

# Arithmetic Intersection Theory on Flag Varieties

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## Abstract

Let  $F$  be the complete flag variety over  $\text{Spec}\mathbb{Z}$  with the tautological filtration  $0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$  of the trivial bundle  $E$  of rank  $n$  over  $F$ . The trivial hermitian metric on  $E(\mathbb{C})$  induces metrics on the quotient line bundles  $L_i(\mathbb{C})$ . Let  $\widehat{c}_1(\overline{L}_i)$  be the first Chern class of  $\overline{L}_i$  in the arithmetic Chow ring  $\widehat{CH}(F)$  and  $\widehat{x}_i = -\widehat{c}_1(\overline{L}_i)$ . Let  $h \in \mathbb{Z}[X_1, \dots, X_n]$  be a polynomial in the ideal  $\langle e_1, \dots, e_n \rangle$  generated by the elementary symmetric polynomials  $e_i$ . We give an effective algorithm for computing the arithmetic intersection  $h(\widehat{x}_1, \dots, \widehat{x}_n)$  in  $\widehat{CH}(F)$ , as the class of a  $SU(n)$ -invariant differential form on  $F(\mathbb{C})$ . In particular we show that all the arithmetic Chern numbers one obtains are rational numbers. The results are true for partial flag varieties and generalize those of Maillot [Ma] for Grassmannians. An ‘arithmetic Schubert calculus’ is established for an ‘invariant arithmetic Chow ring’ which specializes to the Arakelov Chow ring in the Grassmannian case.

## 1 Introduction

Arakelov theory is a way of ‘completing’ a variety defined over the ring of integers of a number field by adding fibers over the archimedean places. In this way one obtains a theory of intersection numbers using an arithmetic degree map; these numbers are generally real valued. The work of Arakelov on arithmetic surfaces has been generalized to higher dimensions by H. Gillet and C. Soulé. This provides a link between number theory and hermitian complex geometry; the road is via arithmetic intersection theory.

One of the difficulties with the higher dimensional theory is a lack of examples where explicit computations are available. The arithmetic Chow ring

of projective space was studied by Gillet and Soulé ([GS2], §5) and arithmetic intersections on the Grassmannian by Maillot [Ma]. In this article we study arithmetic intersection theory on general flag varieties and solve two problems: (i) finding a method to compute products in the arithmetic Chow ring, and (ii) formulating an ‘arithmetic Schubert calculus’ analogous to the geometric case. The Grassmannian case is easier to work with because the fiber at infinity is a hermitian symmetric space. To the author’s knowledge this work is the first to provide explicit calculations when the harmonic forms are not a subalgebra of the space of smooth forms. The question of computing arithmetic intersection numbers on flag manifolds was raised by C. Soulé in his 1995 Santa Cruz lectures [S].

We now describe our results in greater detail. The crucial case is that of complete flags, so we discuss that for simplicity. Let  $F$  denote the complete flag variety over  $\text{Spec}\mathbb{Z}$ , parametrizing over any field  $k$  the complete flags in a  $k$ -vector space of dimension  $n$ . Let  $\overline{E}$  be the trivial vector bundle of rank  $n$  over  $F$  equipped with a trivial hermitian metric on  $E(\mathbb{C})$ . There is a tautological filtration

$$\mathcal{E} : E_0 = 0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

and the metric on  $E$  induces metrics on all the subbundles  $E_i$ . We thus obtain a *hermitian filtration*  $\overline{\mathcal{E}}$  with quotient line bundles  $L_i = E_i/E_{i-1}$ , which are also given induced metrics. Let  $\widehat{CH}(F)$  be the arithmetic Chow ring of  $F$  (see §2.2 and [GS1], 4.2.3) and  $\widehat{x}_i := -\widehat{c}_1(\overline{L}_i)$ , where  $\widehat{c}_1(\overline{L}_i)$  is the arithmetic first Chern class of  $\overline{L}_i$  ([GS2], 2.5).

Let  $h \in \mathbb{Z}[X_1, \dots, X_n]$  be a polynomial in the ideal  $\langle e_1, \dots, e_n \rangle$  generated by the elementary symmetric polynomials  $e_i(X_1, \dots, X_n)$ . Our main result is a computation of the arithmetic intersection  $h(\widehat{x}_1, \dots, \widehat{x}_n)$  in  $\widehat{CH}(F)$ , as a class corresponding to a  $SU(n)$ -invariant differential form on  $F(\mathbb{C})$ . This enables one to reduce the computation of any intersection product in  $\widehat{CH}(F)$  to the level of smooth differential forms; we show how to do this explicitly for products of classes  $\widehat{c}_i(\overline{E}_l/\overline{E}_k)$ . In particular, we obtain the following result: Let  $k_i$ ,  $1 \leq i \leq n$  be nonnegative integers with  $\sum k_i = \dim F = \binom{n}{2} + 1$ . Then the arithmetic Chern number  $\widehat{\text{deg}}(\widehat{x}_1^{k_1} \widehat{x}_2^{k_2} \cdots \widehat{x}_n^{k_n})$  is a rational number.

Let  $CH(\overline{G}_d)$  be the Arakelov Chow ring ([GS1], 5.1) of the Grassmannian  $G_d$  over  $\text{Spec}\mathbb{Z}$  parametrizing  $d$ -planes in  $E$ , with the natural invariant Kähler metric on  $G_d(\mathbb{C})$ . Maillot [Ma] gave a presentation of  $CH(\overline{G}_d)$  and constructed an ‘arithmetic Schubert calculus’ in  $CH(\overline{G}_d)$ . There are difficulties in extending his results to flag varieties, mainly because the Arakelov

Chow group  $CH(\overline{F})$  is not a *subring* of  $\widehat{CH}(F)$ . To overcome this problem we define, for any partial flag variety  $F$ , an ‘invariant arithmetic Chow ring’  $\widehat{CH}_{inv}(F)$ . This subring of  $\widehat{CH}(F)$  specializes to the Arakelov Chow ring if  $F(\mathbb{C})$  is a hermitian symmetric space.

We extend the notion of Bott-Chern forms for short exact sequences to filtered bundles. These forms give relations in  $\widehat{CH}(F)$  (Theorem 2); however they are generally not closed forms, and thus do not represent cohomology classes. This forces us to work on the level of differential forms in order to calculate arithmetic intersections. We compute the Bott-Chern forms on flag varieties  $F$  by using a calculation of the curvature matrices of homogeneous vector bundles on generalized flag manifolds due to Griffiths and Schmid [GrS]. One thus obtains expressions for the Bott-Chern forms in terms of invariant forms on  $F(\mathbb{C})$ .

The Schubert polynomials of Lascoux and Schützenberger provide a convenient basis to describe the product structure of  $\widehat{CH}_{inv}(F)$ . Using them we formulate an ‘arithmetic Schubert calculus’ for flag varieties which generalizes that of Maillot [Ma] for Grassmannians. However explicit general formulas are lacking, as we cannot do these computations using purely cohomological methods.

We note that it is not surprising that Schubert polynomials are the right combinatorial objects for describing the arithmetic Schubert calculus. For we know, by the work of Fulton [Fu2], that these are the only polynomials that satisfy the general degeneracy loci formula in the relative geometric setting (see *loc. cit.*). The analogy between the arithmetic case (over the integers of a number field) and the relative geometric case (over a function field) has been understood for quite some time; the present work provides a further illustration of this fact.

This paper is organized as follows. In §2 we review some preliminary material on Bott-Chern forms, arithmetic intersection theory, flag varieties and Schubert polynomials. In §3 we state the main tool for computing Bott-Chern forms (for any characteristic class) in the case of induced metrics. The definition and construction of the Bott-Chern forms associated to a hermitian filtration is the content of §4. §5 is concerned with the explicit computation of the curvature matrices of the tautological vector bundles over flag varieties  $F$ . In §6 we define the invariant arithmetic Chow ring  $\widehat{CH}_{inv}(F)$ . This subring of  $\widehat{CH}(F)$  is where all the intersections of interest take place. In §7 we give an algorithm for calculating arithmetic intersection numbers on the complete flag variety  $F$ , in particular proving that they are all rational. In §8 we describe the product structure of  $\widehat{CH}_{inv}(F)$  in more

detail, formulating an arithmetic Schubert calculus. Some applications of our results are given in §9. One has the Faltings height of the image of  $F$  under its pluri-Plücker embedding; this is always a rational number. We give a table of the arithmetic Chern numbers for  $F_{1,2,3}$ . Finally §10 shows how to generalize the previous results to partial flag varieties.

This paper is part of the author's 1997 University of Chicago thesis. I wish to thank my advisor William Fulton for many useful conversations and exchanges of ideas. Thanks are also due to Kai Köhler who alerted me to a numerical error in §9.2.

The geometric aspects of this work generalize readily to other semisimple groups. We plan a sequel discussing arithmetic intersection theory on symplectic and orthogonal flag varieties.

## 2 Preliminaries

### 2.1 Bott-Chern forms

The main references for this section are [BC] and [GS2].

Let  $S_n$  denote the symmetric group and consider the coordinate ring  $\mathbb{C}[T_{ij}]$  ( $1 \leq i, j \leq n$ ) of the space  $M_n(\mathbb{C})$  of  $n \times n$  matrices.  $GL_n(\mathbb{C})$  acts on matrices by conjugation; let  $I(n) = \mathbb{C}[T_{ij}]^{GL_n(\mathbb{C})}$  denote the corresponding graded ring of invariants. There is an isomorphism

$$\sigma : I(n) \rightarrow \mathbb{C}[X_1, \dots, X_n]^{S_n}$$

obtained by evaluating an invariant polynomial  $\phi$  on the diagonal matrix  $\text{diag}(X_1, \dots, X_n)$ . We will often identify  $\phi$  with the symmetric polynomial  $\sigma(\phi)$ . We let  $I(n, \mathbb{Q}) = \sigma^{-1}(\mathbb{Q}[X_1, \dots, X_n]^{S_n})$ . For  $A$  an abelian group,  $A_{\mathbb{Q}}$  denotes  $A \otimes_{\mathbb{Z}} \mathbb{Q}$ .

Let  $X$  be a complex manifold, and denote by  $A^{p,q}(X)$  the space of differential forms of type  $(p, q)$  on  $X$ . Let  $A(X) = \bigoplus_p A^{p,p}(X)$  and  $A'(X) \subset A(X)$  be the set of forms  $\varphi$  in  $A(X)$  which can be written as  $\varphi = \partial\eta + \bar{\partial}\eta'$  for some smooth forms  $\eta, \eta'$ . Define  $\tilde{A}(X) = A(X)/A'(X)$ ; note that if  $\omega$  is a closed form in  $A(X)$  then the cup product  $\wedge\omega : \tilde{A}(X) \rightarrow \tilde{A}(X)$  and the operator  $dd^c : \tilde{A}(X) \rightarrow A(X)$  are well defined.

