SCHUBERT POLYNOMIALS AND DEGENERACY LOCUS FORMULAS

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Abstract. In previous work [T6], we employed the approach to Schubert polynomials by Fomin, Stanley, and Kirillov to obtain simple, uniform proofs that the double Schubert polynomials of Lascoux and Schützenberger and Ikeda, Mihalcea, and Naruse represent degeneracy loci for the classical groups in the sense of Fulton. Using this as our starting point, and purely combinatorial methods, we obtain a new proof of the general formulas of [T5], which represent the degeneracy loci coming from any isotropic partial flag variety. Along the way, we also find several new formulas and elucidate the connections between some earlier ones.

0. Introduction

In the 1990s, Fulton [Fu1, Fu2] introduced a notion of degeneracy loci determined by flags of vector bundles associated to any classical Lie group $G$. For any two (isotropic) flags $E_e$ and $F_e$ of subbundles of a (symplectic or orthogonal) vector bundle $E$ over a base variety $M$, and an element $w$ in the Weyl group of $G$, there is a locus $X_w \subset M$ defined by incidence relations between the flags. The degeneracy locus problem is to find a universal polynomial $P_w$ in the Chern classes of the vector bundles involved such that $[X_w] = P_w \cap [M]$. We ask that $P_w$ should be combinatorially explicit and manifestly respect the symmetries (that is, the descent sets) of both $w$ and $w^{-1}$, whenever possible.

When $G$ is the general linear group, the degeneracy locus problem was solved by Buch, Kresch, Yong, and the author [BKTY]. This answer was extended in a type uniform way to the symplectic and orthogonal Lie groups in [T5]. The formulas of [BKTY, T5] rely in part on a theory of Schubert polynomials, which express the classes of the degeneracy loci in terms of the Chern roots of the vector bundles $E_e$ and $F_e$. The desired combinatorial theory of Schubert polynomials, together with its connection to geometry, was established in the papers [LS, L, Fu1] (for Lie type A) and [BH, T2, T3, IMN1, T5] (for Lie types B, C, and D).

In previous work [T6, §7.3], we employed Fomin, Stanley, and Kirillov’s nil-Coxeter algebra approach to Schubert polynomials [FS, FK] to give simple, uniform proofs that the double Schubert polynomials of Lascoux and Schützenberger [LS, L] and Ikeda, Mihalcea, and Naruse [IMN1] represent degeneracy loci of vector bundles, in the above sense. Our main goal in this paper is to begin with the same definition of Schubert polynomials from [T5, T6] and, by purely combinatorial methods, derive the splitting formulas for these polynomials found in [T5, §3 and §6]. The latter results then imply the general degeneracy locus formulas of [T5].

Date: February 18, 2016.

2010 Mathematics Subject Classification. Primary 14M15; Secondary 05E05, 14N15.

The author was supported in part by NSF Grant DMS-1303352.
The proof of the corresponding type A splitting formula from [BKTY] is essentially combinatorial; this is clarified in [T5, §1.4] and §1 of the present paper.

As in [T5], our arguments depend on two key results from [BKT2, BKT3], which state that the single Schubert polynomials indexed by Grassmannian elements of the Weyl groups are represented by (single) \( \theta \) and \( \eta \) polynomials. The original proofs of these theorems used the classical Pieri rules from [BKT1], which were derived geometrically in op. cit. by intersecting Schubert cells. More recent proofs by Ikeda and Matsumura [IM], the author and Wilson [TW, T7], and Anderson and Fulton [AF2] use localization in equivariant cohomology (following [Ar, KK]) or employ other geometric arguments stemming from Kazarian’s work [Ka]. However, the statements of the aforementioned theorems from [BKT2, BKT3] are entirely combinatorial, and it is natural to seek proofs of them within the same framework. The corresponding result in type A is the elementary fact that the symmetric Schubert polynomials are equal to Schur polynomials.

The approach we take here begins by extending Billey and Haiman’s formula [BH, Prop. 4.15] for the single Schubert polynomials indexed by the longest element in the Weyl group of \( G \) to the double Schubert polynomials. From this, we deduce corresponding multi-Pfaffian formulas equivalent to those in [IMN1, Thm. 1.2] and [AF1, AF2] for the (equivariant) Schubert class of a point on the complete flag variety \( G/B \). Following the method of [IM, §8] in the symplectic case, by employing the left divided difference operators, we then derive analogous formulas for the class of a point on symplectic and even orthogonal Grassmannians, which first appeared in Kazarian’s paper [Ka]. The proof is completed by using the arguments found in [TW, T7] to arrive at double \( \theta \) and double \( \eta \) polynomials, and then specializing to obtain their single versions in [BKT2, BKT3].

