SCHUBERT POLYNOMIALS AND ARAKELOV THEORY OF SYMPLECTIC FLAG VARIETIES

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Abstract. Let $\mathfrak{X} = \mathrm{Sp}_{2n}/B$ the flag variety of the symplectic group. We propose a theory of combinatorially explicit Schubert polynomials which represent the Schubert classes in the Borel presentation of the cohomology ring of $\mathfrak{X}$. We use these polynomials to describe the arithmetic Schubert calculus on $\mathfrak{X}$. Moreover, we give a method to compute the natural arithmetic Chern numbers on $\mathfrak{X}$, and show that they are all rational numbers.

0. Introduction

Let $\mathfrak{X} = \mathrm{Sp}_{2n}/B$ be the flag variety for the symplectic group Sp_{2n} of rank n. The cohomology (or Chow) ring of $\mathfrak{X}$ has a standard presentation, due to Borel [Bo], as a quotient ring $\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]/I_n$, where the variables x_i come from the characters of B and I_n is the ideal generated by the invariant polynomials under the action of the Weyl group of Sp_{2n}. On the other hand, the cohomology $\mathrm{H}^*(\mathfrak{X}, \mathbb{Z})$ is a free abelian group with basis given by the classes of the Schubert varieties in $\mathfrak{X}$.

The aim of a theory of *Schubert polynomials* is to provide an explicit and natural set of polynomial representatives for the Schubert classes in the above Borel presentation of the cohomology ring. Using a construction of Bernstein-Gelfand-Gelfand [BGG] and Demazure [D1, D2], one has an algorithm for obtaining a family of polynomials which represent the Schubert classes, by applying divided difference operators to a polynomial T_{w_0} representing the class of a point in $\mathfrak{X}$. For the usual SL_n flag varieties, Lascoux and Schützenberger [LS] observed that one special choice of T_{w_0} leads to polynomials that represent the Schubert classes simultaneously for all sufficiently large n. These type A Schubert polynomials are the most natural ones to use from the point of view of combinatorics and of geometry; they describe the degeneracy loci of maps of vector bundles (see [F1, BKTY, KMS]) and are the prototype for any proposed theory of Schubert polynomials in the other Lie types.

For the purposes of the present paper, our interest in Schubert polynomials is due to their utility in studying the deformations of the cohomology ring of $\mathfrak{X}$ which appear in quantum cohomology and in the extension of Arakelov theory to higher dimensions due to Gillet and Soulé [GS1]. With this latter application in mind, we observed in [T2, T3, T4] that a suitable theory of polynomials should provide a lifting of the Schubert calculus from the quotient ring $\mathbb{Z}[\mathrm{X}_n]/I_n$ to the ring $\mathbb{Z}[\mathrm{X}_n]$ of all polynomials in $\mathrm{X}_n = (\mathrm{x}_1, \ldots, \mathrm{x}_n)$. In addition, one would like to have strong control over which polynomials are contained in the ideal I_n of relations; this was called the *ideal property* in [T3].

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The search for a good theory of Schubert polynomials in types B, C, and D has received much attention in the past (see e.g. [BH, FK, F2, F3, KT1, LP1, LP2]). The best understood theory from the combinatorial point of view appears to be that of Billey and Haiman [BH], whose Schubert polynomials form a $\mathbb{Z}$-basis for a polynomial ring in infinitely many variables. Unfortunately, when expressed in their most explicit form, the Billey-Haiman polynomials are not suitable for the above mentioned applications, because the variables used are not geometrically natural.

This problem is already apparent in the case of the Lagrangian Grassmannian $\mathrm{LG} = \mathrm{Sp}_{2n}/P_n$, where P_n denotes the maximal parabolic subgroup associated to the right end root in the Dynkin diagram of type C_n. The Billey-Haiman Schubert polynomials for the Schubert classes which pull back to $\mathfrak{X}$ from LG coincide with the Schur Q-functions [S], but the latter are not polynomials in the Chern roots of any homogeneous vector bundle over LG. For the geometric applications to degeneracy loci and elsewhere, one may instead use the $\widetilde{Q}$-polynomials of Pragacz and Ratajski [PR]. Indeed, we showed in [T4] and [KT2] that the latter objects control the arithmetic and quantum Schubert calculus on LG, respectively.

In the first part of this article, we introduce a new theory of symplectic Schubert polynomials $\mathfrak{C}_w(\mathrm{X}_n)$, which are to the Billey-Haiman Schubert polynomials what the $\widetilde{Q}$-polynomials are to the Schur Q-functions. The $\mathfrak{C}_w$ are defined as linear combinations of products of $\widetilde{Q}$-polynomials with type A Schubert polynomials, with coefficients given by combinatorially explicit integers which appear in the Billey-Haiman theory. Furthermore, the polynomials $\mathfrak{C}_w$ extend to a $\mathbb{Z}$-basis of the full polynomial ring $\mathbb{Z}[\mathrm{X}_n]$, which has the ideal property mentioned above. Although they represent the Schubert classes in the Borel presentation of $\mathrm{H}^*(\mathfrak{X},\mathbb{Z})$, the $\mathfrak{C}_w$ do not respect the divided difference operator given by the sign change, and thus differ from the previously known type C Schubert polynomials.

In the second half of the paper, we use the polynomials $\mathfrak{C}_w$ to describe the arithmetic Schubert calculus on $\mathfrak{X}$, in its natural smooth Chevalley model over $\operatorname{Spec}\mathbb{Z}$. Arithmetic Schubert calculus is concerned with the multiplicative structure of the Gillet-Soulé arithmetic Chow ring $\widehat{\mathrm{CH}}(\mathfrak{X})$, expressed in terms of certain (carefully chosen!) *arithmetic Schubert classes*. The present study is thus a belated sequel to [T2], which examined arithmetic intersection theory on SL_n flag varieties. As noted in the introduction to [T2], the hermitian differential geometry required to develop the theory for Sp_{2n}/B was available at that time; what was lacking was the theory of Schubert polynomials which is provided here.

The arithmetic scheme $\mathfrak{X}$ parametrizes, over any base field k, all partial flags of subspaces

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n \subset E_{2n} = E$$

with $\dim(E_i) = i$ for each i and E_n isotropic with respect to the skew diagonal symplectic form on E. Let $\overline{E}$ be the trivial vector bundle of rank $2n$ over $\mathfrak{X}$ equipped with a trivial hermitian metric on $E(\mathbb{C})$ compatible with the symplectic form. The metric on $E(\mathbb{C})$ induces metrics on all the subbundles $E_i(\mathbb{C})$, giving a *hermitian filtration* of $\overline{E}$. For $1 \leq i \leq n$, let $\overline{L}_i$ denote the quotient line bundle E_{n+1-i}/E_{n-i} with the induced hermitian metric on $L_i(\mathbb{C})$, and set $\widehat{x}_i = -\widehat{c}_1(\overline{L}_i)$, where $\widehat{c}_1(\overline{L}_i)$ is the arithmetic first Chern class of $\overline{L}_i$.

Let $h \in \mathbb{Z}[\mathrm{X}_n]$ be any polynomial in the ideal I_n. We provide an algorithm to compute the arithmetic intersection $h(\widehat{x}_1,\ldots,\widehat{x}_n)$ in $\widehat{\mathrm{CH}}(\mathfrak{X})$, as the class of an explicit $\mathrm{Sp}(2n)$-invariant differential form on $\mathfrak{X}(\mathbb{C})$. In particular, we show that all arithmetic Chern numbers on $\mathfrak{X}$ involving the $\widehat{x}_i$ are rational numbers (Theorem 2). The key relations in the ring $\widehat{\mathrm{CH}}(\mathfrak{X})$ required for this calculation involve the *Bott-Chern forms* of hermitian fitrations over $\mathfrak{X}$. As in [T2], these differential forms are identified with certain polynomials in the entries of the curvature matrices of the homogeneous vector bundles over $\mathfrak{X}$. Using a computation of Griffiths and Schmid [GrS], the latter entries may be expressed in terms of $\mathrm{Sp}(2n)$-invariant differential forms on $\mathrm{Sp}(2n)$. Finally, our main result (Theorem 3) describes the arithmetic Schubert calculus on $\mathfrak{X}$ using the structure constants for the product of two symplectic Schubert polynomials $\mathfrak{C}_w$ in the polynomial ring $\mathbb{Z}[\mathrm{X}_n]$.

The paper is organized as follows. In §1 we provide some combinatorial preliminaries on $\widetilde{Q}$-polynomials and the Lascoux-Schützenberger and Billey-Haiman Schubert polynomials. We introduce our theory of symplectic Schubert polynomials in §2.2 and derive some of their basic properties in §2.3. Section 3 includes the main facts from the hermitian differential geometry of $\mathfrak{X}(\mathbb{C})$ that we require. The Bott-Chern forms of hermitian filtrations are discussed in §3.1 and the curvature of the relevant homogeneous vector bundles over $\mathfrak{X}$ is computed in §3.2. The arithmetic intersection theory of $\mathfrak{X}$ is studied in §4. Our method for computing arithmetic intersections is explained in §4.3, the arithmetic Schubert calculus is described in §4.4, while §4.5 examines the invariant arithmetic Chow ring of $\mathfrak{X}$. The theory developed in these sections can be used to compute the Faltings height of $\mathfrak{X}$ under its pluri-Plücker embedding in projective space; this application is given in §4.6. Section 4.7 works out the example of Sp_4/B explicitly.

The results of this article on Schubert polynomials and arithmetic intersection theory have analogues for the orthogonal flag varieties. This theory is discussed in detail in [T5]. Part of this project was announced at the Oberwolfach meeting on Arakelov Geometry in September of 2005. I wish to thank the organizers Jean-Benoit Bost, Klaus Künnemann, and Damian Roessler for making this stimulating event possible.

1. Preliminary definitions

1.1. $\widetilde{Q}$- and Q-functions. A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a finite sequence of weakly decreasing nonnegative integers; the set of all partitions is denoted Π. The sum $\sum \lambda_i$ of the parts of λ is the *weight* $|\lambda|$ and the number of (nonzero) parts λ_i is the *length* $\ell(\lambda)$ of λ. We set $\lambda_r = 0$ for any $r > \ell(\lambda)$. A partition is *strict* if all its nonzero parts are distinct. Let $\mathcal{G}_n = \{\lambda \in \Pi \mid \lambda_1 \leq n\}$ and let $\mathcal{D}_n$ be the set of strict partitions in $\mathcal{G}_n$. If $\lambda \in \mathcal{D}_n$, we let λ' denote the partition in $\mathcal{D}_n$ whose parts complement the parts of λ in the set $\{1, \ldots, n\}$.

Let $\mathrm{X} = (\mathrm{x}_1, \mathrm{x}_2, \ldots)$ be a sequence of commuting independent variables. The elementary symmetric functions $e_k = e_k(\mathrm{X})$ are defined by the equation

$$\sum_{k=0}^{\infty} e_k(\mathrm{X}) t^k = \prod_{i=1}^{\infty} (1 + \mathrm{x}_i t).$$

The polynomial ring $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$ is the ring of symmetric functions in the variables X. Following Pragacz and Ratajski [PR], for each partition λ, we define a symmetric polynomial $\widetilde{Q}_\lambda \in \Lambda$ as follows: initially, set $\widetilde{Q}_k = e_k$ for $k \geq 0$. For i, j

nonnegative integers, let

$$\widetilde{Q}_{i,j} = \widetilde{Q}_i\widetilde{Q}_j + 2\sum_{r=1}^{j}(-1)^r\widetilde{Q}_{i+r}\widetilde{Q}_{j-r}.$$

If λ is a partition of length greater than two and m is the least positive integer with $2m \geq \ell(\lambda)$, then set

$$\widetilde{Q}_\lambda = \text{Pfaffian}(\widetilde{Q}_{\lambda_i,\lambda_j})_{1\leq i<j\leq 2m}.$$

These $\widetilde{Q}$-functions are modelled on Schur's Q-functions [S], and enjoy the following properties:

(a) The $\widetilde{Q}_\lambda(\mathrm{X})$ for $\lambda \in \Pi$ form a $\mathbb{Z}$-basis of Λ.

(b) $\widetilde{Q}_{k,k}(\mathrm{X}) = e_k(\mathrm{X}^2) := e_k(\mathrm{x}_1^2, \mathrm{x}_2^2, \ldots)$ for all k.

(c) If $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\lambda^+ = \lambda \cup (k, k) = (\lambda_1, \ldots, k, k, \ldots, \lambda_r)$ then

$$\widetilde{Q}_{\lambda^+} = \widetilde{Q}_{k,k}\widetilde{Q}_\lambda.$$

(d) The coefficients of $\widetilde{Q}_\lambda(\mathrm{X})$ are nonnegative integers.

Let $\Lambda_n = \mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]^{S_n}$ be the ring of symmetric polynomials in $\mathrm{X}_n = (\mathrm{x}_1, \ldots, \mathrm{x}_n)$. Then we have an additional property

(e) If $\lambda_1 > n$, then $\widetilde{Q}_\lambda(\mathrm{X}_n) = 0$. The $\widetilde{Q}_\lambda(\mathrm{X}_n)$ for $\lambda \in \mathcal{G}_n$ form a $\mathbb{Z}$-basis of Λ_n.