Let  $E$  be a rank  $n$  holomorphic vector bundle over  $X$ , equipped with a hermitian metric  $h$ . The pair  $\bar{E} = (E, h)$  is called a *hermitian vector bundle*. A direct sum  $\bar{E}_1 \oplus \bar{E}_2$  of hermitian vector bundles will always mean the orthogonal direct sum  $(E_1 \oplus E_2, h_1 \oplus h_2)$ . Let  $D$  be the hermitian holomorphic connection of  $\bar{E}$ , with curvature  $K = D^2 \in A^{1,1}(X, \text{End}(E))$ .

If  $\phi \in I(n)$  is any invariant polynomial, there is an associated differential form  $\phi(\overline{E}) := \phi(\frac{i}{2\pi}K)$ , defined locally by identifying  $\text{End}(E)$  with  $M_n(\mathbb{C})$ . These differential forms are  $d$  and  $d^c$  closed, have de Rham cohomology class independent of the metric  $h$ , and are functorial under pull back by holomorphic maps (see [BC]). In particular one obtains the *power sum forms*  $p_k(\overline{E})$  with  $p_k = \sum_i X_i^k$  and the *Chern forms*  $c_k(\overline{E})$  with  $c_k = e_k$  the  $k$ -th elementary symmetric polynomial.

Let  $\mathcal{E} : 0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$  be an exact sequence of holomorphic vector bundles on  $X$ . Choose arbitrary hermitian metrics  $h_S, h_E, h_Q$  on  $S, E, Q$  respectively. Let

$$\overline{\mathcal{E}} = (\mathcal{E}, h_S, h_E, h_Q) : 0 \rightarrow \overline{S} \rightarrow \overline{E} \rightarrow \overline{Q} \rightarrow 0. \quad (1)$$

We say that  $\overline{\mathcal{E}}$  is *split* when  $(E, h_E) = (S \oplus Q, h_S \oplus h_Q)$  and  $\mathcal{E}$  is the obvious exact sequence.

Let  $\phi \in I(n)$  be any invariant polynomial. Then there is a unique way to attach to every exact sequence  $\overline{\mathcal{E}}$  a form  $\tilde{\phi}(\overline{\mathcal{E}})$  in  $\tilde{A}(X)$ , called the Bott-Chern form of  $\overline{\mathcal{E}}$ , in such a way that:

- (i)  $dd^c \tilde{\phi}(\overline{\mathcal{E}}) = \phi(\overline{S} \oplus \overline{Q}) - \phi(\overline{E})$ ,
- (ii) For every map  $f : Y \rightarrow X$  of complex manifolds,  $\tilde{\phi}(f^*(\overline{\mathcal{E}})) = f^* \tilde{\phi}(\overline{\mathcal{E}})$ ,
- (iii) If  $\overline{\mathcal{E}}$  is split, then  $\tilde{\phi}(\overline{\mathcal{E}}) = 0$ .

For  $\phi, \psi \in I(n)$  one has the following useful relations in  $\tilde{A}(X)$ :

$$\widetilde{\phi + \psi} = \tilde{\phi} + \tilde{\psi}, \quad \widetilde{\phi\psi} = \tilde{\phi} \cdot \psi(\overline{S} \oplus \overline{Q}) + \phi(\overline{E}) \cdot \tilde{\psi}. \quad (2)$$

## 2.2 Arithmetic intersection theory

We recall here the generalization of Arakelov theory to higher dimensions due to H. Gillet and C. Soulé. For more details see [GS1], [GS2], [SABK].

Let  $X$  be an *arithmetic scheme*, by which we mean a regular scheme, projective and flat over  $\text{Spec}\mathbb{Z}$ . For  $p \geq 0$ , we denote the Chow group of codimension  $p$  cycles on  $X$  modulo rational equivalence by  $CH^p(X)$  and let  $\widehat{CH}(X) = \bigoplus_p CH^p(X)$ .  $\widehat{CH}^p(X)$  will denote the  $p$ -th arithmetic Chow group of  $X$ . Recall that an element of  $\widehat{CH}^p(X)$  is represented by an arithmetic cycle  $(Z, g_Z)$ ; here  $g_Z$  is a Green current for the codimension  $p$  cycle  $Z(\mathbb{C})$ . The functor  $\widehat{CH}^p(X)$  is contravariant in  $X$ ; let  $\widehat{CH}(X) = \bigoplus_p \widehat{CH}^p(X)$ .

The involution of  $X(\mathbb{C})$  induced by complex conjugation is denoted by  $F_\infty$ . Let  $A^{p,p}(X_\mathbb{R})$  be the subspace of  $A^{p,p}(X(\mathbb{C}))$  generated by real forms

$\eta$  such that  $F_\infty^* \eta = (-1)^p \eta$ ; denote by  $\tilde{A}^{p,p}(X_\mathbb{R})$  the image of  $A^{p,p}(X_\mathbb{R})$  in  $\tilde{A}^{p,p}(X(\mathbb{C}))$ . Let  $A(X_\mathbb{R}) = \bigoplus_p A^{p,p}(X_\mathbb{R})$  and  $\tilde{A}(X_\mathbb{R}) = \bigoplus_p \tilde{A}^{p,p}(X_\mathbb{R})$ . We have the following canonical morphisms of abelian groups:

$$\begin{aligned} \zeta : \widehat{CH}^p(X) &\longrightarrow CH^p(X), & [(Z, g_Z)] &\longmapsto [Z], \\ \omega : \widehat{CH}^p(X) &\longrightarrow A^{p,p}(X_\mathbb{R}), & [(Z, g_Z)] &\longmapsto dd^c g_Z + \delta_{Z(\mathbb{C})}, \\ a : \tilde{A}^{p-1,p-1}(X_\mathbb{R}) &\longrightarrow \widehat{CH}^p(X), & \eta &\longmapsto [(0, \eta)]. \end{aligned}$$

For convenience of notation, when we refer to a real differential form  $\eta \in A(X_\mathbb{R})$  as an element of  $\widehat{CH}(X)$ , we shall always mean  $a([\eta])$ , where  $[\eta]$  is the class of  $\eta$  in  $\tilde{A}(X_\mathbb{R})$ . There is an exact sequence

$$CH^{p,p-1}(X) \longrightarrow \tilde{A}^{p-1,p-1}(X_\mathbb{R}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \longrightarrow 0 \quad (3)$$

Here the group  $CH^{p,p-1}(X)$  is the  $E_2^{p,1-p}$  term of the spectral sequence of [Q] §7, used by Quillen to calculate the higher algebraic  $K$ -theory of  $X$  (see also [G]).

One can define a pairing  $\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \rightarrow \widehat{CH}^{p+q}(X)_\mathbb{Q}$  which turns  $\widehat{CH}(X)_\mathbb{Q}$  into a commutative graded unitary  $\mathbb{Q}$ -algebra. The maps  $\zeta, \omega$  are  $\mathbb{Q}$ -algebra homomorphisms. If  $X$  is smooth over  $\mathbb{Z}$  one does not have to tensor with  $\mathbb{Q}$ . We also note the useful identity  $a(x)y = a(x\omega(y))$  for  $x, y \in \widehat{CH}(X)$ .

Choose a Kähler form  $\omega_0$  on  $X(\mathbb{C})$  such that  $F_\infty^* \omega_0 = -\omega_0$  and let  $\mathcal{H}^{p,p}(X_\mathbb{R})$  be the space of harmonic (with respect to  $\omega_0$ )  $(p, p)$  forms on  $X(\mathbb{C})$  invariant under  $F_\infty$ . The  $p$ -th *Arakelov Chow group* of  $\overline{X} = (X, \omega_0)$  is defined by  $CH^p(\overline{X}) = \omega^{-1}(\mathcal{H}^{p,p}(X_\mathbb{R}))$ . The group  $CH(\overline{X})_\mathbb{Q} = \bigoplus_p CH^p(\overline{X})_\mathbb{Q}$  is generally not a subring of  $\widehat{CH}(X)_\mathbb{Q}$ , unless the harmonic forms  $\mathcal{H}^*(X_\mathbb{R})$  are a subring of  $A(X_\mathbb{R})$ . This is true if  $(X(\mathbb{C}), \omega_0)$  is a hermitian symmetric space, such as a complex Grassmannian, but fails for more general flag varieties.

Let  $f : X \rightarrow \text{Spec} \mathbb{Z}$  be the projection. If  $X$  has relative dimension  $d$  over  $\mathbb{Z}$ , then we have an arithmetic degree map  $\widehat{\text{deg}} : \widehat{CH}^{d+1}(X) \rightarrow \mathbb{R}$ , obtained by composing the push-forward  $f_* : \widehat{CH}^{d+1}(X) \rightarrow \widehat{CH}^1(\text{Spec} \mathbb{Z})$  with the isomorphism  $\widehat{CH}^1(\text{Spec} \mathbb{Z}) \xrightarrow{\sim} \mathbb{R}$ . The latter maps the class of  $(0, 2\lambda)$  to the real number  $\lambda$ .

A *hermitian vector bundle*  $\overline{E} = (E, h)$  on an arithmetic scheme  $X$  is an algebraic vector bundle  $E$  on  $X$  with a hermitian metric  $h$  on the induced

holomorphic vector bundle  $E(\mathbb{C})$  on  $X(\mathbb{C})$  satisfying  $F_\infty^*(h) = h$ . There are characteristic classes  $\widehat{\phi}(\overline{E}) \in \widehat{CH}(X)_\mathbb{Q}$  for any  $\phi \in I(n, \mathbb{Q})$ , where  $n = \text{rk}E$ . For example, we have *arithmetic Chern classes*  $\widehat{c}_k(\overline{E}) \in \widehat{CH}^k(X)$ . For the basic properties of these classes, see [GS2], Theorem 4.1.

### 2.3 Flag varieties and Schubert polynomials

Let  $k$  be a field,  $E$  an  $n$ -dimensional vector space over  $k$ . Let

$$\mathfrak{r} = (0 < r_1 < r_2 < \dots < r_m = n)$$

be an increasing  $m$ -tuple of natural numbers. A *flag of type  $\mathfrak{r}$*  is a flag

$$\mathcal{E} : E_0 = 0 \subset E_1 \subset E_2 \subset \dots \subset E_m = E \quad (4)$$

with  $\text{rank}E_i = r_i$ ,  $1 \leq i \leq m$ . Let  $F(\mathfrak{r})$  denote the arithmetic scheme parametrizing flags  $\mathcal{E}$  of type  $\mathfrak{r}$  over any field  $k$ . (4) will also denote the tautological flag of vector bundles over  $F(\mathfrak{r})$ , and we call the resulting filtration of the bundle  $E$  a *filtration of type  $\mathfrak{r}$* .

The above *arithmetic flag variety* is smooth over  $\text{Spec}\mathbb{Z}$  (this follows, for instance, from the criterion in [H], III.10.2). There is an isomorphism  $F(\mathfrak{r})(\mathbb{C}) \simeq SL(n, \mathbb{C})/P$ , where  $P$  is the parabolic subgroup of  $SL(n, \mathbb{C})$  stabilizing a fixed flag. In the extreme case  $m = 2$  (resp.  $m = n$ )  $F(\mathfrak{r})$  is the Grassmannian  $G_d$  parametrizing  $d$ -planes in  $E$  (resp. the complete flag variety  $F$ ). Although the results of this paper are true for any partial flag variety  $F(\mathfrak{r})$ , for simplicity we will work with the complete flag variety  $F$ , leaving the discussion of the general case to §10. The notation for these varieties and the dimension  $n$  will be fixed throughout this paper.

