n \to \infty \) to conclude the proof. \( \square \)

Next we prove Theorem 6.1.
Proof of Theorem 6.1. Since the persistence in $H^{1+s} \times H^s$ was already addressed in [HKZ2], we only need to consider the case $q > 2$. The proof is similar to that of Theorem 2.2 except that we use a different estimate for $\|Au\|_{L^\infty}$ and $\|\omega\|_{L^\infty}$. Recall that
\[
\frac{1}{q} F'(t) + (q - 1) \frac{2}{q^2} \|\nabla(|\Lambda^s \omega|^{q/2})\|_{L^2}^2 \leq C (1 + F(t)) \sqrt{1 + t} + CF(t)(\|Au\|_{L^\infty} + \|\omega\|_{L^\infty}),
\]
where $F(t) = \|u\|_{L^q}^q + \|\Lambda^s \omega\|_{L^q}^q + \|\rho\|_{L^q}^q + \|\Lambda^s \rho\|_{L^q}^q$. By Lemma 6.2, we have
\[
\|\omega\|_{L^\infty} \leq C \sqrt{1 + t}, \quad t \geq 0.
\]
From (4.9), we deduce
\[
\|Au\|_{L^\infty} \leq Cb \|\omega\|_{L^b}^{1-a} \|\Lambda^s \omega\|_{L^r}^a \leq Cb \|\omega\|_{L^q}^{(1-a)/b} \|\omega\|_{L^\infty}^{(1-q/b)(1-a)} \|\Lambda^s \omega\|_{L^r}^q
\]
\[
\leq Cb (1 + \|\omega\|_{L^q} + \|\omega\|_{L^\infty}) \|\Lambda^s \omega\|_{L^r}^2,
\]
where $rs > 2$, $b \geq q$, and $a = 2/(2 + b(s - 2/r))$. Similarly as in the proof of Lemma 4.1, we set $b = q + \log(1 + \|\Lambda^s \omega\|_{L^r})$ in order to get
\[
\|Au\|_{L^\infty} \leq C (1 + \log(1 + \|\Lambda^s \omega\|_{L^r}))(1 + \|\omega\|_{L^q} + \|\omega\|_{L^\infty}).
\]
Therefore, noting that $\|\omega(t)\|_{L^q} \leq C \sqrt{1 + t}$, we obtain
\[
\frac{1}{q} F'(t) + (q - 1) \frac{2}{q^2} \|\nabla(|\Lambda^s \omega|^{q/2})\|_{L^2}^2 \leq C (1 + F(t)) (1 + \log(1 + \|\Lambda^s \omega\|_{L^r})) \sqrt{1 + t}.
\]
We conclude the proof by using Lemma 5.1.

Denoting
\[
\|(u, \rho)\|_X = \max \{\|u\|_{H^{1+s}}, \|u\|_{W^{1+s,q}}, \|\rho\|_{H^s}, \|\rho\|_{W^{s,q}}\},
\]
the global persistence result in the intersection space $X$ reads as follows.

Theorem 6.3. Let $s \in (0, 1)$ and $q \in [2, \infty)$. Assume that $\|(u_0, \rho_0)\|_X \leq M$ where $M$ is an arbitrary positive constant. There exists a unique solution $(u, \rho)$ to the equations (1.1)-(1.3) such that $(u, \rho) \in C([0, \infty), X)$. Moreover,
\[
\|(u(t), \rho(t))\|_X \leq C(M, T), \quad t \in [0, T]
\]
for any fixed $T > 0$.

In the proof, we need the following modification of Lemma 4.1.

Lemma 6.4. Assume that $u \in S(\mathbb{R}^2)^2$. Also, let $0 < s < \infty$ and $1 \leq r \leq \infty$ be such that $rs > 2$. For $2 \leq q < \infty$, with $\omega = \nabla \times u$, we have the inequalities
\[
\|Au\|_{L^\infty} \leq C (1 + \log(1 + \|\Lambda^{1+s} u\|_{L^r}))^{1/2} (1 + \|u\|_{H^2})
\]
and
\[
\|\omega\|_{L^\infty} \leq C (1 + \log(1 + \|\Lambda^s \omega\|_{L^r}))^{1/2} (1 + \|\omega\|_{H^s}).
\]
where $C = C(r, s, q)$. 

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Proof of Lemma 6.4. By (4.9), we have
\[ \|Au\|_{L^\infty} \leq C\|Au\|_{L_b}^{1-a}\|\Lambda^{1+a}u\|_{L^\infty}^{a} \leq C(1 + \|Au\|_{L^p})\|\Lambda^{1+a}u\|_{L^p}^{a} \] (6.21)
where \( b \geq 2 \) and \( a \) is given in (4.8). Also, since as in (4.10),
\[ \|Au\|_{L^p} \leq Cb^{1/2}(\|Au\|_{L^2} + \|\Lambda^2u\|_{L^2}) \leq Cb^{1/2}\|u\|_{H^2} \] (6.22)
we get
\[ \|Au\|_{L^\infty} \leq Cb^{1/2}(1 + \|u\|_{H^2})\|\Lambda^{1+a}u\|_{L^\infty}^{a} \] (6.23)
Then choose \( b = 2 + \log(1 + \|\Lambda^{1+a}u\|_{L^r}) \) and (6.19) follows. The inequality (6.20) is proven analogously. \( \square \)