Suppose that $Y = (y_1, y_2, \ldots)$ is a second sequence of variables and define symmetric functions $q_k(Y)$ by using the generating series

$$\sum_{k=0}^{\infty} q_k(Y)t^k = \prod_{i=1}^{\infty} \frac{1 + y_i t}{1 - y_i t}.$$

Let $\Gamma = \mathbb{Z}[q_1, q_2 \ldots]$ and define a ring homomorphism $\eta : \Lambda \to \Gamma$ by setting $\eta(e_k(\mathrm{X})) = q_k(Y)$ for each $k \geq 1$. Józefiak [Jo] showed that the kernel of η is the ideal generated by the $e_k(\mathrm{X}^2)$ for $k > 0$; it follows that $\eta(\widetilde{Q}_\lambda) = 0$ unless λ is a strict partition. Moreover, if $p_k(\mathrm{X}) = \mathrm{x}_1^k + \mathrm{x}_2^k + \cdots$ denotes the k-th power sum, then we have $\eta(p_k(\mathrm{X})) = 2\,p_k(Y)$, if k is odd, and $\eta(p_k(\mathrm{X})) = 0$, if $k > 0$ is even. For any strict partition λ, the Schur Q-function $Q_\lambda(Y)$ may be defined as the image of $\widetilde{Q}_\lambda(\mathrm{X})$ under η. The Q_λ for strict partitions λ have nonnegative integer coefficients and form a free $\mathbb{Z}$-basis of Γ.

1.2. **Divided differences and type A Schubert polynomials.** Let S_n denote the symmetric group of permutations of the set $\{1, \ldots, n\}$; the elements ϖ of S_n are written in single-line notation as $(\varpi(1), \varpi(2), \ldots, \varpi(n))$ (as usual we will write all mappings on the left of their arguments). Now S_n is the Weyl group for the root system A_{n-1} and is generated by the simple transpositions s_i for $1 \leq i \leq n-1$, where s_i interchanges i and $i+1$ and fixes all other elements of $\{1, \ldots, n\}$.

The hyperoctahedral group W_n is the Weyl group for the root system C_n. The elements of W_n are permutations with a sign attached to each entry, and we will adopt the notation where a bar is written over an element with a negative sign. W_n is an extension of S_n by an element s_0 which acts on the right by

$$(u_1, u_2, \ldots, u_n)s_0 = (\overline{u}_1, u_2, \ldots, u_n).$$

A *reduced word* of $w \in W_n$ is a sequence $a_1 \dots a_r$ of elements in $\{0, 1, \dots, n-1\}$ such that $w = s_{a_1} \cdots s_{a_r}$ and r is minimal (so equal to the length $\ell(w)$ of w). The elements of maximal length in S_n and W_n are

$$\varpi_0 = (n, n-1, \dots, 1) \quad \text{and} \quad w_0 = (\overline{1}, \overline{2}, \dots, \overline{n})$$

respectively.

The group W_n acts on the ring $\mathbb{Z}[\mathrm{X}_n]$ of polynomials in X_n: the transposition s_i interchanges x_i and x_{i+1} for $1 \leq i \leq n-1$, while s_0 replaces x_1 by $-\mathrm{x}_1$ (all other variables remain fixed). The ring of invariants $\mathbb{Z}[\mathrm{X}_n]^{W_n}$ is the ring of polynomials in $\mathbb{Z}[\mathrm{X}_n]$ symmetric in $\mathrm{X}_n^2 = (\mathrm{x}_1^2, \dots, \mathrm{x}_n^2)$. Following [BGG] and [D1] [D2], there are divided difference operators $\partial_i : \mathbb{Z}[\mathrm{X}_n] \to \mathbb{Z}[\mathrm{X}_n]$. For $1 \leq i \leq n-1$ they are defined by

$$\partial_i(f) = (f - s_i f)/(\mathrm{x}_i - \mathrm{x}_{i+1})$$

while

$$\partial_0(f) = (f - s_0 f)/(2\mathrm{x}_1),$$

for any $f \in \mathbb{Z}[\mathrm{X}_n]$. For each $w \in W_n$, define an operator ∂_w by setting

$$\partial_w = \partial_{a_1} \circ \cdots \circ \partial_{a_r}$$

if $a_1 \dots a_r$ is a reduced word of w.

For every permutation $\varpi \in S_n$, Lascoux and Schützenberger [LS] defined a *type A Schubert polynomial* $\mathfrak{S}_\varpi(\mathrm{X}_n) \in \mathbb{Z}[\mathrm{X}_n]$ by

$$\mathfrak{S}_\varpi(\mathrm{X}_n) = \partial_{\varpi^{-1}\varpi_0}\left(\mathrm{x}_1^{n-1}\mathrm{x}_2^{n-2} \cdots \mathrm{x}_{n-1}\right).$$

This definition is stable under the natural inclusion of S_n into S_{n+1}, hence the polynomial $\mathfrak{S}_w$ makes sense for $w \in S_\infty = \cup_{n=1}^\infty S_n$. The set $\{\mathfrak{S}_w\}$ for $w \in S_\infty$ is a free $\mathbb{Z}$-basis of $\mathbb{Z}[\mathrm{X}] = \mathbb{Z}[\mathrm{x}_1, \mathrm{x}_2, \dots]$. Furthermore, the coefficients of $\mathfrak{S}_w$ are nonnegative integers with combinatorial significance (see e.g. [BJS, BKTY]).

1.3. **Billey-Haiman Schubert polynomials.** We regard W_n as a subgroup of W_{n+1} in the obvious way and let W_∞ denote the union $\cup_{n=1}^\infty W_n$. Let $Z = (z_1, z_2, \dots)$ be a third sequence of commuting variables. In their fundamental paper [BH], Billey and Haiman defined a family $\{\mathcal{C}_w\}_{w \in W_\infty}$ of Schubert polynomials of type C, which are a free $\mathbb{Z}$-basis of the ring $\Gamma[Z]$. The expansion coefficients for a product $\mathcal{C}_u\mathcal{C}_v$ in the basis of Schubert polynomials agree with the Schubert structure constants on symplectic flag varieties for sufficiently large n. For every $w \in W_n$ there is a unique expression

$$\mathcal{C}_w = \sum_{\substack{\lambda \text{ strict} \\ \varpi \in S_n}} e^w_{\lambda,\varpi} Q_\lambda(Y) \mathfrak{S}_\varpi(Z) \tag{1}$$

where the coefficients $e^w_{\lambda,\varpi}$ are nonnegative integers. We proceed to give a combinatorial interpretation of these numbers.

A sequence $a = (a_1, \dots, a_m)$ is called *unimodal* if for some r with $0 \leq r \leq m$, we have

$$a_1 > a_2 > \cdots > a_r < a_{r+1} < \cdots < a_m.$$

Let $w \in W_n$ and λ be a Young diagram with r rows such that $|\lambda| = \ell(w)$. A *Kraśkiewicz tableau* for w of shape λ is a filling T of the boxes of λ with nonnegative integers in such a way that

a) If t_i is the sequence of entries in the i-th row of T, reading from left to right, then the row word $t_r \dots t_1$ is a reduced word for w.

b) For each i, t_i is a unimodal subsequence of maximum length in $t_r \dots t_{i+1} t_i$.

Example 1. Let $\lambda \in \mathcal{D}_n$, $\ell = \ell(\lambda)$, and $k = n - \ell = \ell(\lambda')$. The barred permutation

$$w_\lambda = (\overline{\lambda}_1, \dots, \overline{\lambda}_\ell, \lambda'_k, \dots, \lambda'_1)$$

is the *maximal Grassmannian element* of W_n corresponding to λ. There is a unique Kraśkiewicz tableau for w_λ, which has shape λ, and is given as in the following example, for $\lambda = (7, 4, 3)$:

$$\begin{array}{l} 6\ 5\ 4\ 3\ 2\ 1\ 0 \\ 3\ 2\ 1\ 0 \\ 2\ 1\ 0. \end{array}$$

According to [BH, Theorem 3], the polynomial $\mathcal{C}_w$ satisfies

$$\mathcal{C}_w = \sum_{uv=w} F_u(Y) \mathfrak{S}_v(Z),$$

summed over all reduced factorizations $uv = w$ in W_∞ (i.e., such that $\ell(u) + \ell(v) = \ell(w)$) with $v \in S_\infty$. The left factors $F_u(Y)$ are type C Stanley symmetric functions of [BH, FK, La]. In addition, Lam [La] has shown that for any $u \in W_\infty$,

$$F_u(Y) = \sum_\lambda c^u_\lambda \, Q_\lambda(Y)$$

where c^u_λ equals the number of Kraśkiewicz tableaux for u of shape λ. By combining these two facts, we deduce the next result.

Proposition 1 (BH, La)**.** *For every $w \in W_\infty$, the coefficient $e^w_{\lambda,\varpi}$ in* (1) *is equal to the number of Kraśkiewicz tableaux for $w\varpi^{-1}$ of shape λ, if $\ell(w\varpi^{-1}) = \ell(w) - \ell(\varpi)$, and equal to zero otherwise.*

2. Symplectic Schubert polynomials

2.1. **Isotropic flags and Schubert varieties.** Consider the vector space $\mathbb{C}^{2n}$ with its canonical basis $\{e_i\}_{i=1}^{2n}$ of unit coordinate vectors. We define the *skew diagonal symplectic form* $[\ ,\ ]$ on $\mathbb{C}^{2n}$ by setting $[e_i, e_j] = 0$ for $i + j \neq 2n + 1$ and $[e_i, e_{2n+1-i}] = 1$ for $1 \leq i \leq n$. The symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ is the group of linear automorphisms of $\mathbb{C}^{2n}$ preserving the symplectic form. The upper triangular matrices in Sp_{2n} form a Borel subgroup B.

An n-dimensional subspace Σ of $\mathbb{C}^{2n}$ is called *Lagrangian* if the restriction of the symplectic form to Σ vanishes. Let $\mathfrak{X} = \mathrm{Sp}_{2n}/B$ be the variety parametrizing flags of subspaces

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n \subset E_{2n} = \mathbb{C}^{2n}$$

with $\dim E_i = i$ and E_n Lagrangian. Each such flag can be extended to a complete flag $E_\bullet$ in $\mathbb{C}^{2n}$ by letting $E_{n+i} = E_{n-i}^\perp$ for $1 \leq i \leq n$; we will call such a flag a *complete isotropic flag*. The same notation is used to denote the tautological flag $E_\bullet$ of vector bundles over $\mathfrak{X}$.

There is a group monomorphism $\phi : W_n \hookrightarrow S_{2n}$ with image

$$\phi(W_n) = \{\, \varpi \in S_{2n} \mid \varpi(i) + \varpi(2n+1-i) = 2n+1, \quad \text{for all } i \,\}.$$

The map ϕ is determined by setting, for each $w = (w_1, \dots, w_n) \in W_n$ and $1 \leq i \leq n$,

$$\phi(w)(i) = \begin{cases} n + 1 - w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n + \overline{w}_{n+1-i} & \text{otherwise.} \end{cases}$$

Let $F_\bullet$ be a fixed complete isotropic flag. For every $w \in W_n$ define the *Schubert variety* $\mathfrak{X}_w(F_\bullet) \subset \mathfrak{X}$ as the locus of $E_\bullet \in \mathfrak{X}$ such that

$$\dim(E_r \cap F_s) \geq \#\{\, i \leq r \mid \phi(w)(i) > 2n - s \,\} \quad \text{for} \quad 1 \leq r \leq n,\ 1 \leq s \leq 2n.$$

The Schubert class σ_w in $\mathrm{H}^{2\ell(w)}(\mathfrak{X}, \mathbb{Z})$ is the cohomology class which is Poincaré dual to the homology class determined by $\mathfrak{X}_w(F_\bullet)$.

According to Borel [Bo], the cohomology ring $\mathrm{H}^*(\mathfrak{X}, \mathbb{Z})$ is presented as a quotient

$$\mathrm{H}^*(\mathfrak{X}, \mathbb{Z}) \cong \mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]/I_n \tag{2}$$

where I_n is the ideal generated by all positive degree W_n-invariants in $\mathbb{Z}[\mathrm{X}_n]$, that is, $I_n = \langle e_i(\mathrm{X}_n^2),\ 1 \leq i \leq n \rangle$. The inverse of the isomorphism (2) sends the class of x_i to $-c_1(E_{n+1-i}/E_{n-i})$ for each i with $1 \leq i \leq n$.

2.2. **Symplectic Schubert classes and Schubert polynomials.** For every $\lambda \in \mathcal{G}_n$ and $\varpi \in S_n$, define the polynomial $\mathfrak{C}_{\lambda,\varpi} = \mathfrak{C}_{\lambda,\varpi}(\mathrm{X}_n)$ by

$$\mathfrak{C}_{\lambda,\varpi} = \widetilde{Q}_\lambda(\mathrm{X}_n)\mathfrak{S}_\varpi(-\mathrm{X}_n) = (-1)^{\ell(\varpi)}\widetilde{Q}_\lambda(\mathrm{X}_n)\mathfrak{S}_\varpi(\mathrm{X}_n).$$

The products $\widetilde{Q}_\lambda(\mathrm{X}_n)\mathfrak{S}_\varpi(\mathrm{X}_n)$ for $\lambda \in \mathcal{D}_n$ and $\varpi \in S_n$ form a basis for the polynomial ring $\mathbb{Z}[\mathrm{X}_n]$ as a $\mathbb{Z}[\mathrm{X}_n]^{W_n}$-module, which was introduced and studied by Lascoux and Pragacz [LP1]. We observe here that the $\mathfrak{C}_{\lambda,\varpi}(\mathrm{X}_n)$ for $\lambda \in \mathcal{G}_n$ and $\varpi \in S_n$ form a basis of $\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ as an abelian group. Moreover, properties (b) and (c) of $\widetilde{Q}$-polynomials ensure that for $\lambda \in \mathcal{G}_n \smallsetminus \mathcal{D}_n$, we have $\mathfrak{C}_{\lambda,\varpi}(\mathrm{X}_n) \in I_n$.