We now recall the presentation of the Chow ring  $CH(F)$  from [B1] and [BGG]. Define the quotient line bundles  $L_i = E_i/E_{i-1}$ . Consider the polynomial ring  $P_n = \mathbb{Z}[X_1, \dots, X_n]$  and the ideal  $I_n$  generated by the elementary symmetric functions  $e_i(X_1, \dots, X_n)$ . Then  $CH(F) \simeq P_n/I_n$ , where the inverse of this isomorphism sends  $[X_i]$  to  $-c_1(L_i)$ . The ring  $H_n = P_n/I_n$  has a free  $\mathbb{Z}$ -basis consisting of classes of monomials  $X_1^{k_1} X_2^{k_2} \dots X_n^{k_n}$ , where the exponents  $k_i$  satisfy  $k_i \leq n - i$ .

The Schubert polynomials of Lascoux and Schützenberger [LS] are a natural  $\mathbb{Z}$ -basis of  $H_n$ , corresponding to the classes of Schubert varieties in  $CH(F)$ . Our main reference for Schubert polynomials will be Macdonald's notes [M1]; most of what we need is also contained in the survey [M2].

Let  $S_\infty = \cup_n S_n$  and  $P_\infty = \mathbb{Z}[X_1, X_2, \dots]$ ; there is a natural action of  $S_\infty$  on  $P_\infty$  by permuting the variables. For each  $w \in S_\infty$ ,  $\ell(w)$  denotes the *length* of  $w$  and  $\partial_w : P_\infty \rightarrow P_\infty$  the corresponding *divided difference operator*, defined as follows: if  $w = s_i = (i, i+1)$  is the transposition that interchanges  $i$  and  $i+1$ , then

$$\partial_{s_i} f = \partial_i f = \frac{f - s_i f}{X_i - X_{i+1}}.$$

In general if  $w = s_{i_1} s_{i_2} \cdots s_{i_k}$  with  $k = \ell(w)$  then

$$\partial_w := \partial_{i_1} \circ \partial_{i_2} \circ \cdots \circ \partial_{i_k};$$

the resulting operator is independent of the chosen reduced word for  $w$ . For any  $w \in S_n$ , the *Schubert polynomial*  $\mathfrak{S}_w$  is defined by

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}(X_1^{n-1} X_2^{n-2} \cdots X_{n-1}),$$

where  $w_0 = (n, n-1, \dots, 1)$  is the longest permutation in  $S_n$ . This definition is compatible with the natural inclusion  $S_n \subset S_{n+1}$  (with  $w(n+1) = n+1$ ). It follows that  $\mathfrak{S}_w$  is well defined for every  $w \in S_\infty$ , and is a homogeneous polynomial of degree  $\ell(w)$ .

We let  $\Lambda_n = P_n^{S_n}$  be the ring of symmetric polynomials in  $n$  variables. The set  $\{\mathfrak{S}_w \mid w \in S_n\}$  is both a free  $\Lambda_n$ -basis of  $P_n$  and a free  $\mathbb{Z}$ -basis of  $H_n$ . Define

$$S^{(n)} = \{w \in S_\infty \mid w(n+1) < w(n+2) < \cdots\}.$$

Then  $\{\mathfrak{S}_w \mid w \in S^{(n)}\}$  is a free  $\mathbb{Z}$ -basis of  $P_n$  ([M1], (4.13)).

Define a  $\Lambda_n$ -valued scalar product  $\langle \cdot, \cdot \rangle$  on  $P_n$  by  $\langle f, g \rangle = \partial_{w_0}(fg)$ , for  $f, g \in P_n$ . Let  $\{\mathfrak{S}^w\}_{w \in S_n}$  be the  $\Lambda_n$ -basis of  $P_n$  dual to the basis  $\{\mathfrak{S}_w\}$  relative to this product. Then we have ([M1], (5.12))

$$\mathfrak{S}^w(X) = w_0 \mathfrak{S}_{w w_0}(-X). \quad (5)$$

For any  $h \in I_n$  we have a decomposition  $h = \sum_{w \in S_n} \langle h, \mathfrak{S}^w \rangle \mathfrak{S}_w$ , where each  $\langle h, \mathfrak{S}^w \rangle$  is in  $\Lambda_n \cap I_n$ .



### 3 Calculating Bott-Chern forms

Consider the short exact sequence  $\bar{\mathcal{E}}$  in (1) and assume that the metrics on  $\bar{S}$  and  $\bar{Q}$  are induced from the metric on  $E$ . Let  $r, n$  be the ranks of the bundles  $S$  and  $E$ .

For  $\phi \in I(n)$  homogeneous of degree  $k$  we let  $\phi'$  be a  $k$ -multilinear invariant form on  $M_n(\mathbb{C})$  such that  $\phi(A) = \phi'(A, A, \dots, A)$ . Such forms are most easily constructed for the power sums  $p_k$ , by defining

$$p'_k(A_1, A_2, \dots, A_k) = \text{Tr}(A_1 A_2 \cdots A_k).$$

If  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$  is a partition of  $k$ , define  $p_\lambda := \prod_{i=1}^m p_{\lambda_i}$ . For  $p_\lambda$  we can take  $p'_\lambda = \prod p'_{\lambda_i}$ . Since the  $p_\lambda$ 's are an additive  $\mathbb{Q}$ -basis for the ring of symmetric polynomials, we can use the above constructions to find multilinear forms  $\phi'$  for any  $\phi \in I(n)$ . For any two matrices  $A, B \in M_n(\mathbb{C})$  let

$$\phi'(A; B) := \sum_{i=1}^k \phi'(A, \dots, A, B_{(i)}, A, \dots, A),$$

where the index  $i$  means that  $B$  is in the  $i$ -th position.

Consider a local orthonormal frame  $s$  for  $E$  such that the first  $r$  elements generate  $S$ , and let  $K(\bar{S})$ ,  $K(\bar{E})$  and  $K(\bar{Q})$  be the curvature matrices of  $\bar{S}$ ,  $\bar{Q}$  and  $\bar{E}$  with respect to  $s$ . Let  $K_S = \frac{i}{2\pi} K(\bar{S})$ ,  $K_E = \frac{i}{2\pi} K(\bar{E})$  and  $K_Q = \frac{i}{2\pi} K(\bar{Q})$ . Write

$$K_E = \left( \begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array} \right)$$

where  $K_{11}$  is an  $r \times r$  submatrix, and consider the matrices

$$K_0 = \left( \begin{array}{c|c} K_S & 0 \\ \hline K_{21} & K_Q \end{array} \right) \quad \text{and} \quad J_r = \left( \begin{array}{c|c} Id_r & 0 \\ \hline 0 & 0 \end{array} \right).$$

Let  $u$  be a variable and define  $K(u) = uK_E + (1-u)K_0$ . We can then state the main computational

**Proposition 1** *For  $\phi \in I(n)$ , we have*

$$\tilde{\phi}(\bar{\mathcal{E}}) = \int_0^1 \frac{\phi'(K(u); J_r) - \phi'(K(0); J_r)}{u} du.$$

Proposition 1 is essentially a consequence of the work of Bott and Chern [BC], although we have not been able to find this general statement in the literature. For history and a complete proof, see [T].

What will prove most useful to us in the sequel is

**Corollary 1** *For any  $\phi \in I(n, \mathbb{Q})$  the Bott-Chern form  $\tilde{\phi}(\bar{\mathcal{E}})$  is a polynomial in the entries of the matrices  $K_E$ ,  $K_S$  and  $K_Q$  with rational coefficients.*

**Proof.** By the equations (2) it suffices to prove this for  $\phi = p_k$  a power sum. Using the bilinear form  $p'_k$  described previously in Proposition 1 gives

$$\tilde{p}_k(\bar{\mathcal{E}}) = k \int_0^1 \frac{1}{u} \text{Tr}((K(u)^{k-1} - K(0)^{k-1})J_r) du,$$

so the result is clear. □

Define the *harmonic numbers*  $\mathcal{H}_k = \sum_{i=1}^k \frac{1}{i}$ ,  $\mathcal{H}_0 = 0$ . We will need the following useful calculations, which one can deduce from the definition and from Proposition 1:

- (a)  $\tilde{c}_1^k(\bar{\mathcal{E}}) = 0$  for all  $k$  and  $\tilde{c}_p(\bar{\mathcal{E}}) = 0$  for all  $p > \text{rk} E$ .
- (b)  $\tilde{c}_2(\bar{\mathcal{E}}) = c_1(\bar{S}) - \text{Tr} K_{11}$  (see [D], 10.1 and [T]).
- (c) If  $E$  is flat, then  $\tilde{c}_k(\bar{\mathcal{E}}) = \mathcal{H}_{k-1} \sum_{i=0}^{k-1} i c_i(\bar{S}) c_{k-1-i}(\bar{Q})$  ([Ma], Th. 3.4.1).

## 4 Bott-Chern forms for filtered bundles

In this section we will extend the definition of Bott-Chern forms for an exact sequence of bundles to the case of a filtered bundle.

Let  $X$  and  $E$  be as in §2.1, and assume that  $E$  has a filtration of type  $\mathfrak{r}$

$$\mathcal{E} : E_0 = 0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E \tag{6}$$

by complex subbundles  $E_i$ , with  $\mathfrak{r}$  as in §2.3. Let  $Q_i = E_i/E_{i-1}$ ,  $1 \leq i \leq m$  be the quotient bundles. A *hermitian filtration*  $\bar{\mathcal{E}}$  of type  $\mathfrak{r}$  is a filtration (6) together with a choice of hermitian metrics on  $E$  and on each quotient bundle  $Q_i$ . Note that we do not assume that metrics have been chosen on the subbundles  $E_1, \dots, E_m$  or that the metrics on the quotients are induced

from  $E$  in any way. We say that  $\bar{\mathcal{E}}$  is *split* if, when  $E_i$  is given the induced metric from  $E$  for each  $i$ , the sequence

$$\bar{\mathcal{E}}_i : 0 \rightarrow \bar{E}_{i-1} \rightarrow \bar{E}_i \rightarrow \bar{Q}_i \rightarrow 0$$

is split. In this case of course  $\bar{E} = \bigoplus_i \bar{Q}_i$ .