Proof of Theorem 6.3. Since the persistence in \( H^{1+s} \times H^s \) was already addressed in [HKZ2], we only need to consider the case \( q > 2 \). The proof is similar to that of Theorem 2.2 except that we use a different estimate for \( \|Au\|_{L^\infty} \) and \( \|\omega\|_{L^\infty} \). Recall that we have
\[ \frac{1}{q}F'(t) + (q - 1)\frac{2}{q^2}\|\nabla(\Lambda^s\omega)^{1/2})\|_{L^2}^2 \leq C(1 + F(t))\sqrt{1 + t} + CF(t)(\|Au\|_{L^\infty} + \|\omega\|_{L^\infty}), \] (6.24)
where \( F(t) = \|u\|_{L^q} + \|\Lambda^s\omega\|_{L^q} + \|\rho\|_{L^q} + \|\Lambda^s\rho\|_{L^q} \). By Lemma 6.4 and the Calderón-Zygmund inequality, we have
\[ \|Au\|_{L^\infty} \leq C(1 + \log(1 + \|\Lambda^{1+s}u\|_{L^r}))^{1/2}(1 + \|u\|_{H^2}) \leq C(1 + \log(1 + \|\Lambda^s\omega\|_{L^r}))^{1/2}(1 + \|u\|_{H^2}) \] (6.25)
and
\[ \|\omega\|_{L^\infty} \leq C(1 + \log(1 + \|\Lambda^s\omega\|_{L^r}))^{1/2}(1 + \|\omega\|_{H^1}) \leq C(1 + \log(1 + \|\Lambda^s\omega\|_{L^r}))^{1/2}(1 + \|u\|_{H^2}). \] (6.26)
Therefore, we obtain
\[ \frac{1}{q}F'(t) + (q - 1)\frac{2}{q^2}\|\nabla(\Lambda^s\omega)^{1/2})\|_{L^2}^2 \leq C(1 + F(t))\sqrt{1 + t} + CF(t)(1 + \|\Lambda^s\omega\|_{L^r})(1 + \log(1 + \|\Lambda^s\omega\|_{L^r}))^{1/2}. \] (6.27)
From [HKZ2] we recall that
\[ \int_0^T \|u(t)\|_{H^2}^2 dt < \infty. \] (6.28)
Therefore, together with (5.20), we conclude the proof by using Lemma 5.1. \( \square \)

Corollary 6.5. Let \((u_0, \rho_0) \in W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)\) be compactly supported and assume that \( s \) and \( q \) are the same as in the above theorem. There exists a unique solution \((u, \rho)\) to the equations (1.1)–(1.3) such that \((u, \rho) \in C([0, \infty), W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2))\). Moreover,
\[ \|u(t)\|_{W^{1+s,q}(\mathbb{R}^2)}, \|\rho(t)\|_{W^{s,q}(\mathbb{R}^2)} \leq C, \quad t \in [0, T] \] (6.29)
for any fixed \( T > 0 \), where \( C \) depends on the initial data and \( T \).
Proof. Since \((u_0, \rho_0) \in X\) by the assumptions, the assertion follows by applying Theorem 6.3. 

\[\square\]

7 Persistence with periodic boundary conditions

Using the above theorem, we may also get the global persistence in the periodic domain.

**Theorem 7.1.** Let \(\|u_0\|_{W^{1+s, q}(T^2)} \leq M\) where \(M\) is an arbitrary positive constant and where \(0 < s < 1\) and \(q \in [2, \infty)\). Then there exists a unique solution \((u, \rho)\) to the equations (1.1)-(1.3) such that \((u, \rho) \in C([0, \infty), W^{1+s, q}(T^2) \times W^{s, q}(T^2))\). Moreover,

\[
\|u(t)\|_{W^{1+s, q}(T^2)}, \|\rho(t)\|_{W^{s, q}(T^2)} \leq C(M, T), \quad t \in [0, T]
\]

for any fixed \(T > 0\).

**Proof.** Since \(H^{1+s}(T^2) \hookrightarrow W^{1+s, q}(T^2)\) and \(H^{s}(T^2) \hookrightarrow W^{s, q}(T^2)\), the theorem follows by applying Theorem 6.3 provided we can prove Lemma 3.1 for the periodic domain \(T^2\). In fact the Coifman-Mayer estimate still holds for the periodic domain as shown in [Wo]. Then the same argument as in the proof of Lemma 3.1 shows that

\[
\|[\Lambda^s \partial_j, g] f\|_{L^r(T^2)} \leq C\|f\|_{L^r(T^2)} \|\Lambda^{1+s} g\|_{L^{r_1}(T^2)} + C\|\Lambda^s f\|_{L^{r_2}(T^2)} \|g\|_{L^{r_2}(T^2)}
\]

(7.2)

holds, where \(f, g, s, j, q, r_1, r_2, \tilde{r}_1, \tilde{r}_2\) are as in the statement of Lemma 3.1, with the only differences replacing \(\xi \in \mathbb{R}^2\) and \(\eta \in \mathbb{R}^2\) by discrete variables \(m \in \mathbb{T}^2\) and \(n \in \mathbb{T}^2\). 

\[\square\]

Acknowledgments

The authors would like to thank the referee for useful remarks. IK and FW were supported in part by the NSF grant DMS-1311943, while MZ was supported in part by the NSF grant DMS-1109562.

References


A. Larios, E. Lunasin, and E.S. Titi, *Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion*, J. Differential Equations 255 (2013), no. 9, 2636–2654.