Definition 1. For $w \in W_n$, define the symplectic Schubert polynomial $\mathfrak{C}_w = \mathfrak{C}_w(\mathrm{X}_n)$ by

$$\mathfrak{C}_w = \sum_{\substack{\lambda \in \mathcal{D}_n \\ \varpi \in S_n}} e^w_{\lambda,\varpi}\mathfrak{C}_{\lambda,\varpi}(\mathrm{X}_n) \tag{3}$$

where the coefficients $e^w_{\lambda,\varpi}$ are the same as in (1) and Proposition 1.

Theorem 1. *The symplectic Schubert polynomial $\mathfrak{C}_w(\mathrm{X}_n)$ is the unique $\mathbb{Z}$-linear combination of the $\mathfrak{C}_{\lambda,\varpi}(\mathrm{X}_n)$ for $\lambda \in \mathcal{D}_n$ and $\varpi \in S_n$ which represents the Schubert class σ_w in the Borel presentation* (2).

Proof. For each $w \in W_n$, the Billey-Haiman polynomial $\mathcal{C}_w$ represents the Schubert class σ_w in the Borel presentation after a certain change of variables. Recall that a partition is *odd* if all its non-zero parts are odd integers. For each partition μ, let $p_\mu = \prod_i p_{\mu_i}$, where p_r denotes the r-th power sum. The $p_\mu(Y)$ for μ odd form a $\mathbb{Q}$-basis of $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. We therefore have a unique expression

$$\mathcal{C}_w = \sum_{\substack{\mu \text{ odd} \\ \varpi \in S_n}} a^w_{\mu,\varpi}\, p_\mu(Y)\mathfrak{S}_\varpi(Z) \tag{4}$$

in the ring $\Gamma[Z] \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $p_{odd} = (p_1, p_3, p_5, \ldots)$. Define a polynomial $\mathcal{C}_w(p_{odd}(\mathrm{X}), \mathrm{X}_{n-1})$ in the variables $p_k := p_k(\mathrm{X})$ for k odd and $\mathrm{x}_1, \ldots, \mathrm{x}_{n-1}$ by substituting $p_k(Y)$ with $p_k(\mathrm{X})/2$ and z_i with $-\mathrm{x}_i$ in (4). It is shown in [BH, §2] that the polynomial $\mathcal{C}_w(\mathrm{X}_n) := \mathcal{C}_w(p_{odd}(\mathrm{X}_n), \mathrm{X}_{n-1})$ obtained by setting $\mathrm{x}_i = 0$ for all $i > n$ in $\mathcal{C}_w(p_{odd}(\mathrm{X}), \mathrm{X}_{n-1})$

represents the Schubert class σ_w in the Borel presentation (2). Using Józefiak's homomorphism η from §1.1, we see that

$$\sum_{\substack{\lambda \text{ strict} \\ \varpi \in S_n}} e^w_{\lambda,\varpi} \widetilde{Q}_\lambda(\mathrm{X}) \mathfrak{S}_\varpi(-\mathrm{X}_n)$$

differs from $\mathcal{C}_w(p_{odd}(\mathrm{X}), \mathrm{X}_{n-1})$ by an element in the ideal of $\Lambda[\mathrm{X}_{n-1}]$ generated by the $e_i(\mathrm{X}^2)$ for $i > 0$. Since the ideal I_n is generated by the polynomials $e_i(\mathrm{X}_n^2)$, and $\widetilde{Q}_\lambda(\mathrm{X}_n) = 0$ whenever $\lambda_1 > n$, it follows that $\mathfrak{C}_w$ represents the Schubert class σ_w in the presentation (2), as required.

We claim that the $\mathfrak{C}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n \smallsetminus \mathcal{D}_n$ and $\varpi \in S_n$ form a free $\mathbb{Z}$-basis of I_n. To see this, note that if h is an element of I_n then $h(\mathrm{X}_n) = \sum_i e_i(\mathrm{X}_n^2) f_i(\mathrm{X}_n)$ for some polynomials $f_i \in \mathbb{Z}[\mathrm{X}_n]$. Now each f_i is a unique $\mathbb{Z}$-linear combination of the $\mathfrak{C}_{\mu,\varpi}$ for $\mu \in \mathcal{G}_n$ and $\varpi \in S_n$, and properties (b) and (c) of §1.1 give

$$e_i(\mathrm{X}_n^2)\mathfrak{C}_{\mu,\varpi}(\mathrm{X}_n) = \widetilde{Q}_{i,i}(\mathrm{X}_n)\mathfrak{C}_{\mu,\varpi}(\mathrm{X}_n) = \mathfrak{C}_{\mu\cup(i,i),\varpi}(\mathrm{X}_n).$$

We deduce that any $h \in I_n$ lies in the $\mathbb{Z}$-linear span of the $\mathfrak{C}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n \smallsetminus \mathcal{D}_n$ and $\varpi \in S_n$. Since the $\mathfrak{C}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n$ and $\varpi \in S_n$ are linearly independent, this proves the claim and the uniqueness assertion in the theorem. □

We remark that the statement of Theorem 1 may serve as an alternative definition of the symplectic Schubert polynomials $\mathfrak{C}_w(\mathrm{X}_n)$.

2.3. **Properties of symplectic Schubert polynomials.** We give below some basic properties of the polynomials $\mathfrak{C}_w(\mathrm{X}_n)$.

(a) The set

$$\{\mathfrak{C}_w \mid w \in W_n\} \cup \{\mathfrak{C}_{\lambda,\varpi} \mid \lambda \in \mathcal{G}_n \smallsetminus \mathcal{D}_n,\ \varpi \in S_n\}$$

is a free $\mathbb{Z}$-basis of the polynomial ring $\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$. The $\mathfrak{C}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n \smallsetminus \mathcal{D}_n$ and $\varpi \in S_n$ span the ideal I_n of $\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ generated by the $e_i(\mathrm{X}_n^2)$ for $1 \leq i \leq n$.

(b) For every $u, v \in W_n$, we have an equation

$$\mathfrak{C}_u \cdot \mathfrak{C}_v = \sum_{w \in W_n} c_{uv}^w\, \mathfrak{C}_w + \sum_{\substack{\lambda \in \mathcal{G}_n \smallsetminus \mathcal{D}_n \\ \varpi \in S_n}} c_{uv}^{\lambda\varpi}\, \mathfrak{C}_{\lambda,\varpi} \tag{5}$$

in the ring $\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$. The coefficients c_{uv}^w are nonnegative integers, which vanish unless $\ell(w) = \ell(u) + \ell(v)$, and agree with the structure constants in the equation of Schubert classes

$$\sigma_u \cdot \sigma_v = \sum_{w \in W_n} c_{uv}^w\, \sigma_w,$$

which holds in $\mathrm{H}^*(\mathfrak{X}, \mathbb{Z})$. The coefficients $c_{uv}^{\lambda\varpi}$ are integers, some of which may be negative. Equation (5) provides the sought for lifting of the Schubert calculus from the cohomology ring $\mathrm{H}^*(\mathfrak{X}, \mathbb{Z})$ to $\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ discussed in the Introduction.

(c) Stability property: For each $m < n$ let $i = i_{m,n} : W_m \to W_n$ be the natural embedding using the first m components. Then for any $w \in W_m$ we have

$$\mathfrak{C}_{i(w)}(\mathrm{X}_n)\big|_{x_{m+1}=\cdots=x_n=0} = \mathfrak{C}_w(\mathrm{X}_m).$$

(d) For a maximal Grassmannian element $w_\lambda \in W_n$, we have $\mathfrak{C}_w(\mathrm{X}_n) = \widetilde{Q}_\lambda(\mathrm{X}_n)$.

(e) For $\varpi \in S_n$ and $w \in W_n$, we have

$$\partial_\varpi \mathfrak{C}_w = \begin{cases} (-1)^{\ell(\varpi)}\, \mathfrak{C}_{w\varpi} & \text{if } \ell(w\varpi) = \ell(w) - \ell(\varpi), \\ 0 & \text{otherwise.} \end{cases}$$

(f) Let $v_0 = w_0\varpi_0 = (\overline{n}, \overline{n-1}, \ldots, \overline{1})$. Then for every $\varpi \in S_n$, we have

$$\mathfrak{C}_{v_0\varpi}(\mathrm{X}_n) = \widetilde{Q}_{\rho_n}(\mathrm{X}_n)\mathfrak{S}_\varpi(-\mathrm{X}_n),$$

where $\rho_n = (n, n-1, \ldots, 1)$. In particular, for the element $w_0 \in W_n$ of longest length we have

$$\mathfrak{C}_{w_0}(\mathrm{X}_n) = \widetilde{Q}_{\rho_n}(\mathrm{X}_n)\mathfrak{S}_{\varpi_0}(-\mathrm{X}_n) = (-1)^{n(n-1)/2}\mathrm{x}_1^{n-1}\mathrm{x}_2^{n-2}\cdots \mathrm{x}_{n-1}\widetilde{Q}_{\rho_n}(\mathrm{X}_n).$$

Thus $\mathfrak{C}_{w_0}(\mathrm{X}_n)$ agrees with the symplectic Schubert polynomial of Lascoux, Pragacz, and Ratajski [LP1, Appendix A] indexed by w_0. However these two families of polynomials do not coincide, because unlike the Schubert polynomials of loc. cit., the $\mathfrak{C}_w$ do not respect the divided difference operator ∂_0.

(g) To any $w \in W_n$ we associate a strict partition μ whose parts are the absolute values of the negative entries of w. Let $v \in S_n$ be the permutation of maximal length such that there exists a factorization $w = uv$ with $\ell(w) = \ell(u) + \ell(v)$ for some $u \in W_n$. Then $\mathfrak{C}_{\mu,v}$ is the unique homogeneous summand $e^w_{\lambda,\varpi}\mathfrak{C}_{\lambda,\varpi}$ of $\mathfrak{C}_w$ in equation (3) with the weight $|\lambda|$ as small as possible.

(h) Lascoux and Pragacz [LP1, §1] define a $\mathbb{Z}[\mathrm{X}_n]^{W_n}$-linear scalar product

$$\langle\ ,\ \rangle : \mathbb{Z}[\mathrm{X}_n] \times \mathbb{Z}[\mathrm{X}_n] \longrightarrow \mathbb{Z}[\mathrm{X}_n]^{W_n}$$

by

$$\langle f, g\rangle = (-1)^{n(n-1)/2}\partial_{w_0}(fg)$$

for any $f, g \in \mathbb{Z}[\mathrm{X}_n]$. The set of symplectic Schubert polynomials $\{\mathfrak{C}_w(\mathrm{X}_n)\}_{w\in W_n}$ and the polynomials $\{\mathfrak{C}_{\lambda,\varpi}(\mathrm{X}_n)\}_{\lambda\in\mathcal{D}_n,\varpi\in S_n}$ form two bases for the polynomial ring $\mathbb{Z}[\mathrm{X}_n]$ as a $\mathbb{Z}[\mathrm{X}_n^2]$-module. If $\lambda, \mu \in \mathcal{D}_n$ and $\rho, \pi \in S_n$ are such that $\ell(\rho) + \ell(\pi) \leq n(n-1)/2$, then we have the orthogonality relation

$$\langle \mathfrak{C}_{\lambda,\rho}, \mathfrak{C}_{\mu,\pi}\rangle = \begin{cases} 1 & \text{if } \ \mu = \lambda' \ \text{ and } \ \pi = \varpi_0\rho, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Furthermore, if $u, v \in W_n$ are such that $\ell(u) + \ell(v) \leq n^2$, then we have

$$\langle \mathfrak{C}_u, \mathfrak{C}_v\rangle = \begin{cases} 1 & \text{if } \ v = w_0u, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The proof of Theorem 1 established that the $\mathfrak{C}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n \smallsetminus \mathcal{D}_n$ and $\varpi \in S_n$ form a basis of I_n. The remaining statements in properties (a) and (b) follow because the $\mathfrak{C}_w$ for $w \in W_n$ form a basis of $\mathbb{Z}[\mathrm{X}_n]/I_n$. Properties (c), (d), (e), (g) may be derived from the corresponding properties of the Billey-Haiman Schubert polynomials, and (f) is a consequence of (e).