**Theorem 1** *Let  $\phi \in I(n)$  be an invariant polynomial. There is a unique way to attach to every hermitian filtration of type  $\mathfrak{r}$  a form  $\tilde{\phi}(\bar{\mathcal{E}})$  in  $\tilde{A}(X)$  in such a way that:*

(i)  $dd^c \tilde{\phi}(\bar{\mathcal{E}}) = \phi\left(\bigoplus_{i=1}^m \bar{Q}_i\right) - \phi(\bar{E}),$

(ii) *For every map  $f : Y \rightarrow X$  of complex manifolds,  $\tilde{\phi}(f^*(\bar{\mathcal{E}})) = f^* \tilde{\phi}(\bar{\mathcal{E}}),$*

(iii) *If  $\bar{\mathcal{E}}$  is split, then  $\tilde{\phi}(\bar{\mathcal{E}}) = 0.$*

*If  $m = 2$ , i.e. the filtration  $\mathcal{E}$  has length 2, then  $\tilde{\phi}(\bar{\mathcal{E}})$  coincides with the Bott-Chern class  $\tilde{\phi}(0 \rightarrow \bar{Q}_1 \rightarrow \bar{E} \rightarrow \bar{Q}_2 \rightarrow 0)$  defined in §2.1.*

**Proof.** The essential ideas are contained in [GS2], Th. 1.2.2 and sections 7.1.1, 7.1.2., so we will only sketch the argument.

We first show that such forms exist. Given any hermitian filtration  $\mathcal{E}$ , equip each subbundle  $E_i$  with the induced metric from  $\bar{E}$  and consider the exact sequence

$$\bar{\mathcal{E}}_i : 0 \rightarrow \bar{E}_{i-1} \rightarrow \bar{E}_i \rightarrow \bar{Q}_i \rightarrow 0.$$

If  $\tilde{\phi}(\bar{\mathcal{E}})$  and  $\tilde{\psi}(\bar{\mathcal{E}})$  have already been defined then the equations

$$\widetilde{\phi + \psi}(\bar{\mathcal{E}}) = \tilde{\phi}(\bar{\mathcal{E}}) + \tilde{\psi}(\bar{\mathcal{E}})$$

$$\widetilde{\phi\psi}(\bar{\mathcal{E}}) = \tilde{\phi}(\bar{\mathcal{E}})\psi\left(\bigoplus_{i=1}^m \bar{Q}_i\right) + \phi(\bar{E})\tilde{\psi}(\bar{\mathcal{E}})$$

can be used to define  $\widetilde{\phi + \psi}$  and  $\widetilde{\phi\psi}$  (see [GS2], Prop. 1.3.1 for the case  $m = 2$ ). Therefore it suffices to construct the Bott-Chern classes  $\tilde{p}_k$  for the power sums  $p_k$ . For this we simply let

$$\tilde{p}_k(\bar{\mathcal{E}}) = \sum_{i=1}^m \tilde{p}_k(\bar{\mathcal{E}}_i). \quad (7)$$

Since the  $\tilde{p}_k(\bar{\mathcal{E}}_i)$  are functorial and additive on orthogonal direct sums, it is clear that (7) satisfies (i)-(iii). The construction for  $m = 2$  gives the classes of §2.1.

We will use a separate construction of the total Chern forms  $\tilde{c}(\overline{\mathcal{E}})$ : For each  $i$  with  $1 \leq i \leq m-1$ , let  $\overline{\mathcal{Q}}_i$  be the sequence

$$0 \rightarrow 0 \rightarrow \bigoplus_{j=i+1}^m \overline{\mathcal{Q}}_j \rightarrow \bigoplus_{j=i+1}^m \overline{\mathcal{Q}}_j \rightarrow 0,$$

and let  $\overline{\mathcal{E}}_i^+ = \overline{\mathcal{E}}_i \oplus \overline{\mathcal{Q}}_i$ . Let  $\overline{\mathcal{E}}_m^+ = \overline{\mathcal{E}}_m$ . To each exact sequence  $\overline{\mathcal{E}}_i^+$  we associate a Bott-Chern form  $\tilde{c}(\overline{\mathcal{E}}_i^+)$ . It follows from [GS2], Prop. 1.3.2 that

$$\tilde{c}(\overline{\mathcal{E}}_i^+) = \tilde{c}(\overline{\mathcal{E}}_i \oplus \overline{\mathcal{Q}}_i) = \tilde{c}(\overline{\mathcal{E}}_i) c\left(\bigoplus_{j=i+1}^m \overline{\mathcal{Q}}_j\right) = \tilde{c}(\overline{\mathcal{E}}_i) \wedge \bigwedge_{j=i+1}^m c(\overline{\mathcal{Q}}_j).$$

It is easy to see that  $\tilde{c}(\overline{\mathcal{E}}) := \sum_{i=1}^m \tilde{c}(\overline{\mathcal{E}}_i^+)$  satisfies (i)-(iii).

To prove that the form  $\tilde{\phi}(\overline{\mathcal{E}})$  is unique, one constructs a deformation of the filtration  $\overline{\mathcal{E}}$  to the split filtration, as in [GS2], §7.1.2. Let  $\overline{\mathcal{O}}(1)$  be the canonical line bundle on  $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$  with its natural Fubini-Study metric and let  $\sigma$  be a section of  $\overline{\mathcal{O}}(1)$  vanishing only at  $\infty$ . Let  $p_1, p_2$  be the projections from  $X \times \mathbb{P}^1$  to  $X, \mathbb{P}^1$  respectively. We denote by  $E, E_i, Q_i$  and  $\mathcal{O}(1)$  the bundles  $p_1^*E, p_1^*E_i, p_1^*Q_i$ , and  $p_2^*\mathcal{O}(1)$  on  $X \times \mathbb{P}^1$ . For a bundle  $F$  on  $X \times \mathbb{P}^1$  we let  $F(k) := F \otimes \mathcal{O}(1)^k$ .

For each  $i \leq m-1$ , we map  $E_i(m-1-i)$  to  $E_{i+1}(m-1-i)$  by the inclusion of  $E_i \hookrightarrow E_{i+1}$  and to  $E_i(m-i)$  by  $\text{id}_{E_i} \otimes \sigma$ . For  $1 \leq j \leq m$  let

$$\tilde{E}_k := \left( \bigoplus_{i=1}^k E_i(m-i) \right) / \left( \bigoplus_{i=1}^{k-1} E_i(m-1-i) \right).$$

Setting  $\tilde{E} := \tilde{E}_m$  we get a filtration of type  $\mathfrak{r}$  over  $X \times \mathbb{P}^1$ :

$$\tilde{\mathcal{E}} : 0 \subset \tilde{E}_1 \subset \tilde{E}_2 \subset \cdots \subset \tilde{E}_m = \tilde{E}.$$

The quotients of this filtration are  $\tilde{Q}_i = \tilde{E}_i / \tilde{E}_{i-1} = Q_i(m-i)$ ,  $1 \leq i \leq m$ .

For  $z \in \mathbb{P}^1$ , denote by  $i_z : X \rightarrow X \times \mathbb{P}^1$  the map given by  $i_z(x) = (x, z)$ . When  $z \neq \infty$ ,  $i_z^* \tilde{E} \simeq E$ , while  $i_\infty^* \tilde{E} \simeq \bigoplus_{i=1}^m Q_i$ . Using a partition of unity we can choose hermitian metrics  $\tilde{h}_i$  on  $\tilde{Q}_i$  and  $\tilde{h}$  on  $\tilde{E}$  such that the isomorphisms  $i_z^* \tilde{Q}_i \simeq Q_i$ ,  $i_0^* \tilde{E} \simeq E$  and  $i_\infty^* \tilde{E} \simeq \bigoplus_{i=1}^m Q_i$  all become isometries.

Then one shows (as in [BiGS], Th. 1.2.9 and [GS2], 1.2.3) that  $\tilde{\phi}(\bar{\mathcal{E}})$  is uniquely determined in  $\tilde{A}(X)$  by the formula

$$\tilde{\phi}(\bar{\mathcal{E}}) = - \int_{\mathbb{P}^1} \phi(\tilde{E}, \tilde{h}) \log |z|^2.$$

□

**Remark.** Gillet and Soulé used  $\widetilde{ch}(\bar{\mathcal{E}})$  to give an explicit description of the Beilinson regulator map on  $K_1(X)$ , where  $X$  is an arithmetic scheme ([GS2], 7.1).

It is easy to prove that analogues of the properties of Bott-Chern forms for short exact sequences ([GS2], §1.3) are true for the above generalization to filtered bundles. In particular the formulas (2) take the form:

$$\widetilde{\phi + \psi} = \tilde{\phi} + \tilde{\psi}, \quad \widetilde{\phi\psi} = \tilde{\phi} \cdot \psi\left(\bigoplus_{i=1}^m \bar{Q}_i\right) + \phi(\bar{E}) \cdot \tilde{\psi}. \quad (8)$$

for any  $\phi, \psi \in I(n)$ . Using Theorem 1 and the same argument as in the proof of Theorem 4.8(ii) in [GS2], we obtain

**Theorem 2** *Let*

$$\bar{\mathcal{E}} : 0 \subset \bar{E}_1 \subset \bar{E}_2 \subset \cdots \subset \bar{E}_m = \bar{E}$$

*be a hermitian filtration on an arithmetic scheme  $X$ , with quotient bundles  $Q_i$ , and let  $\phi \in I(n, \mathbb{Q})$ . Then*

$$\widehat{\phi}\left(\bigoplus_{i=1}^m \bar{Q}_i\right) - \widehat{\phi}(\bar{E}) = a(\tilde{\phi}(\bar{\mathcal{E}})).$$

Assume that the subbundles  $E_i$  are given metrics induced from  $E$  and the quotient bundles  $Q_i$  are given the metrics induced from the exact sequences  $\bar{\mathcal{E}}_i$ . Define matrices  $K_{E_i} = \frac{i}{2\pi} K(\bar{E}_i)$  and  $K_{Q_i} = \frac{i}{2\pi} K(\bar{Q}_i)$  as in §3. Then the constructions in Theorem 1 and Corollary 1 immediately imply

**Corollary 2** *For any  $\phi \in I(n, \mathbb{Q})$  the Bott-Chern form  $\tilde{\phi}(\bar{\mathcal{E}})$  is a polynomial in the entries of the matrices  $K_{E_i}$  and  $K_{Q_i}$ ,  $1 \leq i \leq m$ , with rational coefficients.*

## 5 Curvature of homogeneous vector bundles

Let  $G = SL(n, \mathbb{C})$ ,  $K = SU(n)$  and  $P$  be a parabolic subgroup of  $G$ , with Lie algebras  $\mathfrak{g}$ ,  $\mathfrak{k}$  and  $\mathfrak{p}$  respectively. Complex conjugation of  $\mathfrak{g}$  with respect to  $\mathfrak{k}$  is given by the map  $\tau$  with  $\tau(A) = -\bar{A}^t$ . We let  $\mathfrak{v} = \mathfrak{p} \cap \tau(\mathfrak{p})$  and  $\mathfrak{n}$  be the unique maximal nilpotent ideal of  $\mathfrak{p}$ , so that  $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{n} \oplus \tau(\mathfrak{n})$ .