The result of our efforts is a straightforward and type uniform combinatorial proof of the equivariant Giambelli and degeneracy locus formulas of [Ka, I, IN, IMN1, IM, T5, T7, TW]. Along the way, we obtain several new formulas, and illuminate the connections between some earlier ones. In particular, we define in §2.3 the reverse double Schubert polynomials \( \tilde{G}_w(Y,Z) \), which provide a bridge between the type A and type C theories. Furthermore, in type D, inspired by [I, IN, IMN1], we use Ivanov’s double (or factorial) Schur \( P \)-functions [Iv1, Iv2, Iv3], and obtain in §4.2 new results about them, which are suitable for our purposes.

Our earlier paper [T6] contained a detailed exposition of the various ingredients that went into [T5], revealing the author’s perspective on the subject of degeneracy loci, as it stood in 2009. We revisit some of that material here, for completeness, but focus in §1 – §4 on what is required for our new combinatorial proofs of the main results. In §5, we discuss how to use geometrization to translate the theorems of the previous sections into Chern class formulas, and provide some detailed remarks on the history of this problem, which supplement the ones contained in [T6].

In a recent preprint, Anderson and Fulton [AF2] use Young’s raising operators and algebro-geometric arguments to define multi-\( \theta \) and multi-\( \eta \) polynomials, which extend the double \( \theta \) and \( \eta \) polynomials of [TW, T7, W] even further. The resulting degeneracy locus formulas and their proofs are an important contribution to the theory of \( \theta \) and \( \eta \) polynomials, but hold only for certain special elements \( w \) of the Weyl group of \( G \). We leave the task of including them within the present algebraic and combinatorial framework to future research.
This article is organized as follows. In §1 we review the type A theory, culminating in the relevant splitting formulas for type A double Schubert polynomials from [BPTY]. In §2, we discuss flagged Schur polynomials, type A duality, and introduce the reverse double Schubert polynomials $\tilde{\mathcal{S}}(Y,Z)$. These objects are used in §3 and §4, which provide the corresponding theory in Lie types C and D. Finally, in §5 we discuss the history of the geometrization of the single and double Schubert polynomials, focusing on the symplectic case.

My work on this paper was inspired by my participation in the conference in Algebraic Geometry ‘IMPANGA 15’ on 12-18 April 2015 in Bedlewo, Poland. I am grateful to the organizing committee for the invitation and for bringing together many researchers in the area in such a pleasant and stimulating atmosphere. I also thank Andrew Kresch for his helpful comments on the manuscript.

1. The type A theory

1.1. Schubert polynomials and divided differences. The Weyl group $S_n$ of permutations of \{1, \ldots, n\} is generated by the $n - 1$ simple transpositions $s_i = (i, i + 1)$. A reduced word of a permutation $\varpi$ in $S_\infty$ is a sequence $a_1 \cdots a_\ell$ of positive integers such that $\varpi = s_{a_1} \cdots s_{a_\ell}$ and $\ell$ is minimal, so (by definition) equal to the length $\ell(\varpi)$ of $\varpi$. We say that $\varpi$ has a descent at position $r$ if $\ell(\varpi s_r) < \ell(\varpi)$, where $s_r$ is the simple reflection indexed by $r$.

Our main references for type A Schubert polynomials are [LS, M2, FS]. We recall from [FS] their construction using the nilCoxeter algebra $\mathcal{N}_n$ of the symmetric group $S_n$. By definition, $\mathcal{N}_n$ is the free associative algebra with unit generated by the elements $u_1, \ldots, u_{n-1}$ modulo the relations

\[
\begin{align*}
    u_i^2 &= 0, & i & \geq 1; \\
    u_i u_j &= u_j u_i, & |i - j| & \geq 2; \\
    u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1}, & i & \geq 1.
\end{align*}
\]

For any $\varpi \in S_n$, choose a reduced word $a_1 \cdots a_\ell$ for $\varpi$ and define $u_\varpi := u_{a_1} \cdots u_{a_\ell}$. Then the $u_\varpi$ for $\varpi \in S_n$ are well defined and form a free $\mathbb{Z}$-basis of $\mathcal{N}_n$. We denote the coefficient of $u_\varpi$ in $\mathcal{N}_n$ in the expansion of the element $\xi \in \mathcal{N}_n$ by $\langle \xi, \varpi \rangle$; in other words, $\xi = \sum_{\varpi \in S_n} \langle \xi, \varpi \rangle u_\varpi$, for all $\xi \in \mathcal{N}_n$.

Let $t$ be an indeterminate and define

\[
A_i(t) := (1 + tu_{n-1})(1 + tu_{n-2}) \cdots (1 + tu_i);
\]

\[
\tilde{A}_i(t) := (1 - tu_i)(1 - tu_{i+1}) \cdots (1 - tu_{n-1}).
\]

Suppose that $Y = (y_1, y_2, \ldots)$ and $Z = (z_1, z_2, \ldots)$ are two infinite sequences of commuting independent variables. For any $\varpi \in S_n$, the double Schubert polynomial $\mathcal{S}_\varpi(Y,Z)$ of Lascoux and Schützenberger [LS, L] is given by the prescription

\[
\mathcal{S}_\varpi(Y,Z) := \langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1)A_1(y_1) \cdots A_{n-1}(y_{n-1}), \varpi \rangle.
\]

The polynomial $\mathcal{S}_\varpi(Y) := \mathcal{S}_\varpi(Y,0)$ is the single Schubert polynomial. The definition (1) implies that $\mathcal{S}_\varpi(Y)$ has nonnegative integer coefficients, which admit a combinatorial interpretation (compare with [BJS, Thm. 1.1]).