Equation (6) is proved by factoring the divided difference operator $\partial_{w_0} = \partial_{v_0}\partial_{\varpi_0}$, where $v_0 = w_0\varpi_0$, noting that

$$\langle \mathfrak{C}_{\lambda,\rho}(\mathrm{X}_n), \mathfrak{C}_{\mu,\pi}(\mathrm{X}_n)\rangle = (-1)^{n(n-1)/2}\partial_{v_0}(\widetilde{Q}_\lambda(\mathrm{X}_n)\widetilde{Q}_\mu(\mathrm{X}_n))\partial_{\varpi_0}(\mathfrak{S}_\rho(-\mathrm{X}_n)\mathfrak{S}_\pi(-\mathrm{X}_n)),$$

and using the corresponding properties of the inner products defined by ∂_{v_0} and ∂_{ϖ_0}, which are derived in [LP1, (10)] and [M, (5.4)], respectively. The relation (7) may be deduced from (6) and property (g) above, or by using the fact that the Schubert classes $\sigma_u = \mathfrak{C}_u(\mathrm{X}_n)$ and $\sigma_v = \mathfrak{C}_v(\mathrm{X}_n)$ satisfy

$$\int_{\mathfrak{X}} \sigma_u \cdot \sigma_v = \delta_{u,w_0 v}.$$

Example 2. The list of all symplectic Schubert polynomials $\mathfrak{C}_w$ for $w \in W_3$ is given in Table 1. These polynomials are displayed according to the eight orbits of the symmetric group S_3 on W_3. The Schubert polynomials in each orbit are easily computed from the unique one of highest degree by applying type A divided difference operators, using property (e). The reader should compare this table with [BH, Table 2] and [LP1, Appendix A].

Example 3. Let $n = 6$ and consider the (non-maximal) Grassmannian element $w = 126\overline{5}\overline{4}3$. The symplectic Schubert polynomial for w is given by

$$\begin{aligned}\mathfrak{C}_w = {} & \widetilde{Q}_{651}\,\mathfrak{S}_{145236} - (\widetilde{Q}_{65} + \widetilde{Q}_{641})\,\mathfrak{S}_{245136} - \widetilde{Q}_{65}\,\mathfrak{S}_{146235} \\ & + (\widetilde{Q}_{64} + \widetilde{Q}_{541})\,\mathfrak{S}_{345126} + \widetilde{Q}_{64}\,\mathfrak{S}_{246135} - \widetilde{Q}_{54}\,\mathfrak{S}_{346125}.\end{aligned}$$

The corresponding Billey-Haiman Schubert polynomial is given by

$$\begin{aligned}\mathcal{C}_w = {} & Q_{871} + (Q_{87} + Q_{861})\,\mathfrak{S}_{124356} + (Q_{86} + Q_{851})\,\mathfrak{S}_{134256} + (Q_{86} + Q_{761})\,\mathfrak{S}_{125346} \\ & + (Q_{85} + Q_{841})\,\mathfrak{S}_{234156} + (Q_{85} + Q_{751} + Q_{76})\,\mathfrak{S}_{135246} + Q_{76}\,\mathfrak{S}_{126345} \\ & + (Q_{84} + Q_{75} + Q_{741})\,\mathfrak{S}_{235146} + (Q_{75} + Q_{651})\,\mathfrak{S}_{145236} + Q_{75}\,\mathfrak{S}_{136245} \\ & + (Q_{74} + Q_{65} + Q_{641})\,\mathfrak{S}_{245136} + Q_{74}\,\mathfrak{S}_{236145} + Q_{65}\,\mathfrak{S}_{146235} \\ & + (Q_{64} + Q_{541})\,\mathfrak{S}_{345126} + Q_{64}\,\mathfrak{S}_{246135} + Q_{54}\,\mathfrak{S}_{346125}.\end{aligned}$$

3. Hermitian differential geometry

3.1. **Bott-Chern forms.** In this section X denotes a complex manifold, and $A^{p,q}(X)$ is the space of $\mathbb{C}$-valued smooth differential forms of type (p,q) on X. Let $A(X) = \bigoplus_p A^{p,p}(X)$ and $A'(X) \subset A(X)$ be the set of forms φ in $A(X)$ which can be written as $\varphi = \partial\eta + \overline{\partial}\eta'$ for some smooth forms η, η'. Define $\widetilde{A}(X) = A(X)/A'(X)$. Observe that the operator $dd^c : \widetilde{A}(X) \to A(X)$ is well defined, as is the cup product $\wedge\omega : \widetilde{A}(X) \to \widetilde{A}(X)$ for any closed form ω in $A(X)$.

A *hermitian vector bundle* on X is a pair $\overline{E} = (E, h)$ consisting of a holomorphic vector bundle E over X and a hermitian metric h on E. Let D be the hermitian holomorphic connection of $\overline{E}$, with curvature $K = D^2 \in A^{1,1}(X, \mathrm{End}(E))$, and let n denote the rank of E. For each integer k with $1 \leq k \leq n$, there is an associated Chern form $c_k(\overline{E}) := \mathrm{Tr}(\bigwedge^k(\frac{i}{2\pi}K)) \in A^{k,k}(X)$, defined locally by identifying $\mathrm{End}(E)$ with $M_n(\mathbb{C})$. We also have the total Chern form $c(\overline{E}) = 1 + \sum_{k=1}^n c_k(\overline{E})$. These differential forms are d and d^c closed, and their classes in the de Rham cohomology of X are the usual Chern classes of E.

Let $\mathfrak{r} = (1 \leq r_1 < r_2 < \ldots < r_m = n)$ be an increasing sequence of natural numbers. A *hermitian filtration* $\overline{\mathcal{E}}$ *of type* $\mathfrak{r}$ is a filtration

$$\mathcal{E} :\ 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E \tag{8}$$

of E by subbundles E_i with $\mathrm{rank}(E_i) = r_i$ for $1 \leq i \leq m$, together with a choice of hermitian metrics on E and on each quotient bundle $Q_i = E_i/E_{i-1}$. We say that

TABLE 1. Symplectic Schubert polynomials for $w \in W_3$

w	$\mathfrak{C}_w(\mathrm{X}_3) = \sum e^w_{\lambda,\varpi}\,\widetilde{Q}_\lambda(\mathrm{X}_3)\,\mathfrak{S}_\varpi(-\mathrm{X}_3)$
$123 = 1$	1
$213 = s_1$	$\widetilde{Q}_1 - \mathfrak{S}_{213}$
$132 = s_2$	$\widetilde{Q}_1 - \mathfrak{S}_{132}$
$231 = s_1s_2$	$\widetilde{Q}_2 - \widetilde{Q}_1\,\mathfrak{S}_{132} + \mathfrak{S}_{231}$
$312 = s_2s_1$	$\widetilde{Q}_2 - \widetilde{Q}_1\,\mathfrak{S}_{213} + \mathfrak{S}_{312}$
$321 = s_1s_2s_1$	$\widetilde{Q}_3 + \widetilde{Q}_{21} - \widetilde{Q}_2\,\mathfrak{S}_{213} - \widetilde{Q}_2\,\mathfrak{S}_{132} + \widetilde{Q}_1\,\mathfrak{S}_{312} + \widetilde{Q}_1\,\mathfrak{S}_{231} - \mathfrak{S}_{321}$
$\overline{1}23 = s_0$	$\widetilde{Q}_1$
$2\overline{1}3 = s_0s_1$	$\widetilde{Q}_2 - \widetilde{Q}_1\,\mathfrak{S}_{213}$
$\overline{1}32 = s_0s_2$	$2\,\widetilde{Q}_2 - \widetilde{Q}_1\,\mathfrak{S}_{132}$
$23\overline{1} = s_0s_1s_2$	$\widetilde{Q}_3 - \widetilde{Q}_2\,\mathfrak{S}_{132} + \widetilde{Q}_1\,\mathfrak{S}_{231}$
$3\overline{1}2 = s_0s_2s_1$	$\widetilde{Q}_3 + \widetilde{Q}_{21} - 2\,\widetilde{Q}_2\,\mathfrak{S}_{213} + \widetilde{Q}_1\,\mathfrak{S}_{312}$
$32\overline{1} = s_0s_1s_2s_1$	$\widetilde{Q}_{31} - \widetilde{Q}_3\,\mathfrak{S}_{213} - \widetilde{Q}_3\,\mathfrak{S}_{132} - \widetilde{Q}_{21}\,\mathfrak{S}_{132} + \widetilde{Q}_2\,\mathfrak{S}_{312} + 2\,\widetilde{Q}_2\,\mathfrak{S}_{231} - \widetilde{Q}_1\,\mathfrak{S}_{321}$
$\overline{2}13 = s_1s_0$	$\widetilde{Q}_2$
$1\overline{2}3 = s_1s_0s_1$	$\widetilde{Q}_3 - \widetilde{Q}_2\,\mathfrak{S}_{213}$
$\overline{2}31 = s_1s_0s_2$	$\widetilde{Q}_3 + \widetilde{Q}_{21} - \widetilde{Q}_2\,\mathfrak{S}_{132}$
$13\overline{2} = s_1s_0s_1s_2$	$-\widetilde{Q}_3\,\mathfrak{S}_{132} + \widetilde{Q}_2\,\mathfrak{S}_{231}$
$3\overline{2}1 = s_1s_0s_2s_1$	$\widetilde{Q}_{31} - \widetilde{Q}_3\,\mathfrak{S}_{213} - \widetilde{Q}_{21}\,\mathfrak{S}_{213} + \widetilde{Q}_2\,\mathfrak{S}_{312}$
$31\overline{2} = s_1s_0s_1s_2s_1$	$-\widetilde{Q}_{31}\,\mathfrak{S}_{132} + \widetilde{Q}_3\,\mathfrak{S}_{312} + \widetilde{Q}_3\,\mathfrak{S}_{231} + \widetilde{Q}_{21}\,\mathfrak{S}_{231} - \widetilde{Q}_2\,\mathfrak{S}_{321}$
$\overline{3}12 = s_2s_1s_0$	$\widetilde{Q}_3$
$1\overline{3}2 = s_2s_1s_0s_1$	$-\widetilde{Q}_3\,\mathfrak{S}_{213}$
$\overline{3}21 = s_2s_1s_0s_2$	$\widetilde{Q}_{31} - \widetilde{Q}_3\,\mathfrak{S}_{132}$
$12\overline{3} = s_2s_1s_0s_1s_2$	$\widetilde{Q}_3\,\mathfrak{S}_{231}$
$2\overline{3}1 = s_2s_1s_0s_2s_1$	$-\widetilde{Q}_{31}\,\mathfrak{S}_{213} + \widetilde{Q}_3\,\mathfrak{S}_{312}$
$21\overline{3} = s_2s_1s_0s_1s_2s_1$	$\widetilde{Q}_{31}\,\mathfrak{S}_{231} - \widetilde{Q}_3\,\mathfrak{S}_{321}$
$\overline{21}3 = s_0s_1s_0$	$\widetilde{Q}_{21}$
$\overline{12}3 = s_0s_1s_0s_1$	$\widetilde{Q}_{31} - \widetilde{Q}_{21}\,\mathfrak{S}_{213}$
$\overline{2}3\overline{1} = s_0s_1s_0s_2$	$\widetilde{Q}_{31} - \widetilde{Q}_{21}\,\mathfrak{S}_{132}$
$\overline{1}3\overline{2} = s_0s_1s_0s_1s_2$	$-\widetilde{Q}_{31}\,\mathfrak{S}_{132} + \widetilde{Q}_{21}\,\mathfrak{S}_{231}$
$3\overline{21} = s_0s_1s_0s_2s_1$	$\widetilde{Q}_{32} - \widetilde{Q}_{31}\,\mathfrak{S}_{213} + \widetilde{Q}_{21}\,\mathfrak{S}_{312}$
$3\overline{12} = s_0s_1s_0s_1s_2s_1$	$-\widetilde{Q}_{32}\,\mathfrak{S}_{132} + \widetilde{Q}_{31}\,\mathfrak{S}_{312} + \widetilde{Q}_{31}\,\mathfrak{S}_{231} - \widetilde{Q}_{21}\,\mathfrak{S}_{321}$
$\overline{31}2 = s_0s_2s_1s_0$	$\widetilde{Q}_{31}$
$\overline{13}2 = s_0s_2s_1s_0s_1$	$-\widetilde{Q}_{31}\,\mathfrak{S}_{213}$
$\overline{3}2\overline{1} = s_0s_2s_1s_0s_2$	$\widetilde{Q}_{32} - \widetilde{Q}_{31}\,\mathfrak{S}_{132}$
$\overline{1}2\overline{3} = s_0s_2s_1s_0s_1s_2$	$\widetilde{Q}_{31}\,\mathfrak{S}_{231}$
$2\overline{31} = s_0s_2s_1s_0s_2s_1$	$-\widetilde{Q}_{32}\,\mathfrak{S}_{213} + \widetilde{Q}_{31}\,\mathfrak{S}_{312}$
$2\overline{13} = s_0s_2s_1s_0s_1s_2s_1$	$\widetilde{Q}_{32}\,\mathfrak{S}_{231} - \widetilde{Q}_{31}\,\mathfrak{S}_{321}$
$\overline{32}1 = s_1s_0s_2s_1s_0$	$\widetilde{Q}_{32}$
$\overline{23}1 = s_1s_0s_2s_1s_0s_1$	$-\widetilde{Q}_{32}\,\mathfrak{S}_{213}$
$\overline{3}1\overline{2} = s_1s_0s_2s_1s_0s_2$	$-\widetilde{Q}_{32}\,\mathfrak{S}_{132}$
$\overline{2}1\overline{3} = s_1s_0s_2s_1s_0s_1s_2$	$\widetilde{Q}_{32}\,\mathfrak{S}_{231}$
$1\overline{32} = s_1s_0s_2s_1s_0s_2s_1$	$\widetilde{Q}_{32}\,\mathfrak{S}_{312}$
$1\overline{23} = s_1s_0s_2s_1s_0s_1s_2s_1$	$-\widetilde{Q}_{32}\,\mathfrak{S}_{321}$
$\overline{321} = s_0s_1s_0s_2s_1s_0$	$\widetilde{Q}_{321}$
$\overline{231} = s_0s_1s_0s_2s_1s_0s_1$	$-\widetilde{Q}_{321}\,\mathfrak{S}_{213}$
$\overline{312} = s_0s_1s_0s_2s_1s_0s_2$	$-\widetilde{Q}_{321}\,\mathfrak{S}_{132}$
$\overline{213} = s_0s_1s_0s_2s_1s_0s_1s_2$	$\widetilde{Q}_{321}\,\mathfrak{S}_{231}$
$\overline{132} = s_0s_1s_0s_2s_1s_0s_2s_1$	$\widetilde{Q}_{321}\,\mathfrak{S}_{312}$
$\overline{123} = s_0s_1s_0s_2s_1s_0s_1s_2s_1$	$-\widetilde{Q}_{321}\,\mathfrak{S}_{321}$

$\overline{\mathcal{E}}$ is *split* if, when E_i is given the induced metric from E for each i, the sequences

$$\overline{\mathcal{E}}_i \;:\; 0 \to \overline{E}_{i-1} \to \overline{E}_i \to \overline{Q}_i \to 0$$

are split, for $1 \leq i \leq m$. In this case we have an orthogonal splitting $\overline{E} = \bigoplus_{i=1}^{m} \overline{Q}_i$.