Let  $\mathfrak{h} = \{\text{diag}(z_1, \dots, z_n) \mid \sum z_i = 0\}$  be the Cartan subalgebra of diagonal matrices in  $\mathfrak{g}$ . The set of roots  $\Delta = \{z_i - z_j \mid 1 \leq i \neq j \leq n\}$  is a subset of  $\mathfrak{h}^*$ . We denote the root  $z_i - z_j$  by the pair  $ij$ , and fix a system of positive roots  $\Delta_+ = \{ij \mid i > j\}$ . The adjoint representation of  $\mathfrak{h}$  on  $\mathfrak{g}$  determines a decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ . Here the root space  $\mathfrak{g}^\alpha = \mathbb{C}e_\alpha$ ,

where  $e_\alpha = e_{ij} = E_{ij}$  is the matrix with 1 at the  $ij$ -th entry and zeroes elsewhere. Set  $\bar{e}_{ij} = \tau(e_{ij}) = -E_{ji}$ .

Let  $V = K \cap P$  and consider the complex manifold  $X = G/P = K/V$ . Let  $p : K \rightarrow X$  be the quotient map, and let  $\Psi \subset \Delta_+$  be such that  $\mathfrak{n} = \sum_{\alpha \in \Psi} \mathfrak{g}^{-\alpha}$ . For  $\alpha, \beta \in \Psi$ , the equations

$$\begin{aligned} \omega^\alpha(e_\beta) &= \delta_{\alpha\beta}, & \omega^\alpha(\bar{e}_\beta) &= 0, & \omega^\alpha(\mathfrak{v}) &= 0 \\ \bar{\omega}^\alpha(e_\beta) &= 0, & \bar{\omega}^\alpha(\bar{e}_\beta) &= \delta_{\alpha\beta}, & \bar{\omega}^\alpha(\mathfrak{v}) &= 0 \end{aligned}$$

define elements of the dual space  $\mathfrak{g}^*$ , which we shall regard as left invariant complex one-forms on  $K$ . A given differential form  $\eta$  on  $X$  pulls back to

$$p^*\eta = \sum_{a,b} f_{ab} \omega^{\alpha_1} \wedge \dots \wedge \omega^{\alpha_r} \wedge \bar{\omega}^{\beta_1} \wedge \dots \wedge \bar{\omega}^{\beta_s} \quad (9)$$

on  $K$ , with coefficients  $f_{ab} \in C^\infty(K)$ . Conversely, every  $V$ -invariant element of  $C^\infty(K) \otimes \wedge \tau(\mathfrak{n})^* \otimes \wedge \mathfrak{n}^*$  is the pullback to  $K$  of a differential form on  $X$ . A form  $\eta$  on  $X$  is of  $(p, q)$  type precisely when every summand on the right hand side of (9) involves  $p$  unbarred and  $q$  barred terms.

**Definition 1**  $\text{Inv}_{\mathbb{R}}(X)$  (respectively  $\text{Inv}_{\mathbb{Q}}(X)$ ) denotes the ring of  $K$ -invariant forms in the  $\mathbb{R}$ -subalgebra (respectively  $\mathbb{Q}$ -subalgebra) of  $A(X)$  generated by  $\{\frac{i}{2\pi} \omega^\alpha \wedge \bar{\omega}^\beta \mid \alpha, \beta \in \Psi\}$ .

Suppose now that  $\pi : V \rightarrow GL(E_0)$  is an irreducible unitary representation of  $V$  on a complex vector space  $E_0$ .  $\pi$  defines a homogeneous vector bundle  $\bar{E} = K \times_V E_0 \rightarrow X$  which has a  $K$ -invariant hermitian metric.

Extend  $\pi$  to a unique holomorphic representation  $\pi : P \rightarrow GL(E_0)$ , and denote the induced representation of  $\mathfrak{p}$  by the same letter. Then  $\overline{E} = G \times_P E_0$  is a holomorphic hermitian vector bundle over  $X$  which gives a complex structure to  $K \times_V E_0$ .

In [GrS], equation (4.4)<sub>X</sub>, Griffiths and Schmid calculate the  $K$ -invariant curvature matrix  $K(\overline{E})$  explicitly in terms of the above data. Their result is

$$K(\overline{E}) = \sum_{\alpha, \beta \in \Psi} \pi([e_\alpha, e_{-\beta}]_{\mathfrak{v}}) \otimes \omega^\alpha \wedge \overline{\omega}^\beta. \quad (10)$$

The invariant differential forms giving the Chern classes of homogeneous line bundles were given by Borel [B2]; see the introduction to [GrS] for more references.

Let  $Y = F(\mathbb{C}) \simeq SL(n, \mathbb{C})/B = SU(n)/S(U(1)^n)$  be the complex flag variety and let  $\overline{E}$  denote the trivial hermitian vector bundle over  $Y$ , with the tautological hermitian filtration

$$\overline{\mathcal{E}} : 0 \subset \overline{E}_1 \subset \overline{E}_2 \subset \cdots \subset \overline{E}_n = \overline{E}$$

with quotient line bundles  $\overline{L}_i$  and all metrics induced from the metric on  $\overline{E}$ . All of these bundles are homogeneous, and we want to use equation (10) to compute their curvature matrices. Note that (10) applies directly only to the line bundles  $\overline{L}_i$ , as the higher rank bundles are not given by irreducible representations of the torus  $S(U(1)^n)$ . We can avoid this problem by considering the Grassmannian  $Y_k = G_k(\mathbb{C}) = SU(n)/S(U(k) \times U(n-k))$  and the natural projection  $\rho : Y \rightarrow Y_k$ . Now (10) can be applied to the universal bundle  $\overline{E}_k$  over  $Y_k$  and the curvature matrix  $K(\overline{E}_k)$  pulls back via  $\rho$  to the required matrix over  $Y$ . In fact by projecting to a partial flag variety one can compute the curvature matrix of any quotient bundle  $E_l/E_k$ . The representations  $\pi$  of  $V$  inducing these bundles are the obvious ones in each case. What remains is a straightforward application of equation (10), so we will describe the answer without belaboring the details.

We have defined differential forms  $\omega^{ij}, \overline{\omega}^{ij}$  on  $K = SU(n)$  which we identify with corresponding forms on  $Y$ . With this notation, we can state (compare [GrS], (4.13)<sub>X</sub>) :

**Proposition 2** *Let  $k < l$  and consider the vector bundle  $Q_{lk} = E_l/E_k$  over  $F(\mathbb{C})$ . Let the curvature matrix of  $\overline{Q}_{lk}$  with its induced metric be  $\Theta = \{\Theta_{\alpha\beta}\}_{k+1 \leq \alpha, \beta \leq l}$ . Then*

$$\Theta_{\alpha\beta} = \sum_{i \leq k} \omega^{\alpha i} \wedge \overline{\omega}^{\beta i} - \sum_{j > l} \omega^{j\alpha} \wedge \overline{\omega}^{j\beta}.$$

For notational convenience we let  $\omega_{ij} := \gamma\omega^{ji}$ ,  $\bar{\omega}_{ij} := \gamma\bar{\omega}^{ji}$  and  $\Omega_{ij} := \omega_{ij} \wedge \bar{\omega}_{ij}$ , ( $i < j$ ), where  $\gamma$  is a constant such that  $\gamma^2 = \frac{i}{2\pi}$ . Then we have

**Corollary 3**

$$c_1(\bar{L}_k) = \sum_{i < k} \Omega_{ik} - \sum_{j > k} \Omega_{kj}$$

$$K_{\bar{E}_k} = \frac{i}{2\pi} K(\bar{E}_k) = - \left\{ \sum_{j > k} \omega_{\alpha j} \wedge \bar{\omega}_{\beta j} \right\}_{1 \leq \alpha, \beta \leq k}$$

Let  $\Omega := \bigwedge_{i < j} \Omega_{ij}$ . To compute classical intersection numbers on the flag

variety using the differential forms in Corollary 3 it suffices to know  $\int_Y \Omega$ . If  $x_i = -c_1(\bar{L}_i)$ , it is well known that  $\eta = \mathfrak{S}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$  is dual to the class of a point in  $Y$ ; thus  $\int_Y \eta = 1$ . An easy calculation shows

$$\text{that } \eta = \prod_{k=1}^{n-1} k! \cdot \Omega, \text{ thus } \int_Y \Omega = \prod_{k=1}^{n-1} \frac{1}{k!}.$$

## 6 Invariant arithmetic Chow rings

It is well known that the arithmetic variety  $F$  has a cellular decomposition in the sense of [Fu1], Ex. 1.9.1. It follows that one can use the excision exact sequence for the groups  $CH^{*,*}(F)$  (see [G], §8) to show that  $CH^{p,p-1}(F) = 0$  (compare [Ma], Lemma 4.0.6). Therefore the exact sequence (3) summed over  $p$  gives

$$0 \longrightarrow \tilde{A}(F_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}(F) \xrightarrow{\zeta} CH(F) \longrightarrow 0. \quad (11)$$

Recall that  $\tilde{A}(F_{\mathbb{R}}) = \text{Ker}\zeta$  is an ideal of  $\widehat{CH}(F)$  whose  $\widehat{CH}(F)$ -module structure is given as follows: if  $\alpha \in \widehat{CH}(F)$  and  $\eta \in \tilde{A}(F_{\mathbb{R}})$ , then  $\alpha \cdot \eta = \omega(\alpha) \wedge \eta$ .  $\tilde{A}(F_{\mathbb{R}})$  is not a square zero ideal, but its product is induced by  $\theta \cdot \eta = (dd^c\theta) \wedge \eta$ . This product is well defined and commutative ([GS1], 4.2.11).

We equip  $E(\mathbb{C})$  with a trivial hermitian metric. This metric induces metrics on all the  $L_i$ , which thus become hermitian line bundles  $\bar{L}_i$ . Recall



from §2.3 that  $CH(F)$  has a free  $\mathbb{Z}$ -basis of monomials in the Chern classes  $c_1(L_i)$ . The unique map of abelian groups

$$\epsilon : CH(F) \rightarrow \widehat{CH}(F)$$

sending  $\prod c_1(L_i)^{k_i}$  to  $\prod \widehat{c}_1(\overline{L}_i)^{k_i}$  when  $k_i \leq n - i$  for all  $i$  is then a splitting of (11). Thus we have an isomorphism of abelian groups

$$\widehat{CH}(F) \simeq CH(F) \oplus \widetilde{A}(F_{\mathbb{R}}). \quad (12)$$

As an analogue of the Arakelov Chow ring we define an *invariant arithmetic Chow ring*  $\widehat{CH}_{inv}(F)$  as follows. Let  $\text{Inv}^{p,p}(F_{\mathbb{R}})$  be the group of  $(p, p)$ -forms  $\eta$  in  $\text{Inv}_{\mathbb{R}}(F(\mathbb{C}))$  satisfying  $F_{\infty}^* \eta = (-1)^p \eta$ , and set  $\text{Inv}(F_{\mathbb{R}}) = \bigoplus_p \text{Inv}^{p,p}(F_{\mathbb{R}})$ . Let  $\widehat{\text{Inv}}(F_{\mathbb{R}}) \subset \widetilde{A}(F_{\mathbb{R}})$  be the image of  $\text{Inv}(F_{\mathbb{R}})$  in  $\widetilde{A}(F_{\mathbb{R}})$ . Define the rings  $\text{Inv}(F_{\mathbb{Q}})$  and  $\widehat{\text{Inv}}(F_{\mathbb{Q}})$  similarly, replacing  $\mathbb{R}$  by  $\mathbb{Q}$  in the above.

**Definition 2** *The invariant arithmetic Chow ring  $\widehat{CH}_{inv}(F)$  is the subring of  $\widehat{CH}(F)$  generated by  $\epsilon(CH(F))$  and  $a(\widehat{\text{Inv}}(F_{\mathbb{R}}))$ .*

Suppose that  $x, y \in CH(F)$  and view  $x$  and  $y$  as elements of  $\widehat{CH}(F)$  using the inclusion  $\epsilon$ . Theorem 5 of §8 implies that under the isomorphism (12),  $xy \in CH(F) \oplus \widehat{\text{Inv}}(F_{\mathbb{Q}})$ . It follows that there is an exact sequence of abelian groups

$$0 \longrightarrow \widehat{\text{Inv}}(F_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}_{inv}(F) \xrightarrow{\zeta} CH(F) \longrightarrow 0 \quad (13)$$

which splits as before, giving

**Theorem 3** *There is an isomorphism of abelian groups*

$$\widehat{CH}_{inv}(F) \simeq CH(F) \oplus \widehat{\text{Inv}}(F_{\mathbb{R}}).$$

**Remark 1:** One can define another ‘invariant arithmetic Chow ring’

$$\widehat{CH}'_{inv}(F) := \omega^{-1}(\text{Inv}(F_{\mathbb{R}})),$$

where  $\omega$  is the ring homomorphism defined in §2.2. There is a natural inclusion  $\widehat{CH}_{inv}(F) \hookrightarrow \widehat{CH}'_{inv}(F)$ ; we do not know if these two rings coincide.

**Remark 2:** The arithmetic Chern classes of the natural homogeneous vector bundles over  $F$  are all contained in the ring  $\widehat{CH}_{inv}(F)$ . In fact one need not use real coefficients for this; it suffices to take  $CH(F) \oplus \widetilde{\text{Inv}}(F_{\mathbb{Q}})$  with the induced product from  $\widehat{CH}(F)$ . As there are bounds on the denominators that occur, it follows that *the subring of  $\widehat{CH}(F)$  generated by  $\epsilon(CH(F))$  is a finitely generated abelian group.* However it seems that this group is too small to contain the characteristic classes of all the vector bundles of interest.

## 7 Calculating arithmetic intersections

In this section we describe an effective procedure for computing arithmetic intersection numbers on the complete flag variety  $F$ . One has a tautological hermitian filtration of the trivial bundle  $\overline{E}$  over  $F$

$$\overline{\mathcal{E}} : 0 \subset \overline{E}_1 \subset \overline{E}_2 \subset \cdots \subset \overline{E}_n = \overline{E}$$

as in §5. Recall that the inverse of the isomorphism  $CH(F) \simeq P_n/I_n$  sends  $[X_i]$  to  $-c_1(L_i)$ . Let  $x_i = -c_1(\overline{L}_i)$  and  $\hat{x}_i = -\hat{c}_1(\overline{L}_i)$  for  $1 \leq i \leq n$ .

If  $\phi \in \Lambda_n \otimes_{\mathbb{Z}} \mathbb{Q}$  is a homogeneous symmetric polynomial of positive degree then  $\phi$  defines a characteristic class. Theorem 2 applied to the hermitian filtration  $\overline{\mathcal{E}}$  shows that

$$\phi(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) = (-1)^{\deg \phi} \tilde{\phi}(\overline{\mathcal{E}}) \quad (14)$$

in the arithmetic Chow ring  $\widehat{CH}(F)$ . In particular for  $\phi = e_i$  an elementary symmetric polynomial this gives

$$e_i(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) = (-1)^i \tilde{c}_i(\overline{\mathcal{E}}).$$

Let  $h$  be a homogeneous polynomial in the ideal  $I_n$ . We will give an algorithm for computing  $h(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$  as a class in  $\widetilde{\text{Inv}}(F_{\mathbb{Q}})$ :

**Step 1:** Decompose  $h$  as a sum  $h = \sum e_i f_i$  for some polynomials  $f_i$ . More canonically one may use the equality

$$h = \sum_{w \in S_n} \langle h, \mathfrak{S}^w \rangle \mathfrak{S}_w$$

from §2.3. Since  $a(x)y = a(x\omega(y))$  in  $\widehat{CH}(F)$  and  $\omega(f_i(\hat{x}_1, \dots, \hat{x}_n)) = f_i(x_1, \dots, x_n)$ , we have

$$h(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) = \sum_{i=1}^n (-1)^i \tilde{c}_i(\overline{\mathcal{E}}) f_i(x_1, x_2, \dots, x_n) =$$

$$= \sum_{w \in S_n} (-1)^{\deg h + \ell(w)} \langle \widetilde{h, \mathfrak{S}^w} \rangle(\overline{\mathcal{E}}) \mathfrak{S}_w(x_1, x_2, \dots, x_n).$$

(the signs in the second equality come from a combination of (5), (14) and the fact that  $\mathfrak{S}_w$  is homogeneous of degree  $\ell(w)$ ).

**Step 2:** By Corollary 2, we may express the forms  $\tilde{c}_i(\overline{\mathcal{E}})$  and  $\langle \widetilde{h, \mathfrak{S}^w} \rangle(\overline{\mathcal{E}})$  as polynomials in the entries of the matrices  $K_{E_i}$  and  $K_{L_i} = c_1(\overline{L}_i)$  with rational coefficients. In practice this may be done recursively for the Chern forms  $\tilde{c}_i$  as follows: Use equation (7) and the construction in Corollary 1 to obtain the power sum forms  $\tilde{p}_i(\overline{\mathcal{E}})$ , then apply the formulas (8) to Newton's identity

$$p_i - c_1 p_{i-1} + c_2 p_{i-2} - \dots + (-1)^i i c_i = 0.$$

On the other hand Corollary 3 gives explicit expressions for all the above curvature matrices in terms of differential forms on  $F(\mathbb{C})$ . Thus we obtain formulas for  $\tilde{c}_i(\overline{\mathcal{E}})$  and  $\langle \widetilde{h, \mathfrak{S}^w} \rangle(\overline{\mathcal{E}})$  in terms of these forms. For example, using the notation of §5, we have

**Proposition 3**  $\tilde{c}_1(\overline{\mathcal{E}}) = 0$  and  $\tilde{c}_2(\overline{\mathcal{E}}) = - \sum_{i < j} \Omega_{ij}$ .

**Proof.** Use (7), properties (a) and (b) at the end of §3, and the identity  $2c_2 = c_1^2 - p_2$ .  $\square$

**Step 3:** Substitute the forms obtained in Step 2 into the formulas given in Step 1. Note that the result is the class of a form in  $\text{Inv}(F_{\mathbb{Q}})$  since all the ingredients are functorial for the natural  $U(n)$  action on  $F(\mathbb{C})$ .

In particular, if  $k_i$  are nonnegative integers with  $\sum k_i = \dim F = \binom{n}{2} + 1$ , the monomial  $X_1^{k_1} \dots X_n^{k_n}$  is in the ideal  $I_n$ . If  $X_1^{k_1} \dots X_n^{k_n} = \sum e_i f_i$ , then we have

$$\widehat{x}_1^{k_1} \widehat{x}_2^{k_2} \dots \widehat{x}_n^{k_n} = \sum_i (-1)^i \tilde{c}_i(\overline{\mathcal{E}}) f_i(x_1, \dots, x_n).$$

Now if  $\Omega = \bigwedge \Omega_{ij}$  is defined as in §5, we have shown that

$$\tilde{c}_i(\overline{\mathcal{E}}) f_i(x_1, \dots, x_n) = r_i \Omega$$

for some rational number  $r_i$ . Therefore

$$\widehat{\deg}(\widehat{x}_1^{k_1} \widehat{x}_2^{k_2} \dots \widehat{x}_n^{k_n}) = \frac{1}{2} \sum_i (-1)^i r_i \int_{F(\mathbb{C})} \Omega = \frac{1}{2} \sum_i (-1)^i r_i \prod_{k=1}^{n-1} \frac{1}{k!}.$$

Of course this equation implies

**Theorem 4** *The arithmetic Chern number  $\widehat{\deg}(\widehat{x}_1^{k_1}\widehat{x}_2^{k_2}\cdots\widehat{x}_n^{k_n})$  is a rational number.*

**Remark.** For  $a < b$  let  $\overline{Q}_{b,a} = E_b/E_a$ , equipped with the induced metric. Then one can show that any intersection number  $\widehat{\deg}(\prod_i \widehat{c}_{m_i}(\overline{Q}_{b_i a_i})^{k_i})$  for  $\sum k_i m_i (b_i - a_i) = \dim F$  is rational. This is done by using the hermitian filtrations

$$0 \subset \overline{Q}_{a+1,a} \subset \overline{Q}_{a+2,a} \subset \cdots \subset \overline{Q}_{b,a}$$

and Theorem 2 to reduce the problem to the intersections occurring in Theorem 4. To compute arithmetic intersections of the form  $(0, \eta) \cdot (0, \eta')$  with  $\eta, \eta' \in \widetilde{\text{Inv}}(F_{\mathbb{Q}})$ , we need to know the value of  $dd^c \eta$ . For this one may use the Maurer-Cartan structure equations on  $SU(n)$  (see [GrS], Chp. 1); all such intersections lie in  $\widetilde{\text{Inv}}(F_{\mathbb{Q}})$ .

Although there is an effective algorithm for computing arithmetic Chern numbers, explicit general formulas seem difficult to obtain. There are some general facts we can deduce for those intersections that pull back from Grassmannians, for instance that  $\widehat{x}_1^{n+1} = \widehat{x}_n^{n+1} = 0$ . There is also a useful symmetry in these intersections:

**Proposition 4**  $\widehat{x}_1^{k_1}\widehat{x}_2^{k_2}\cdots\widehat{x}_n^{k_n} = \widehat{x}_n^{k_1}\widehat{x}_{n-1}^{k_2}\cdots\widehat{x}_1^{k_n}$ , for all integers  $k_i \geq 0$ .