In [T2, Theorem 1] we showed that there is a unique way to attach to every hermitian filtration of type $\mathfrak{r}$ a form $\widetilde{c}(\overline{\mathcal{E}})$ in $\widetilde{A}(X)$ in such a way that:

(i) $dd^c\widetilde{c}(\overline{\mathcal{E}}) = c(\bigoplus_{i=1}^{m} \overline{Q}_i) - c(\overline{E})$,

(ii) For every map $f : Y \to X$ of complex manifolds, $\widetilde{c}(f^*(\overline{\mathcal{E}})) = f^*\widetilde{c}(\overline{\mathcal{E}})$,

(iii) If $\overline{\mathcal{E}}$ is *split*, then $\widetilde{c}(\overline{\mathcal{E}}) = 0$.

The differential form $\widetilde{c}(\overline{\mathcal{E}})$ is called the *Bott-Chern form* of the hermitian filtration $\mathcal{E}$ corresponding to the total Chern class. If $m = 2$, i.e., if the filtration $\mathcal{E}$ has length two, then $\widetilde{c}(\overline{\mathcal{E}})$ coincides with the Bott-Chern class $\widetilde{c}(0 \to \overline{Q}_1 \to \overline{E} \to \overline{Q}_2 \to 0)$ defined in [BC, GS2].

Suppose we are given a hermitian vector bundle $\overline{E}$ and a filtration $\mathcal{E}$ of E by subbundles as in (8). Assume that the subbundles E_i are given metrics induced from the hermitian metric on E and that the quotient bundles Q_i are then given the metrics induced from the exact sequences $\overline{\mathcal{E}}_i$. Consider a local holomorphic orthonormal frame for E such that the first r_i elements generate E_i, and let $K(\overline{E}_i)$ and $K(\overline{Q}_i)$ be the curvature matrices of $\overline{E}_i$ and $\overline{Q}_i$ with respect to the chosen frame. Let $K_{E_i} = \frac{i}{2\pi}K(\overline{E}_i)$ and $K_{Q_i} = \frac{i}{2\pi}K(\overline{Q}_i)$. Then we have the following result.

Proposition 2 (T2)**.** *The Bott-Chern form $\widetilde{c}(\overline{\mathcal{E}})$ is a polynomial in the entries of the matrices K_{E_i} and K_{Q_i}, $1 \leq i \leq m$, with* rational *coefficients.*

The polynomial in Proposition 2 may be computed by an explicit algorithm. In fact, one has the equation

$$\widetilde{c}(\overline{\mathcal{E}}) = \sum_{i=2}^{m} \widetilde{c}(\overline{\mathcal{E}}_i) \wedge c(\overline{Q}_{i+1}) \wedge \cdots \wedge c(\overline{Q}_m)$$

and the Bott-Chern forms $\widetilde{c}(\overline{\mathcal{E}}_i)$ can be evaluated using the formulas in [T2, §3]. In particular, we have $\widetilde{c}_1(\overline{\mathcal{E}}) = 0$ and $\widetilde{c}_2(\overline{\mathcal{E}}) = \sum_{i=2}^{m} \widetilde{c}_2(\overline{\mathcal{E}}_i)$, while $\widetilde{c}_p(\overline{\mathcal{E}}) = 0$ for $p > \mathrm{rank}(E)$.

Suppose that $\overline{E}$ is flat and $m = 2$, so that $\overline{\mathcal{E}}$ is equivalent to a short exact sequence

$$\overline{\mathcal{E}} \;:\; 0 \to \overline{E}_1 \to \overline{E} \to \overline{Q}_1 \to 0$$

of hermitian vector bundles, with metrics induced from $\overline{E}$. In this situation, according to [T1, Prop. 3], we have

$$\widetilde{c}_k(\overline{\mathcal{E}}) = (-1)^{k-1}\mathcal{H}_{k-1}p_{k-1}(\overline{Q}_1). \tag{9}$$

Here $p_r(\overline{Q}_1) = \mathrm{Tr}((K_{Q_1})^r)$ denotes the r-th power sum form of $\overline{Q}_1$, while $\mathcal{H}_0 = 0$ and $\mathcal{H}_r = 1 + \frac{1}{2} + \cdots + \frac{1}{r}$ is a harmonic number for $r > 0$.

3.2. **Curvature of homogeneous vector bundles.** To simplify the notation in this section, we will redefine the group $\mathrm{Sp}_{2n}(\mathbb{C})$ using the *standard symplectic form* $[\ ,\ ]'$ on $\mathbb{C}^{2n}$ whose matrix $[e_i, e_j]'_{i,j}$ on unit coordinate vectors is $\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$, where Id_n denotes the $n\times n$ identity matrix. Let $\mathfrak{X} = \mathrm{Sp}_{2n}/B$ be the symplectic flag variety and $E_\bullet$ its tautological complete isotropic flag of vector bundles. We equip the trivial vector bundle $E_{2n} = \mathbb{C}^{2n}_{\mathfrak{X}}$ with the trivial hermitian metric h compatible with the symplectic form $[\ ,\ ]'$ on $\mathbb{C}^{2n}$. More precisely, view the quaternion algebra $\mathbb{H}$ as $\mathbb{C}\oplus j\mathbb{C}$ and $\mathbb{C}^{2n} = \mathbb{C}^n \oplus j\mathbb{C}^n$ as a right $\mathbb{H}$-module. Then the metric h may be defined by the equation $h(v,w) = -[vj, w]'$ (see for example [FH, §7.2]).

The metric on E induces metrics on all the subbundles E_i and the quotient line bundles $Q_i = E_i/E_{i-1}$, for $1 \leq i \leq n$. Our goal here is to compute the $\mathrm{Sp}(2n)$-invariant curvature matrices of the homogeneous vector bundles $\overline{E}_i$ and $\overline{Q}_i$ for $1 \leq i \leq n$. Following [GrS, §4] and [T2, §5], this may be done by pulling back these matrices of $(1,1)$-forms from $\mathfrak{X}$ to the compact Lie group $\mathrm{Sp}(2n)$, where their entries may be expressed in terms of the basic invariant forms on $\mathrm{Sp}(2n)$.

The Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{C})$ is given by

$$\mathfrak{sp}(2n,\mathbb{C}) = \{(A,B,C) \mid A,B,C \in M_n(\mathbb{C}),\ B, C \text{ symmetric}\},$$

where (A,B,C) denotes the matrix $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$. Complex conjugation of the algebra $\mathfrak{sp}(2n,\mathbb{C})$ with respect to the Lie algebra of $\mathrm{Sp}(2n)$ is given by the map τ with $\tau(A) = -\overline{A}^t$. The Cartan subalgebra $\mathfrak{h}$ consists of all matrices of the form $\{(\mathrm{diag}(t_1,\ldots,t_n),0,0) \mid t_i \in \mathbb{C}\}$, where $\mathrm{diag}(t_1,\ldots,t_n)$ denotes a diagonal matrix. Consider the set of roots

$$R = \{\pm t_i \pm t_j \mid i \neq j\} \cup \{\pm 2t_i\} \subset \mathfrak{h}^*$$

and a system of positive roots

$$R^+ = \{t_i - t_j \mid i < j\} \cup \{t_p + t_q \mid p \leq q\},$$

where the indices run from 1 to n. We use ij to denote a positive root in the first set and pq for a positive root in the second. The corresponding basis vectors are $e_{ij} = (E_{ij},0,0)$, $e^{pq} = (0, E_{pq}+E_{qp}, 0)$ for $p<q$, and $e^{pp} = (0,E_{pp},0)$, where E_{ij} is the matrix with 1 as the ij-th entry and zeroes elsewhere.

Define $\overline{e}_{ij} = \tau(e_{ij})$, $\overline{e}^{pq} = \tau(e^{pq})$, and consider the linearly independent set

$$\mathcal{B}' = \{e_{ij},\, \overline{e}_{ij},\, e^{pq},\, \overline{e}^{pq} \mid i<j,\ p \leq q\}.$$

The adjoint representation of $\mathfrak{h}$ on $\mathfrak{sp}(2n,\mathbb{C})$ gives a root space decomposition

$$\mathfrak{sp}(2n,\mathbb{C}) = \mathfrak{h} \oplus \sum_{i<j}(\mathbb{C}\, e_{ij} \oplus \mathbb{C}\,\overline{e}_{ij}) \oplus \sum_{p\leq q}(\mathbb{C}\, e^{pq} \oplus \mathbb{C}\,\overline{e}^{pq}).$$

Extend $\mathcal{B}'$ to a basis $\mathcal{B}$ of $\mathfrak{sp}(2n,\mathbb{C})$ and let $\mathcal{B}^*$ denote the dual basis of $\mathfrak{sp}(2n,\mathbb{C})^*$. Let ω^{ij}, $\overline{\omega}^{ij}$, ω_{pq}, $\overline{\omega}_{pq}$ be the vectors in $\mathcal{B}^*$ which are dual to e_{ij}, $\overline{e}_{ij}$, e^{pq}, $\overline{e}^{pq}$, respectively; we regard these elements as left invariant complex one-forms on $\mathrm{Sp}(2n)$. If $p>q$ we agree that $\omega_{pq} = \omega_{qp}$ and $\overline{\omega}_{pq} = \overline{\omega}_{qp}$. Finally, define $\omega_{ij} = \gamma\omega^{ij}$, $\overline{\omega}_{ij} = \gamma\overline{\omega}^{ij}$, $\omega^{pq} = \gamma\omega_{pq}$, and $\overline{\omega}^{pq} = \gamma\overline{\omega}_{pq}$, where γ is a constant such that $\gamma^2 = \frac{i}{2\pi}$, and set $\Omega_{ij} = \omega_{ij}\wedge\overline{\omega}_{ij}$ and $\Omega^{pq} = \omega^{pq}\wedge\overline{\omega}^{pq}$.

If $\pi : \mathrm{Sp}(2n) \to \mathfrak{X}$ denotes the quotient map, the pullbacks of the aforementioned curvature matrices under π can now be written explicitly, following [GrS, $(4.13)_X$] and [T2, §5]. The result is recorded in the following proposition.

Proposition 3. *For every k with $1 \leq k \leq n$ we have*

$$c_1(\overline{Q}_k) = \sum_{i<k} \Omega_{ik} - \sum_{j>k} \Omega_{kj} - \sum_{p=1}^{n} \Omega^{pk}$$

and $K_{E_k} = \{\Theta_{\alpha\beta}\}_{1\leq\alpha,\beta\leq k}$, where

$$\Theta_{\alpha\beta} = -\sum_{j>k} \omega_{\alpha j} \wedge \overline{\omega}_{\beta j} - \sum_{p=1}^{n} \omega^{p\alpha} \wedge \overline{\omega}^{p\beta}.$$

Let $\Omega = \bigwedge_{i<j} \Omega_{ij} \wedge \bigwedge_{p\leq q} \Omega^{pq}$. Since the class of a point in $\mathfrak{X}$ is Poincaré dual to $\prod_{k=1}^{n} c_1(\overline{Q}_k^*)^{2n-2k+1}$ (see e.g. [PR, Cor. 5.6]) we deduce that $\int_{\mathfrak{X}} \Omega = \prod_{k=1}^{n} \frac{1}{(2k-1)!}$.

4. Arithmetic intersection theory on Sp_{2n}/B

4.1. **Symplectic flag varieties over** $\mathrm{Spec}\,\mathbb{Z}$**.** For the rest of this paper, $\mathfrak{X}$ will denote the Chevalley scheme over $\mathbb{Z}$ for the homogeneous space Sp_{2n}/B described in §2.1. The scheme $\mathfrak{X}$ parametrizes complete isotropic flags $E_\bullet$ of a $2n$-dimensional vector space E equipped with the skew diagonal symplectic form, over any base field. The arithmetic symplectic flag variety $\mathfrak{X}$ is smooth over $\mathrm{Spec}\,\mathbb{Z}$, and has a decomposition into Schubert cells induced by the Bruhat decomposition of Sp_{2n} (see e.g. [Ja, §13.3] for details).