**Proof.** This is a consequence of the involution  $\nu : F(\mathbb{C}) \rightarrow F(\mathbb{C})$  sending

$$\overline{\mathcal{E}} : 0 \subset \overline{E}_1 \subset \overline{E}_2 \subset \cdots \subset \overline{E}_n = \overline{E}$$

to

$$\overline{\mathcal{E}}^\perp : 0 = \overline{E}^\perp \subset \overline{E}_{n-1}^\perp \subset \overline{E}_{n-2}^\perp \subset \cdots \subset 0^\perp = \overline{E}.$$

Over  $\text{Spec}\mathbb{Z}$ ,  $\nu$  corresponds to the map of flag varieties sending  $E_i$  to the quotient  $E/E_i$ . If  $\widehat{x}_i^\perp$  are the arithmetic Chern classes obtained from  $\overline{\mathcal{E}}^\perp$ , then using the split exact sequences  $0 \rightarrow \overline{E}_i \rightarrow \overline{E} \rightarrow \overline{E}_i^\perp \rightarrow 0$  we obtain

$$\widehat{x}_i^\perp = -\widehat{c}_1(\overline{E}_i^\perp) = -\widehat{c}_1(\overline{E}_{n-i}^\perp) + \widehat{c}_1(\overline{E}_{n+1-i}^\perp) = \widehat{c}_1(\overline{E}_{n-i}) - \widehat{c}_1(\overline{E}_{n+1-i}) = \widehat{x}_{n+1-i}.$$

Since  $\nu$  is an isomorphism, the result follows.  $\square$

## 8 Arithmetic Schubert calculus

Let  $P_n, I_n, \Lambda_n$  and  $S^{(n)}$  be as in §2.3. The Chow ring  $CH(F)$  is isomorphic to the quotient  $H_n = P_n/I_n$ . Recall that  $H_n$  has a natural basis of Schubert polynomials  $\{\mathfrak{S}_w \mid w \in S_n\}$ , and that the  $\mathfrak{S}_w$  for  $w \in S^{(n)}$  form a free  $\mathbb{Z}$ -basis of  $P_n$ . We let  $T_n = S^{(n)} \setminus S_n$ . The key property of Schubert polynomials that we require for the ‘arithmetic Schubert calculus’ is described in

**Lemma 1** *If  $w \in T_n$ , then  $\mathfrak{S}_w \in I_n$ . In fact we have a decomposition*

$$\mathfrak{S}_w = \sum_{v \in S_n} \langle \mathfrak{S}_w, \mathfrak{S}^v \rangle \mathfrak{S}_v,$$

where  $\langle \mathfrak{S}_w, \mathfrak{S}^v \rangle \in \Lambda_n \cap I_n$ .

**Proof.** Assume first that  $w(1) > w(2) > \dots > w(n)$ , so that  $w$  is *dominant*. Then by [M1], (4.7) we have

$$\mathfrak{S}_w = X_1^{w(1)-1} X_2^{w(2)-1} \dots X_n^{w(n)-1}.$$

If  $w \notin S_n$  then clearly  $w(1) > n$ , so  $X_1^{w(1)-1} \in I_n$  and thus  $\mathfrak{S}_w \in I_n$ .

If  $w \in T_n$  is arbitrary, form  $w' \in T_n$  by rearranging  $(w(1), w(2), \dots, w(n))$  in decreasing order and letting  $w'(i) = w(i)$  for  $i > n$ . We have shown that  $\mathfrak{S}_{w'} \in I_n$ . There is an element  $v \in S_n$  such that  $wv = w'$  and  $\ell(v) = \ell(w') - \ell(w)$ . Note that since  $\partial_v$  is  $\Lambda_n$ -linear,  $\partial_v I_n \subset I_n$ . Therefore ([M1], (4.2)):  $\mathfrak{S}_w = \partial_v \mathfrak{S}_{wv} = \partial_v \mathfrak{S}_{w'} \in I_n$ .

The decomposition claimed now follows, as in §2.3. □

It is well known that there is an equality in  $P_\infty$

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_\infty} c_{uv}^w \mathfrak{S}_w, \tag{15}$$

where the  $c_{uv}^w$  are nonnegative integers that vanish whenever  $\ell(w) \neq \ell(u) + \ell(v)$  ([M1], (A.6)). A particular case of this is *Monk’s formula*: if  $s_k$  denotes the transposition  $(k, k+1)$ , then

$$\mathfrak{S}_{s_k} \mathfrak{S}_w = \sum_t \mathfrak{S}_{wt}$$

summed over all transpositions  $t = (i, j)$  such that  $i \leq k < j$  and  $\ell(wt) = \ell(w) + 1$  ([M1], (4.15'')).

We now express arithmetic intersections in  $\widehat{CH}(F)$  using the basis of Schubert polynomials. Lemma 1 is the main reason why this basis facilitates our task. This property (for Schur functions) also plays a crucial role in the arithmetic Schubert calculus for Grassmannians (see §10 and [Ma], Th. 5.2.1).

For each  $w \in S^{(n)}$ , let  $\widehat{\mathfrak{S}}_w = \mathfrak{S}_w(\widehat{x}_1, \dots, \widehat{x}_n)$ . If  $w \in T_n$  then Lemma 1 and the discussion in §7 imply that  $\widehat{\mathfrak{S}}_w \in \widetilde{\text{Inv}}(F_{\mathbb{Q}})$ ; we denote these classes by  $\widetilde{\mathfrak{S}}_w$ . We have

$$\widetilde{\mathfrak{S}}_w = \sum_{v \in S_n} (-1)^{\ell(v) + \ell(w)} \langle \widetilde{\mathfrak{S}}_w, \widetilde{\mathfrak{S}}^v \rangle (\bar{\mathcal{E}}) \mathfrak{S}_v(x_1, \dots, x_n).$$

We can now describe the multiplication in  $\widehat{CH}_{inv}(F)$ :

**Theorem 5** *Any element of  $\widehat{CH}_{inv}(F)$  can be expressed uniquely in the form  $\sum_{w \in S_n} a_w \widehat{\mathfrak{S}}_w + \eta$ , where  $a_w \in \mathbb{Z}$  and  $\eta \in \widetilde{\text{Inv}}(F_{\mathbb{R}})$ . For  $u, v \in S_n$  we have*

$$\widehat{\mathfrak{S}}_u \cdot \widehat{\mathfrak{S}}_v = \sum_{w \in S_n} c_{uv}^w \widehat{\mathfrak{S}}_w + \sum_{w \in T_n} c_{uv}^w \widetilde{\mathfrak{S}}_w, \quad (16)$$

$$\widehat{\mathfrak{S}}_u \cdot \eta = \mathfrak{S}_u(x_1, \dots, x_n) \wedge \eta, \quad \text{and} \quad \eta \cdot \eta' = (dd^c \eta) \wedge \eta',$$

where  $\widetilde{\mathfrak{S}}_w \in \widetilde{\text{Inv}}(F_{\mathbb{Q}})$ ,  $\eta, \eta' \in \widetilde{\text{Inv}}(F_{\mathbb{R}})$  and the  $c_{uv}^w$  are as in (15).

**Proof.** The first statement is a corollary of Theorem 3. Equation (16) follows immediately from the formal identity (15) and our definition of  $\widehat{\mathfrak{S}}_w$ . The rest is a consequence of properties of the multiplication in  $\widehat{CH}(F)$  discussed in §6 and §7.  $\square$

**Remark.** It is interesting to note that we also have, for  $u, v \in T_n$ ,

$$\widetilde{\mathfrak{S}}_u \cdot \widetilde{\mathfrak{S}}_v = (dd^c \widetilde{\mathfrak{S}}_u) \wedge \widetilde{\mathfrak{S}}_v = \sum_{w \in T_n} c_{uv}^w \widetilde{\mathfrak{S}}_w$$

in  $\widetilde{\text{Inv}}(F_{\mathbb{Q}})$ .

Applying (16) when  $\mathfrak{S}_u = \mathfrak{S}_{s_k}$  is a special Schubert class gives

**Corollary 4** (*Arithmetic Monk Formula*):

$$\widehat{\mathfrak{S}}_{s_k} \cdot \widehat{\mathfrak{S}}_w = \sum_s \widehat{\mathfrak{S}}_{ws} + \sum_t \widetilde{\mathfrak{S}}_{wt},$$

where the first sum is over all transpositions  $s = (i, j) \in S_n$  such that  $i \leq k < j$  and  $\ell(ws) = \ell(w) + 1$ , and the second over all transpositions  $t = (i, n+1)$  with  $i \leq k$  and  $w(i) > w(j)$  for all  $j$  with  $i < j \leq n$ .

## 9 Examples

### 9.1 Heights

The flag variety  $F$  has a natural pluri-Plücker embedding  $j : F \hookrightarrow \mathbb{P}_{\mathbb{Z}}^N$ .  $j$  is defined as the composition of a product of Plücker embeddings followed by a Segre embedding; if  $Q_i = E/E_i$ , then  $j$  is associated to the line bundle  $Q = \bigotimes_{i=1}^{n-1} \det(Q_i)$ . Let  $\overline{\mathcal{O}}(1)$  denote the canonical line bundle over projective space, equipped with its canonical metric (so that  $c_1(\overline{\mathcal{O}}(1))$  is the Fubini-Study form). The *height* of  $F$  relative to  $\overline{\mathcal{O}}(1)$  (see [Fa], [BoGS], [S]) is defined by

$$h_{\overline{\mathcal{O}}(1)}(F) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}}(1))^{\binom{n}{2}+1} | F).$$

The pullback  $j^*(\mathcal{O}(1)) = Q$  is an isometry when  $Q$  is equipped with the canonical metric given by tensoring the induced metrics on the determinants of the  $\overline{Q}_i$  (this follows from the definition of the Segre and Plücker embeddings). We deduce that

$$j^*(\widehat{c}_1(\overline{\mathcal{O}}(1))) = \widehat{c}_1(\overline{Q}) = - \sum_{i=1}^{n-1} \widehat{c}_1(\overline{E}_i) = \sum_{i=1}^{n-1} (n-i) \widehat{x}_i = \sum_{i=1}^{n-1} \widehat{\mathfrak{S}}_{s_i},$$

and therefore

$$h_{\overline{\mathcal{O}}(1)}(F) = \widehat{\deg}(\widehat{c}_1(\overline{Q})^{\binom{n}{2}+1} | F) = \widehat{\deg}((\sum_{i=1}^{n-1} \widehat{\mathfrak{S}}_{s_i})^{\binom{n}{2}+1}).$$

Now Theorems 4 and 5 immediately imply

**Corollary 5** *The height  $h_{\overline{\mathcal{O}}(1)}(F)$  is a rational number.*

### 9.2 Intersections in $F_{1,2,3}$

In this section we calculate the arithmetic intersection numbers for the classes  $\widehat{x}_i$  in  $\widehat{CH}(F)$  when  $n = 3$ , so  $F = F_{1,2,3}$ .