There is a tautological complete isotropic flag of vector bundles

$$E_\bullet :\ 0 = E_0 \subset E_1 \subset \cdots \subset E_{2n} = E$$

over $\mathfrak{X}$. For each i with $1 \leq i \leq 2n$ we let $\mathcal{E}_i$ denote the short exact sequence

$$\mathcal{E}_i \ :\ 0 \to E_{i-1} \to E_i \to Q_i \to 0.$$

If $\mathrm{CH}(\mathfrak{X})$ is the Chow ring of algebraic cycles on $\mathfrak{X}$ modulo rational equivalence, then the class map induces an isomorphism $\mathrm{CH}(\mathfrak{X}) \cong \mathrm{H}^*(\mathfrak{X}(\mathbb{C}), \mathbb{Z})$, following [F4, Ex. 19.1.11] and [KM, Lem. 6]. We deduce that there is a ring presentation $\mathrm{CH}(\mathfrak{X}) \cong \mathbb{Z}[\mathrm{X}_n]/I_n$. The relations $e_i(\mathrm{X}_n^2)$ in I_n come from the Whitney sum formula applied to the filtration $E_\bullet$. This gives a Chern class equation $\prod_{i=1}^{2n}(1 + c_1(Q_i)) = c(E)$ in $\mathrm{CH}(\mathfrak{X})$, which maps to the identity $\prod_{i=1}^{2n}(1 - \mathrm{x}_i^2) = 1$, since E is a trivial bundle.

We have an isomorphism of abelian groups

$$\mathrm{CH}(\mathfrak{X}) \cong \bigoplus_{w\in W_n} \mathbb{Z}\,\mathfrak{C}_w(\mathrm{X}_n)$$

where the polynomial $\mathfrak{C}_w(\mathrm{X}_n)$ represents the class of the codimension $\ell(w)$ Schubert scheme $\mathfrak{X}_w$ in $\mathfrak{X}$. The latter is defined as the closure of the corresponding Schubert cell, so that $\mathfrak{X}_w(\mathbb{C})$ is given as in §2.1.

4.2. **The arithmetic Chow group.** For $p \geq 0$ we let $\widehat{\mathrm{CH}}^p(\mathfrak{X})$ denote the p-th arithmetic Chow group of $\mathfrak{X}$, in the sense of Gillet and Soulé [GS1]. The elements in $\widehat{\mathrm{CH}}^p(\mathfrak{X})$ are represented by arithmetic cycles (Z, g_Z), where Z is a codimension p cycle on $\mathfrak{X}$ and g_Z is a current of type $(p-1, p-1)$ such that the current $dd^c g_Z + \delta_{Z(\mathbb{C})}$ is represented by a smooth differential form on $\mathfrak{X}(\mathbb{C})$.

We let F_∞ be the involution of $\mathfrak{X}(\mathbb{C})$ induced by complex conjugation. Let $A^{p,p}(\mathfrak{X}_{\mathbb{R}})$ be the subspace of $A^{p,p}(\mathfrak{X}(\mathbb{C}))$ generated by real forms η such that $F_\infty^*\eta = (-1)^p\eta$; denote by $\widetilde{A}^{p,p}(\mathfrak{X}_{\mathbb{R}})$ the image of $A^{p,p}(\mathfrak{X}_{\mathbb{R}})$ in $\widetilde{A}^{p,p}(\mathfrak{X}(\mathbb{C}))$. Let $A(\mathfrak{X}_{\mathbb{R}}) = \bigoplus_p A^{p,p}(\mathfrak{X}_{\mathbb{R}})$ and $\widetilde{A}(\mathfrak{X}_{\mathbb{R}}) = \bigoplus_p \widetilde{A}^{p,p}(\mathfrak{X}_{\mathbb{R}})$.

Since the homogeneous space $\mathfrak{X}$ admits a cellular decomposition, it follows as in e.g. [KM] that for each p, there is an exact sequence

$$0 \longrightarrow \widetilde{A}^{p-1,p-1}(\mathfrak{X}_{\mathbb{R}}) \xrightarrow{a} \widehat{\mathrm{CH}}^p(\mathfrak{X}) \xrightarrow{\zeta} \mathrm{CH}^p(\mathfrak{X}) \longrightarrow 0, \tag{10}$$

where the maps a and ζ are defined by

$$a(\eta) = (0,\eta) \qquad \text{and} \qquad \zeta(Z, g_Z) = Z.$$

Summing (10) over all p gives the sequence

$$0 \longrightarrow \widetilde{A}(\mathfrak{X}_{\mathbb{R}}) \xrightarrow{a} \widehat{\mathrm{CH}}(\mathfrak{X}) \xrightarrow{\zeta} \mathrm{CH}(\mathfrak{X}) \longrightarrow 0. \tag{11}$$

We equip $E(\mathbb{C})$ with the trivial hermitian metric compatible with the skew diagonal symplectic form $[\ ,\ ]$ on $\mathbb{C}^{2n}$. This metric induces metrics on (the complex points of) all the vector bundles E_i and the line bundles $L_i = E_{n+1-i}/E_{n-i}$, for $1 \leq i \leq n$. We thus obtain hermitian vector bundles $\overline{E}_i$ and line bundles $\overline{L}_i$, together with their arithmetic Chern classes $\widehat{c}_k(\overline{E}_i) \in \widehat{\mathrm{CH}}^k(\mathfrak{X})$ and $\widehat{c}_1(\overline{L}_i) \in \widehat{\mathrm{CH}}^1(\mathfrak{X})$, according to [GS2]. Set $\widehat{x}_i = -\widehat{c}_1(\overline{L}_i)$ and for any $w \in W_n$, define

$$\widehat{\mathfrak{C}}_w := \mathfrak{C}_w(\widehat{x}_1, \ldots, \widehat{x}_n) \in \widehat{\mathrm{CH}}^{\ell(w)}(\mathfrak{X}).$$

The unique map of abelian groups

$$\epsilon\ :\ \mathrm{CH}(\mathfrak{X}) \to \widehat{\mathrm{CH}}(\mathfrak{X}) \tag{12}$$

sending the Schubert class $\mathfrak{C}_w(\mathrm{X}_n)$ to $\widehat{\mathfrak{C}}_w$ for all $w \in W_n$ is then a splitting of (11). We thus obtain an isomorphism of abelian groups

$$\widehat{\mathrm{CH}}(\mathfrak{X}) \cong \mathrm{CH}(\mathfrak{X}) \oplus \widetilde{A}(\mathfrak{X}_{\mathbb{R}}). \tag{13}$$

4.3. **Computing arithmetic intersections.** We now describe an effective procedure for computing arithmetic Chern numbers on the symplectic flag variety $\mathfrak{X}$, parallel to [T2, §7]. Let $c_k(\overline{E}_i)$ and $c_1(\overline{L}_i)$ be the Chern forms of $\overline{E_i(\mathbb{C})}$ and $\overline{L_i(\mathbb{C})}$, respectively. In the sequel we will identify these with their images in $\widehat{\mathrm{CH}}(\mathfrak{X})$ under the inclusion a. Let $x_i = -c_1(\overline{L}_i)$ for $1 \leq i \leq n$.

We begin with the short exact sequence

$$\overline{\mathcal{E}}_{\mathrm{LG}}\ :\ 0 \to \overline{E}_n \to \overline{E} \to \overline{E}_n^* \to 0$$

where E_n denotes the tautological Lagrangian subbundle of E over $\mathfrak{X}$. By [GS2, Theorem 4.8(ii)], we have an equation

$$\widehat{c}(\overline{E}_n)\,\widehat{c}(\overline{E}_n^*) = 1 + \widetilde{c}(\overline{\mathcal{E}}_{\mathrm{LG}}) \tag{14}$$

in $\widehat{\mathrm{CH}}(\mathfrak{X})$. Consider the hermitian filtration

$$\overline{\mathcal{E}}\ :\ 0 = \overline{E}_0 \subset \overline{E}_1 \subset \cdots \subset \overline{E}_n.$$

According to [T2, Theorem 2], we have an equation

$$\prod_{i=1}^n (1 - \widehat{x}_i) = \widehat{c}(\overline{E}_n) + \widetilde{c}(\overline{\mathcal{E}}). \tag{15}$$

If $\widetilde{c}(\overline{\mathcal{E}}) = \sum_i \alpha_i$ with $\alpha_i \in \widetilde{A}^{i,i}(\mathfrak{X}_{\mathbb{R}})$ for each i, then define $\widetilde{c}(\overline{\mathcal{E}}^*) = \sum_i (-1)^{i+1}\alpha_i$. We obtain the dual equation

$$\prod_{i=1}^{n}(1+\widehat{x}_i) = \widehat{c}(\overline{E}_n^*) + \widetilde{c}(\overline{\mathcal{E}}^*). \tag{16}$$

The abelian group $\widetilde{A}(\mathfrak{X}_{\mathbb{R}}) = \mathrm{Ker}\zeta$ is an ideal of $\widehat{\mathrm{CH}}(\mathfrak{X})$ such that for any hermitian vector bundle $\overline{F}$ over $\mathfrak{X}$ and $\eta, \eta' \in \widetilde{A}(\mathfrak{X}_{\mathbb{R}})$, we have

$$\widehat{c}_k(\overline{F})\cdot\eta = c_k(\overline{F})\wedge\eta \qquad \text{and} \qquad \eta\cdot\eta' = (dd^c\eta)\wedge\eta'. \tag{17}$$

We now multiply (15) with (16) and combine the result with (14) to obtain

$$\prod_{i=1}^{n}(1-\widehat{x}_i^2) = 1 + \widetilde{c}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*), \tag{18}$$

where

$$\widetilde{c}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) = \widetilde{c}(\overline{\mathcal{E}}_{\mathrm{LG}}) + \widetilde{c}(\overline{\mathcal{E}})\wedge c(\overline{E}_n^*) + \widetilde{c}(\overline{\mathcal{E}}^*)\wedge c(\overline{E}_n) + (dd^c\widetilde{c}(\overline{\mathcal{E}}))\wedge\widetilde{c}(\overline{\mathcal{E}}^*). \tag{19}$$

Using (9) and Proposition 2, we can express the differential form $\widetilde{c}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*)$ as a polynomial in the entries of the matrices K_{E_i} and K_{L_i} with rational coefficients. On the other hand, Proposition 3 gives explicit formulas for all these curvature matrices in terms of $\mathrm{Sp}(2n)$-invariant differential forms on $\mathfrak{X}(\mathbb{C})$. Note that since we are using the skew diagonal symplectic form to define the Lie groups in this section, the formulas in §3.2 have to be modified accordingly. For the matrix realization of the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ in this case, one may consult e.g. [GW, §1.2, §2.3], while the basis elements of $\mathfrak{h}$ should be ordered as in [BH, (2.20)]. The indices (i,j) and (p,q) in Proposition 3 are then replaced by $(n+1-j, n+1-i)$ and $(n+1-q, n+1-p)$, respectively. Recalling that $L_i = E_{n+1-i}/E_{n-i}$, we have the identities

$$\begin{aligned}
x_1 &= -\Omega_{12} - \Omega_{13} - \cdots - \Omega_{1n} + \Omega^{11} + \Omega^{12} + \cdots + \Omega^{1n} \\
x_2 &= \Omega_{12} - \Omega_{23} - \cdots - \Omega_{2n} + \Omega^{12} + \Omega^{22} + \cdots + \Omega^{2n} \\
&\qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots \\
x_n &= \Omega_{1n} + \Omega_{2n} + \cdots + \Omega_{n-1,n} + \Omega^{1n} + \Omega^{2n} + \cdots + \Omega^{nn}
\end{aligned}$$

in $A^{1,1}(\mathfrak{X}_{\mathbb{R}})$. We also deduce the next result.

Proposition 4. *We have* $\widetilde{c}_1(\overline{\mathcal{E}}) = \widetilde{c}_1(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) = 0$, $\widetilde{c}_2(\overline{\mathcal{E}}) = -\sum_{i<j}\Omega_{ij}$, *and*

$$\widetilde{c}_2(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) = -2\sum_{i<j}\Omega_{ij} - 2\sum_{p<q}\Omega^{pq} - \sum_{p=1}^{n}\Omega^{pp}.$$

Proof. We have $\widetilde{c}_2(\overline{\mathcal{E}}) = \sum_{i=2}^{n}\widetilde{c}_2(\overline{\mathcal{E}}_i)$, where $\overline{\mathcal{E}}_i$ is the short exact sequence

$$\overline{\mathcal{E}}_i \ : \ 0 \to \overline{E}_{i-1} \to \overline{E}_i \to \overline{L}_{n+1-i} \to 0.$$

For each i, write

$$K_{E_i} = \left(\begin{array}{c|c} K_{11}^i & K_{12}^i \\ \hline K_{21}^i & K_{22}^i \end{array}\right)$$

where K_{11}^i is an $(i-1)\times(i-1)$ submatrix. According to [T1, Cor. 1], we then have $\widetilde{c}_2(\overline{\mathcal{E}}_i) = c_1(\overline{E}_{i-1}) - \operatorname{Tr} K_{11}^i$. Therefore

$$\widetilde{c}_2(\overline{\mathcal{E}}) = \sum_{i=2}^{n} \big(c_1(\overline{E}_{i-1}) - \operatorname{Tr} K_{11}^i\big) = c_1(\overline{E}_1) - \operatorname{Tr} K_{11}^n + \sum_{i=2}^{n-1} \operatorname{Tr} K_{22}^i.$$

Using Proposition 3, we obtain

$$c_1(\overline{E}_1) = -\sum_{j<n} \Omega_{jn} - \sum_{p=1}^{n} \Omega^{pn},$$

$$-\operatorname{Tr} K_{11}^n = \sum_{p=1}^{n}\sum_{q=1}^{n-1} \Omega^{p,n+1-q}$$

and

$$\operatorname{Tr} K_{22}^i = -\sum_{j>i} \Omega_{n+1-j,n+1-i} - \sum_{p=1}^{n} \Omega^{p,n+1-i}$$

for $2 \le i \le n-1$. The claimed computation of $\widetilde{c}_2(\overline{\mathcal{E}})$ follows by adding these equations. We deduce from (9) that $\widetilde{c}_2(\overline{\mathcal{E}}_{\mathrm{LG}}) = -\mathcal{H}_1 p_1(\overline{E}_n^*) = c_1(\overline{E}_n)$, while clearly $\widetilde{c}_1(\overline{\mathcal{E}}^*) = \widetilde{c}_1(\overline{\mathcal{E}}) = 0$ and $\widetilde{c}_2(\overline{\mathcal{E}}^*) = \widetilde{c}_2(\overline{\mathcal{E}})$. Therefore, equation (19) gives

$$\widetilde{c}_2(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) = \widetilde{c}_2(\overline{\mathcal{E}}_{\mathrm{LG}}) + 2\,\widetilde{c}_2(\overline{\mathcal{E}}) = c_1(\overline{E}_n) + 2\,\widetilde{c}_2(\overline{\mathcal{E}}).$$

Finally, $c_1(\overline{E}_n) = \operatorname{Tr} K_{E_n}$, and the latter is computed using Proposition 3 again. □

Let $h(\mathrm{X}_n)$ be a homogeneous polynomial in the ideal I_n of §2.1. We give an effective algorithm to compute the arithmetic intersection $h(\widehat{x}_1, \ldots, \widehat{x}_n)$ as a class in $\widetilde{A}(\mathfrak{X}_{\mathbb{R}})$. First, we decompose h as a sum $h(\mathrm{X}_n) = \sum_i e_i(\mathrm{X}_n^2) f_i(\mathrm{X}_n)$ for some polynomials f_i. Equation (18) implies that

$$e_i(\widehat{x}_1^2, \ldots, \widehat{x}_n^2) = (-1)^i\, \widetilde{c}_{2i}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) \tag{20}$$

for $1 \le i \le n$. Using this and (17) we see that

$$h(\widehat{x}_1, \widehat{x}_2, \ldots \widehat{x}_n) = \sum_{i=1}^{n} (-1)^i\, \widetilde{c}_{2i}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) \wedge f_i(x_1, x_2, \ldots, x_n) \tag{21}$$

in $\widehat{\mathrm{CH}}(\mathfrak{X})$. Now, thanks to the previous analysis, we can write the right hand side of (21) as a polynomial in the x_i and the entries of the matrices K_{E_i} for $1 \le i \le n$, with rational coefficients, which is (the class of) an explicit $\mathrm{Sp}(2n)$-invariant differential form in $\widetilde{A}(\mathfrak{X}_{\mathbb{R}})$.

In particular, if k_i are nonnegative integers with $\sum k_i = \dim \mathfrak{X} = n^2+1$, the monomial $\mathrm{x}_1^{k_1}\cdots \mathrm{x}_n^{k_n}$ lies in the ideal I_n. If $\mathrm{x}_1^{k_1}\cdots \mathrm{x}_n^{k_n} = \sum_{i=1}^n e_i(\mathrm{X}_n^2) f_i(\mathrm{X}_n)$, then we have

$$\widehat{x}_1^{k_1}\widehat{x}_2^{k_2}\cdots \widehat{x}_n^{k_n} = \sum_{i=1}^{n} (-1)^i\, \widetilde{c}_{2i}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) \wedge f_i(x_1, \ldots, x_n).$$

Now if the top invariant form Ω is defined as in §3.2, we have shown that

$$\widetilde{c}_{2i}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) \wedge f_i(x_1, \ldots, x_n) = r_i\, \Omega$$

for some rational number r_i. Therefore the arithmetic degree [GS1] of the above monomial satisfies

$$\widehat{\deg}(\widehat{x}_1^{k_1}\widehat{x}_2^{k_2}\cdots\widehat{x}_n^{k_n}) = \frac{1}{2}\sum_{i=1}^n(-1)^i\,r_i\int_{\mathfrak{X}(\mathbb{C})}\Omega = \frac{1}{2}\prod_{k=1}^n\frac{1}{(2k-1)!}\sum_{i=1}^n(-1)^i\,r_i.$$

We deduce the following analogue of [T2, Theorem 4].

Theorem 2. *For any nonnegative integers $k_1,\ldots,k_n$ with $\sum k_i = n^2+1$, the arithmetic Chern number $\widehat{\deg}(\widehat{x}_1^{k_1}\widehat{x}_2^{k_2}\cdots\widehat{x}_n^{k_n})$ is a rational number.*

4.4. **Arithmetic Schubert calculus.** For any partition $\lambda\in\mathcal{G}_n$ and $\varpi\in S_n$, define

$$\widehat{\mathfrak{C}}_{\lambda,\varpi} = \mathfrak{C}_{\lambda,\varpi}(\widehat{x}_1,\ldots,\widehat{x}_n).$$

If $\lambda\in\mathcal{G}_n\smallsetminus\mathcal{D}_n$, let r_λ be the largest repeated part of λ, and let $\overline{\lambda}$ be the partition obtained from λ by deleting two of the parts r_λ. For instance, if $\lambda = (8,7,7,7,6,3,3,2)$, then $\overline{\lambda} = (8,7,6,3,3,2)$.

If $\lambda\in\mathcal{G}_n\smallsetminus\mathcal{D}_n$, then properties (b), (c) in §1.1, (17), and (20) imply that

$$\widehat{\mathfrak{C}}_{\lambda,\varpi} = \widehat{\mathfrak{C}}_{\overline{\lambda},\varpi}\widetilde{Q}_{r_\lambda,r_\lambda}(\widehat{x}_1^2,\ldots,\widehat{x}_n^2) = (-1)^{r_\lambda}\mathfrak{C}_{\overline{\lambda},\varpi}(x_1,\ldots,x_n)\wedge\widetilde{c}_{2r_\lambda}(\overline{\mathcal{E}},\overline{\mathcal{E}}^*).$$

Since $\widehat{\mathfrak{C}}_{\lambda,\varpi}\in a(\widetilde{A}(\mathfrak{X}_\mathbb{R}))$ whenever $\lambda\in\mathcal{G}_n\smallsetminus\mathcal{D}_n$, we will denote these classes by $\widetilde{\mathfrak{C}}_{\lambda,\varpi}$.

The next theorem computes arbitrary arithmetic intersections in $\widehat{\mathrm{CH}}(\mathfrak{X})$ with respect to the splitting (13) induced by (12), using the basis of symplectic Schubert polynomials.

Theorem 3. *Any element of the arithmetic Chow ring $\widehat{CH}(\mathfrak{X})$ can be expressed uniquely in the form $\sum_{w\in W_n} a_w\widehat{\mathfrak{C}}_w+\eta$, where $a_w\in\mathbb{Z}$ and $\eta\in\widetilde{A}(\mathfrak{X}_\mathbb{R})$. For $u,v\in W_n$ we have*

$$\widehat{\mathfrak{C}}_u\cdot\widehat{\mathfrak{C}}_v = \sum_{w\in W_n}c_{uv}^w\,\widehat{\mathfrak{C}}_w + \sum_{\substack{\lambda\in\mathcal{G}_n\smallsetminus\mathcal{D}_n\\ \varpi\in S_n}}c_{uv}^{\lambda\varpi}\,\widetilde{\mathfrak{C}}_{\lambda,\varpi}, \tag{22}$$

$$\widehat{\mathfrak{C}}_u\cdot\eta = \mathfrak{C}_u(x_1,\ldots,x_n)\wedge\eta, \qquad \textit{and} \qquad \eta\cdot\eta' = (dd^c\eta)\wedge\eta',$$

where η, $\eta'\in\widetilde{A}(\mathfrak{X}_\mathbb{R})$ and the integers c_{uv}^w, $c_{uv}^{\lambda\varpi}$ are as in (5).

Proof. The first statement is a consequence of the splitting (13). Equation (22) follows from the formal identity (5) and our definitions of $\widehat{\mathfrak{C}}_w$ and $\widetilde{\mathfrak{C}}_{\lambda,\varpi}$. The remaining assertions are derived immediately from the structure equations (17). □

4.5. **The invariant arithmetic Chow ring.** The arithmetic Chow group $\widehat{\mathrm{CH}}(\mathfrak{X})$ is not finitely generated, as it contains the infinite dimensional real vector space $\widetilde{A}(\mathfrak{X}_\mathbb{R})$ as a subgroup. Following [T2, §6], we can work equally well with a finite dimensional variant of $\widehat{\mathrm{CH}}(\mathfrak{X})$, obtained by replacing the space $A(\mathfrak{X}_\mathbb{R})$ by a certain subspace of the space of all $\mathrm{Sp}(2n)$-invariant differential forms on $\mathfrak{X}(\mathbb{C})$.

Recall the notation introduced in §3.2 and §4.2. Let $\mathrm{Inv}(\mathfrak{X}_\mathbb{R})$ denote the ring of $\mathrm{Sp}(2n)$-invariant forms in the $\mathbb{R}$-subalgebra of $A(\mathfrak{X}(\mathbb{C}))$ generated by the differential forms $\omega_1\wedge\overline{\omega}_2$ for all ω_1,ω_2 in the set $\{\omega_{ij},\,\omega^{pq}\mid i<j,\ p\leq q\}$. Define $\widetilde{\mathrm{Inv}}(\mathfrak{X}_\mathbb{R})\subset\widetilde{A}(\mathfrak{X}_\mathbb{R})$ to be the image of $\mathrm{Inv}(\mathfrak{X}_\mathbb{R})$ in $\widetilde{A}(\mathfrak{X}_\mathbb{R})$.

Definition 2. The invariant arithmetic Chow ring $\widehat{\mathrm{CH}}_{\mathrm{inv}}(\mathfrak{X})$ is the subring of $\widehat{\mathrm{CH}}(\mathfrak{X})$ generated by $\epsilon(\mathrm{CH}(\mathfrak{X}))$ and $a(\widetilde{\mathrm{Inv}}(\mathfrak{X}_\mathbb{R}))$, where ϵ is the splitting map (12).

There is an exact sequence of abelian groups

$$0 \longrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{X}_{\mathbb{R}}) \xrightarrow{a} \widehat{\mathrm{CH}}_{\mathrm{inv}}(\mathfrak{X}) \xrightarrow{\zeta} \mathrm{CH}(\mathfrak{X}) \longrightarrow 0$$

which splits under ϵ, giving an isomorphism of abelian groups

$$\widehat{\mathrm{CH}}_{\mathrm{inv}}(\mathfrak{X}) \simeq \mathrm{CH}(\mathfrak{X}) \oplus \widetilde{\mathrm{Inv}}(\mathfrak{X}_{\mathbb{R}}).$$

Theorem 3 may be refined to an analogous statement for the invariant arithmetic Chow ring $\widehat{\mathrm{CH}}_{\mathrm{inv}}(\mathfrak{X})$, by replacing the group $\widetilde{A}(\mathfrak{X}_{\mathbb{R}})$ with $\widetilde{\mathrm{Inv}}(\mathfrak{X}_{\mathbb{R}})$ throughout.