Over  $F$  we have 3 exact sequences

$$\overline{\mathcal{E}}_i : 0 \rightarrow \overline{E}_{i-1} \rightarrow \overline{E}_i \rightarrow \overline{L}_i \rightarrow 0 \quad 1 \leq i \leq 3.$$

We adopt the notation of §5 and define  $\Omega_{ij} = \omega_{ij} \wedge \overline{\omega}_{ij}$ . Then Corollary 3 gives

$$x_1 = \Omega_{12} + \Omega_{13}, \quad x_2 = -\Omega_{12} + \Omega_{23}, \quad x_3 = -\Omega_{13} - \Omega_{23},$$

$$K_{E_2} = - \begin{pmatrix} \Omega_{13} & \omega_{13} \wedge \bar{\omega}_{23} \\ \omega_{23} \wedge \bar{\omega}_{13} & \Omega_{23} \end{pmatrix}.$$

We refer now to the properties of the forms  $\tilde{c}_k$  mentioned at the end of §3. By property (a)  $\tilde{c}(\bar{\mathcal{E}}_1) = 0$ , while (b) gives  $\tilde{c}(\bar{\mathcal{E}}_2) = -\Omega_{12}$ . Property (c) applied to  $\bar{\mathcal{E}}_3$  gives  $\tilde{c}(\bar{\mathcal{E}}_3) = -\Omega_{13} - \Omega_{23} + 3\Omega_{13}\Omega_{23}$ . Using the construction of the Bott-Chern form for the total Chern class given in the proof of Theorem 1, we find that

$$\tilde{c}(\bar{\mathcal{E}}) = -\Omega_{12} - \Omega_{13} - \Omega_{23} - \Omega_{12}\Omega_{13} - \Omega_{12}\Omega_{23} + 3\Omega_{13}\Omega_{23}. \quad (17)$$

Notice that this expression for  $\tilde{c}(\bar{\mathcal{E}})$  is not unique as a class in  $\widetilde{\text{Inv}}(F_{\mathbb{R}})$ . For instance, we can add the exact form

$$c_1(\bar{L}_1)c_1(\bar{L}_2) - c_2(\bar{E}_2) = \Omega_{12}\Omega_{23} - \Omega_{12}\Omega_{13} - \Omega_{13}\Omega_{23}$$

to get

$$\tilde{c}(\bar{\mathcal{E}}) = -\Omega_{12} - \Omega_{13} - \Omega_{23} - 2\Omega_{12}\Omega_{13} + 2\Omega_{13}\Omega_{23}. \quad (18)$$

The Bott-Chern form (18) is the key to computing any intersection number  $\widehat{\text{deg}}(\hat{x}_1^{k_1}\hat{x}_2^{k_2}\hat{x}_3^{k_3})$ , following the prescription of §7. (Of course we can just as well use (17), with the same results.) For example, since

$$x_1^4 = x_1^3e_1 - x_1^2e_2 + x_1e_3,$$

we have

$$\hat{x}_1^4 = x_1^2(\Omega_{12} + \Omega_{13} + \Omega_{23}) + x_1(2\Omega_{12}\Omega_{13} - 2\Omega_{13}\Omega_{23}) = 2\Omega - 2\Omega = 0.$$

On the other hand, a similar calculation for  $\hat{x}_2^4$  gives

$$\hat{x}_2^4 = -x_2^2\tilde{c}_2(\bar{\mathcal{E}}) - x_2\tilde{c}_3(\bar{\mathcal{E}}) = -2\Omega + 4\Omega = 2\Omega.$$

$$\text{Thus } \widehat{\text{deg}}(\hat{x}_2^4) = \int_{F(\mathbb{C})} \Omega = \frac{1}{2}.$$

The following is a table of all the intersection numbers  $\widehat{\text{deg}}(\hat{x}_1^{k_1}\hat{x}_2^{k_2}\hat{x}_3^{k_3})$  (multiplied by 4):

$k_1k_2k_3$	$4 \widehat{\text{deg}}$	$k_1k_2k_3$	$4 \widehat{\text{deg}}$	$k_1k_2k_3$	$4 \widehat{\text{deg}}$
400	0	004	0	040	2
310	5	013	5	121	4
301	-5	103	-5	202	7
220	-3	022	-3		
211	-2	112	-2		
130	-1	031	-1		



Note that the numbers in the first two columns are equal, in agreement with Proposition 4. We can use the table to compute the height of  $F$  in its pluri-Plücker embedding in  $\mathbb{P}_{\mathbb{Z}}^8$ :

$$h_{\overline{\mathcal{O}}(1)}(F_{1,2,3}) = \widehat{\deg}((2\hat{x}_1 + \hat{x}_2)^4) = \frac{41}{2}.$$

## 10 Partial flag varieties

In this final section we show how to generalize the previous work to partial flags  $F(\mathfrak{r})$ . Our results may thus be regarded as an extension of those of Maillot [Ma] in the Grassmannian case.

As usual we have a tautological filtration of type  $\mathfrak{r}$

$$\mathcal{E} : 0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E \quad (19)$$

of the trivial bundle over  $F(\mathfrak{r})$ , with quotient bundles  $Q_i$ . Equip  $E(\mathbb{C})$  with the trivial hermitian metric, inducing metrics on all the above bundles. The calculations of §5 apply equally well to  $X_{\mathfrak{r}} = F(\mathfrak{r})(\mathbb{C})$ . Proposition 2 describes the curvature matrices of all the relevant homogeneous vector bundles, and one can compute classical intersection numbers on  $X_{\mathfrak{r}}$  in a similar fashion.

Call a permutation  $w \in S_{\infty}$  an  $\mathfrak{r}$ -permutation if  $w(i) < w(i+1)$  for all  $i$  not contained in  $\{r_1, \dots, r_{m-1}\}$ . Let  $S_{n,\mathfrak{r}}$  and  $T_{n,\mathfrak{r}}$  be the set of  $\mathfrak{r}$ -permutations in  $S_n$  and  $T_n$ , respectively. For such  $w$  one knows (see [Fu2], §8) that the Schubert polynomial  $\mathfrak{S}_w$  is symmetric in the variables in each of the groups

$$X_1, \dots, X_{r_1}; X_{r_1+1}, \dots, X_{r_2}; \dots; X_{r_{m-2}+1}, \dots, X_{r_{m-1}}. \quad (20)$$

The product group  $H = \prod_{i=1}^m S_{r_i - r_{i-1}}$  acts on  $P_n$ , the factors for  $i < m$  by permuting the variables in the corresponding group of (20), while  $S_{n-r_{m-1}}$  permutes the remaining variables  $X_{r_{m-1}+1}, \dots, X_n$ . If  $P_{n,\mathfrak{r}} = P_n^H$  is the ring of invariants and  $I_{n,\mathfrak{r}} = P_{n,\mathfrak{r}} \cap I_n$ , then  $CH(F(\mathfrak{r})) \simeq P_{n,\mathfrak{r}}/I_{n,\mathfrak{r}}$ . The set of Schubert polynomials  $\mathfrak{S}_w$  for all  $w \in S_{n,\mathfrak{r}}$  is a free  $\mathbb{Z}$ -basis for  $P_{n,\mathfrak{r}}/I_{n,\mathfrak{r}}$ .

Let  $w \in S_{n,\mathfrak{r}}$ . If we regard each of the groups of variables (20) as the Chern roots of the bundles  $Q_1, Q_2, \dots, Q_{m-1}$ , it follows that we may write  $\mathfrak{S}_w$  as a polynomial  $\mathfrak{S}_{w,\mathfrak{r}}$  in the Chern classes of the  $Q_i$ ,  $1 \leq i \leq m-1$ . The class of  $\mathfrak{S}_{w,\mathfrak{r}}$  in  $CH(F(\mathfrak{r}))$  is that of corresponding Schubert variety in  $F(\mathfrak{r})$

(see loc. cit. for the relative case). By putting ‘hats’ on all the quotient bundles involved (with their induced metrics as in §8) we obtain classes  $\widehat{\mathfrak{S}}_{w,\mathfrak{r}}$  in  $\widehat{CH}_{inv}(F(\mathfrak{r}))$ .

The analysis of §6 remains valid; the map  $\epsilon$  can be defined by  $\epsilon(\mathfrak{S}_{w,\mathfrak{r}}) = \widehat{\mathfrak{S}}_{w,\mathfrak{r}}$ . In particular we have an invariant arithmetic Chow ring  $\widehat{CH}_{inv}(F(\mathfrak{r}))$  for which Theorem 3 holds. If  $F(\mathfrak{r}) = G_d$  is a Grassmannian over  $\text{Spec}\mathbb{Z}$ , then  $\widehat{CH}_{inv}(G_d)$  coincides with the Arakelov Chow ring  $CH(\overline{G}_d)$ , where  $G_d(\mathbb{C})$  is given its natural invariant Kähler metric, as in [Ma].

Suppose that  $\mathfrak{r}'$  is a refinement of  $\mathfrak{r}$ , so we have a projection  $p: F(\mathfrak{r}') \rightarrow F(\mathfrak{r})$ . In this case there are natural inclusions  $\widetilde{\text{Inv}}(F(\mathfrak{r})_{\mathbb{R}}) \hookrightarrow \widetilde{\text{Inv}}(F(\mathfrak{r}')_{\mathbb{R}})$  and  $CH(F(\mathfrak{r})) \hookrightarrow CH(F(\mathfrak{r}'))$ . Applying the five lemma to the two exact sequences (13) shows that the pullback  $p^*: \widehat{CH}_{inv}(F(\mathfrak{r})) \hookrightarrow \widehat{CH}_{inv}(F(\mathfrak{r}'))$  is an injection. Note however that this is not compatible with the splitting of Theorem 3.

One can compute arithmetic intersections in  $\widehat{CH}_{inv}(F(\mathfrak{r}))$  as in §7. Applying Theorem 2 to the filtration (19) (with induced metrics as above) gives the key relation required for the calculation. In particular we see that all the arithmetic Chern numbers are rational, as is the Faltings height of  $F(\mathfrak{r})$  in its natural pluri-Plücker embedding. Corollary 5 thus generalizes the corresponding result of Maillot mentioned in §1.

There is an arithmetic Schubert calculus in  $\widehat{CH}_{inv}(F(\mathfrak{r}))$  analogous to that for complete flags. The analogue of Lemma 1 is true, that is  $\mathfrak{S}_w \in I_{n,\mathfrak{r}}$  if  $w \in T_{n,\mathfrak{r}}$  (this is an easy consequence of Lemma 1 itself). It follows that for  $w \in T_{n,\mathfrak{r}}$ ,  $\widehat{\mathfrak{S}}_{w,\mathfrak{r}}$  is a class  $\widetilde{\mathfrak{S}}_{w,\mathfrak{r}} \in \widetilde{\text{Inv}}(F(\mathfrak{r})_{\mathbb{Q}})$ . The analogue of (16) in this context is

$$\widehat{\mathfrak{S}}_{u,\mathfrak{r}} \cdot \widehat{\mathfrak{S}}_{v,\mathfrak{r}} = \sum_{w \in S_{n,\mathfrak{r}}} c_{uv}^w \widehat{\mathfrak{S}}_{w,\mathfrak{r}} + \sum_{w \in T_{n,\mathfrak{r}}} c_{uv}^w \widetilde{\mathfrak{S}}_{w,\mathfrak{r}} \quad (21)$$

where  $u, v \in S_{n,\mathfrak{r}}$  and the numbers  $c_{uv}^w$  are as in (15). The remaining statements of Theorem 5 require no further change.

**Remark.** Equation (21) is not a direct generalization of the analogous statement in [Ma], Theorem 5.2.1. However one can reformulate Maillot’s results using the classes  $\widehat{c}_*(\overline{S})$  instead of  $\widehat{c}_*(\overline{Q} - \overline{E})$  (notation as in [Ma], §5.2). With this modification, the arithmetic Schubert calculus described above (for  $m = 2$ ) and that in [Ma] coincide. In the Grassmannian case  $\mathfrak{S}_{w,\mathfrak{r}}$  is a Schur polynomial and there are explicit formulas for  $\widetilde{\mathfrak{S}}_{w,\mathfrak{r}}$  in terms of harmonic forms on  $G_d(\mathbb{C})$  (as in loc. cit.).

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