4.6. **Height computation.** The flag variety $\mathfrak{X}$ has a natural pluri-Plücker embedding j in projective space. The morphism j may be defined as a composite

$$\mathfrak{X} \hookrightarrow F_{\mathrm{SL}} \overset{\iota}{\hookrightarrow} \mathbb{P}^N$$

where $F_{\mathrm{SL}} = \mathrm{SL}_{2n}/P$ denotes the variety parametrizing all partial flags

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n \subset E_{2n} = E$$

with $\dim(E_i) = i$ for each i, and ι is a composition of a product of Plücker embeddings followed by a Segre embedding. Observe that j is the embedding given by the line bundle $Q = \bigotimes_{i=1}^n \det(E/E_i)$. Let $\overline{\mathcal{O}}(1)$ denote the canonical line bundle over the projective space $\mathbb{P}^N$, equipped with its canonical metric (so that $c_1(\overline{\mathcal{O}}(1))$ is the Fubini-Study form). The *Faltings height* of $\mathfrak{X}$ relative to $\overline{\mathcal{O}}(1)$ (see [GS1, Fa, BoGS]) is given by

$$h_{\overline{\mathcal{O}}(1)}(\mathfrak{X}) = \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{O}}(1))^{n^2+1} |\ \mathfrak{X}\right).$$

The pullback $j^*(\mathcal{O}(1)) = Q$ is an isometry when $Q(\mathbb{C})$ is equipped with the canonical metric given by tensoring the induced metrics on the determinants of the quotient bundles E/E_i for $1 \leq i \leq n$. The short exact sequences

$$\overline{\mathcal{E}}_i \ : \ 0 \to \overline{E}_{i-1} \to \overline{E}_i \to \overline{L}_{n+1-i} \to 0$$

satisfy $\widetilde{c}_1(\overline{\mathcal{E}}_i) = 0$, and hence $\widehat{c}_1(\overline{E}_i) = \widehat{c}_1(\overline{E}_{i-1}) - \widehat{x}_{n+1-i}$. It follows by induction that $\widehat{c}_1(\overline{E}_i) = -\widehat{x}_{n+1-i} - \cdots - \widehat{x}_n$, for $1 \leq i \leq n$. We deduce that

$$j^*(\widehat{c}_1(\overline{\mathcal{O}}(1))) = \widehat{c}_1(\overline{Q}) = -\sum_{i=1}^n \widehat{c}_1(\overline{E}_i) = \sum_{i=1}^n i\, \widehat{x}_i$$

and therefore

$$h_{\overline{\mathcal{O}}(1)}(\mathrm{Sp}_{2n}/B) = \widehat{\deg}\left(\widehat{c}_1(\overline{Q})^{n^2+1} |\ \mathfrak{X}\right) = \widehat{\deg}\left(\left(\sum_{i=1}^n i\, \widehat{x}_i\right)^{n^2+1}\right). \tag{23}$$

We conclude from Theorem 2 and (23) that the height $h_{\overline{\mathcal{O}}(1)}(\mathrm{Sp}_{2n}/B)$ is a rational number. One may also derive this fact from the height formula in [KK].

4.7. **An example:** Sp_4/B**.** In this section we compute the arithmetic intersection numbers for the classes $\widehat{x}_i$ in $\widehat{\mathrm{CH}}(\mathfrak{X})$ when $n = 2$, so that $\mathfrak{X}$ is the variety parametrizing partial flags $0 \subset E_1 \subset E_2 \subset E_4 = E$ with E_2 Lagrangian.

Consider the differential forms

$$\xi_1 = c_1(\overline{E}_2^*) = \Omega^{11} + 2\,\Omega^{12} + \Omega^{22}$$

and

$$\xi_2 = c_2(\overline{E}_2^*) = \Omega^{11}\Omega^{22} + 2\,\Omega^{11}\Omega^{12} + 2\,\Omega^{12}\Omega^{22}.$$

Notice that $\xi_1^2 = 2\,\xi_2$. Over $\mathfrak{X}$ we have the filtrations of hermitian vector bundles

$$\overline{\mathcal{E}}_{\mathrm{LG}}:\ 0 \subset \overline{E}_2 \subset \overline{E} \qquad \text{and} \qquad \overline{\mathcal{E}}:\ 0 \subset \overline{E}_1 \subset \overline{E}_2.$$

Equation (9) gives

$$\begin{aligned}\widetilde{c}(\overline{\mathcal{E}}_{\mathrm{LG}}) &= p_1(\overline{E}_2) + \mathcal{H}_3\, p_3(\overline{E}_2) = c_1(\overline{E}_2) + \mathcal{H}_3\left(c_1^3(\overline{E}_2) - 3c_1(\overline{E}_2)c_2(\overline{E}_2)\right)\\ &= -\xi_1 - \frac{11}{6}\xi_1^3 + \frac{11}{2}\xi_1\xi_2 = -\xi_1 + \frac{11}{6}\xi_1\xi_2\end{aligned}$$

and therefore

$$\widetilde{c}(\overline{\mathcal{E}}_{\mathrm{LG}}) = -\Omega^{11} - 2\,\Omega^{12} - \Omega^{22} + 11\,\Omega^{11}\Omega^{12}\Omega^{22}.$$

On the other hand, Proposition 4 gives $\widetilde{c}(\overline{\mathcal{E}}) = \widetilde{c}(\overline{\mathcal{E}}^*) = -\Omega_{12}$. Using the Maurer-Cartan structure equations for $\mathrm{Sp}_{2n}(\mathbb{C})$, we find that

$$d\omega_{12} = \overline{\partial}\omega_{12} = \frac{1}{\gamma}(\omega^{11}\wedge\overline{\omega}^{12} + \omega^{12}\wedge\overline{\omega}^{22})$$

$$d\overline{\omega}_{12} = \partial\overline{\omega}_{12} = -\frac{1}{\gamma}(\omega^{22}\wedge\overline{\omega}^{12} + \omega^{12}\wedge\overline{\omega}^{11})$$

and hence

$$dd^c(\Omega_{12}) = \gamma^2\partial\overline{\partial}(\omega_{12}\wedge\overline{\omega}_{12}) = \gamma^2(\overline{\partial}\omega_{12}\wedge\partial\overline{\omega}_{12}) = \Omega^{12}(\Omega^{11}+\Omega^{22}).$$

We deduce from the above calculations and (19) that

$$\begin{aligned}&\widetilde{c}(\overline{\mathcal{E}},\overline{\mathcal{E}}^*) = -\xi_1 - 2\,\Omega_{12} - 2\,\Omega_{12}\,\xi_2 + 11\,\Omega^{11}\Omega^{12}\Omega^{22} + \Omega_{12}\Omega^{12}(\Omega^{11}+\Omega^{22})\\ &= -\xi_1 - 2\,\Omega_{12} - 2\,\Omega_{12}\Omega^{11}\Omega^{22} - 3\,\Omega_{12}\Omega^{11}\Omega^{12} - 3\,\Omega_{12}\Omega^{12}\Omega^{22} + 11\,\Omega^{11}\Omega^{12}\Omega^{22}.\end{aligned}$$

Let a and b be nonnegative integers with $a+b=5$. The Bott-Chern form $\widetilde{c}(\overline{\mathcal{E}},\overline{\mathcal{E}}^*)$ is the key to computing any arithmetic intersection $\widehat{x}_1^a\widehat{x}_2^b$, following the algorithm of §4.3. The result will be a multiple of the class of $\Omega = \Omega_{12}\Omega^{11}\Omega^{12}\Omega^{22}$ in the arithmetic Chow group $\widehat{\mathrm{CH}}^5(\mathfrak{X})$. For instance, we compute that

$$\widehat{x}_1^3\widehat{x}_2^2 = \widehat{x}_1\cdot e_2(\widehat{x}_1^2,\widehat{x}_2^2) = (-\Omega_{12}+\Omega^{11}+\Omega^{12})\wedge\widetilde{c}_4(\overline{\mathcal{E}},\overline{\mathcal{E}}^*) = -16\,\Omega,$$

while

$$\widehat{x}_1^2\widehat{x}_2^3 = \widehat{x}_2\cdot e_2(\widehat{x}_1^2,\widehat{x}_2^2) = (\Omega_{12}+\Omega^{12}+\Omega^{22})\wedge\widetilde{c}_4(\overline{\mathcal{E}},\overline{\mathcal{E}}^*) = 6\,\Omega.$$

We similarly find

$$\widehat{x}_1^5 = 10\,\Omega, \qquad \widehat{x}_1^4\widehat{x}_2 = -8\,\Omega, \qquad \widehat{x}_1\widehat{x}_2^4 = 26\,\Omega, \qquad \widehat{x}_2^5 = 0.$$

For the Faltings height of $\mathfrak{X}$, we conclude that

$$h_{\overline{\mathcal{O}}(1)}(\mathrm{Sp}_4/B) = \widehat{\deg}\left((\widehat{x}_1+2\widehat{x}_2)^5\right) = \widehat{\deg}(1850\,\Omega) = 925\int_{\mathfrak{X}(\mathbb{C})}\Omega = \frac{925}{6}.$$

Kaiser and Köhler have proved a cohomological formula for the height of generalized flag varieties [KK, Thm. 8.1]. One can check that in the case of Sp_4/B, their result agrees with the above computation.

References

[BGG] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand : *Schubert cells and cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), no. 3, 1–26.

[BH] S. Billey and M. Haiman : *Schubert polynomials for the classical groups*, J. Amer. Math. Soc. **8** (1995), 443–482.

[BJS] S. Billey, W. Jockusch and R. P. Stanley : *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), 345–374.

[Bo] A. Borel : *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.

[BoGS] J.-B. Bost, H. Gillet and C. Soulé : *Heights of projective varieties and positive Green forms*, J. Amer. Math. Soc. **7** (1994), 903-1027.

[BC] R. Bott and S. S. Chern : *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections*, Acta Math. **114** (1968), 71–112.

[BKTY] A. S. Buch, A. Kresch, H. Tamvakis and A. Yong, *Schubert polynomials and quiver formulas*, Duke Math. J. **122** (2004), 125–143.

[D1] M. Demazure : *Invariants symétriques des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287–301.

[D2] M. Demazure : *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88.

[Fa] G. Faltings : *Diophantine approximation on abelian varieties*, Ann. of Math. **133** (1991), 549-576.

[FK] S. Fomin and A. N. Kirillov : *Combinatorial B_n-analogs of Schubert polynomials*, Trans. Amer. Math. Soc. **348** (1996), 3591–3620.

[F1] W. Fulton : *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), 381-420.

[F2] W. Fulton : *Schubert varieties in flag bundles for the classical groups*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 241–262, Israel Math. Conf. Proc. **9**, Bar-Ilan Univ., Ramat Gan, 1996.

[F3] W. Fulton : *Determinantal formulas for orthogonal and symplectic degeneracy loci*, J. Differential Geom. **43** (1996), 276–290.

[F4] W. Fulton : *Intersection Theory*, Second edition, Ergebnisse der Math. **2**, Springer-Verlag, Berlin, 1998.

[FH] W. Fulton and J. Harris : *Representation Theory. A First Course*, Graduate Texts in Mathematics **129**, Springer-Verlag, New York, 1991.

[GS1] H. Gillet and C. Soulé : *Arithmetic intersection theory*, Publ. math., I.H.E.S. **72** (1990), 94–174.

[GS2] H. Gillet and C. Soulé : *Characteristic classes for algebraic vector bundles with hermitian metrics, I, II*, Annals Math. **131** (1990), 163–203 and 205–238.

[GW] R. Goodman and N. Wallach : *Representations and Invariants of the Classical Groups*, Encyclopedia of Mathematics and its Applications **68**, Cambridge University Press, Cambridge, 1998.

[GrS] P. Griffiths and W. Schmid : *Locally homogeneous complex manifolds*, Acta Math. **123** (1969), 253–302.

[Ja] J. C. Jantzen : *Representations of Algebraic Groups*, Second edition, American Mathematical Society, Providence, RI, 2003.

[Jo] T. Józefiak : *Schur Q-functions and cohomology of isotropic Grassmannians*, Math. Proc. Camb. Phil. Soc. **109** (1991), 471–478.

[KK] C. Kaiser and K. Köhler : *A fixed point formula of Lefschetz type in Arakelov geometry III: representations of Chevalley schemes and heights of flag varieties*, Invent. Math. **147** (2002), 633-669.

[KMS] A. Knutson, E. Miller, and M. Shimozono : *Four positive formulae for type A quiver polynomials*, Invent. Math. **166** (2006), 229–325.

[KT1] A. Kresch and H. Tamvakis : *Double Schubert polynomials and degeneracy loci for the classical groups*, Ann. Inst. Fourier **52** (2002), 1681–1727.

[KT2] A. Kresch and H. Tamvakis : *Quantum cohomology of the Lagrangian Grassmannian*, J. Alg. Geometry **12** (2003), 777–810.

[KM] K. Künnemann and V. Maillot : *Théorèmes de Lefschetz et de Hodge arithmétiques pour les variétés admettant une décomposition cellulaire*, Regulators in analysis, geometry and number theory, Progr. Math. **171**, 197–205, Birkhäuser, Boston, 2000.

[La] T. K. Lam : *B_n Stanley symmetric functions*, Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994). Discrete Math. **157** (1996), 241–270.

[LP1] A. Lascoux and P. Pragacz : *Operator calculus for $\widetilde{Q}$-polynomials and Schubert polynomials*, Adv. Math. **140** (1998), 1–43.

[LP2] A. Lascoux and P. Pragacz : *Orthogonal divided differences and Schubert polynomials, $\widetilde{P}$-functions, and vertex operators*, Michigan Math. J. **48** (2000), 417–441.

[LS] A. Lascoux and M.-P. Schützenberger : *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 447–450.

[M] I. G. Macdonald : *Notes on Schubert polynomials*, Publ. LACIM **6**, Univ. de Québec à Montréal, Montréal, 1991.

[PR] P. Pragacz and J. Ratajski : *Formulas for Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach*, Compositio Math. **107** (1997), 11–87.

[S] I. Schur : *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. reine angew. Math. **139** (1911), 155–250.

[T1] H. Tamvakis : *Bott-Chern forms and arithmetic intersections*, Enseign. Math. **43** (1997), 33–54.

[T2] H. Tamvakis : *Arithmetic intersection theory on flag varieties*, Math. Ann. **314** (1999), 641–665.

[T3] H. Tamvakis : *Schubert calculus on the arithmetic Grassmannian*, Duke Math. J. **98** (1999), 421–443.

[T4] H. Tamvakis : *Arakelov theory of the Lagrangian Grassmannian*, J. reine angew. Math. **516** (1999), 207–223.

[T5] H. Tamvakis : *Schubert polynomials and Arakelov theory of orthogonal flag varieties*, Math. Z., to appear.

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