For each $n \geq 1$, there is an injective group homomorphism $i_n : S_n \to S_{n+1}$, defined by adjoining the fixed point $n+1$, and we let $S_\infty := \cup_n S_n$. The polynomials
\( \mathcal{S}_{\varpi}(Y, Z) \) have an important stability property under the inclusions \( i_n \) of the Weyl groups, namely, if \( \varpi \in S_n \), then we have

\[
\mathcal{S}_{i_n(\varpi)}(Y, Z) = \mathcal{S}_{\varpi}(Y, Z).
\]

The stability property implies that \( \mathcal{S}_{\varpi}(Y, Z) \) is well defined for all \( \varpi \in S_\infty \).

For any \( i \geq 1 \), there are right and left divided difference operators \( \partial^i_\varpi \) and \( \partial_i^\varpi \) which act on the polynomial ring \( \mathbb{Z}[Y, Z] \). We define an action of \( S_\infty \) on \( \Gamma[Y, Z] \) by letting the simple transpositions \( s_i \) act by interchanging \( y_i \) and \( y_{i+1} \) and leaving all the remaining variables fixed. Define \( \partial_i^\varpi \) on \( \mathbb{Z}[Y, Z] \) by

\[
\partial_i^\varpi f := \frac{f - s_i f}{y_i - y_{i+1}}.
\]

Consider the ring involution \( \omega : \mathbb{Z}[Y, Z] \to \mathbb{Z}[Y, Z] \) determined by \( \omega(y_j) = -z_j \) and \( \omega(z_j) = -y_j \) for each \( j \), and set \( \partial_i^\omega := \omega \partial_i^\varpi \omega \) for each \( i \geq 1 \).

Both the single and double Schubert polynomials may be characterized by their compatibility with the divided difference operators. In fact, the polynomials \( \mathcal{S}_{\varpi} \) for \( \varpi \in S_\infty \) are the unique family of elements of \( \mathbb{Z}[Y, Z] \) satisfying the equations

\[
(2) \quad \partial_i^\varpi \mathcal{S}_{\varpi} = \begin{cases} \mathcal{S}_{\varpi s_i} & \text{if } \ell(\varpi s_i) < \ell(\varpi), \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\partial_i^\varpi \mathcal{S}_{\varpi} = \begin{cases} \mathcal{S}_{s_i \varpi} & \text{if } \ell(s_i \varpi) < \ell(\varpi), \\ 0 & \text{otherwise}, \end{cases}
\]

for all \( i \geq 1 \), together with the condition that the constant term of \( \mathcal{S}_{\varpi} \) is 1 if \( \varpi = 1 \), and 0 otherwise. This characterization theorem is straightforward to prove directly from the definition (1), following [FS, Thm. 2.2].

The above result has two important consequences. The first is that \( \mathcal{S}_{\varpi}(Y, Z) \) is symmetric in \( y_i, y_{i+1} \) (respectively in \( z_j, z_{j+1} \)) if and only if \( \varpi_i < \varpi_{i+1} \) (respectively \( \varpi_j^- < \varpi_{j+1}^- \)). In other words, the descents of \( \varpi \) and \( \varpi^- \) determine the symmetries of the polynomial \( \mathcal{S}_{\varpi}(Y, Z) \). A second consequence is that the double Schubert polynomials \( \mathcal{S}_{\varpi}(Y, Z) \) represent the universal Schubert classes in type A flag bundles, and therefore degeneracy loci of vector bundles, in the sense of [Fu1].

Let \( \varpi_0 = (n, n-1, \ldots, 1) \) be the longest element of \( S_n \). According to [FS, Lemma 2.1], for all commuting variables \( s, t \) and indices \( i \), we have \( A_i(s)A_i(t) = A_i(t)A_i(s) \). Since \( \tilde{A}_i(t) = A_i(t)^{-1} \), we also have \( A_i(s)\tilde{A}_i(t) = \tilde{A}_i(t)A_i(s) \). Fomin and Stanley [FS, Cor. 4.4] use this fact and the definition (1) to show that

\[
(3) \quad \mathcal{S}_{\varpi_0}(Y, Z) = \prod_{i+j \leq n} (y_i - z_j).
\]

1.2. Schur polynomials. Following [LS, M2], the product in (3) may be written in the form of a multi-Schur determinant. Furthermore, by applying divided differences to \( \mathcal{S}_{\varpi_0} \) and using the equations (2), one can express more general Schubert polynomials \( \mathcal{S}_{\varpi} \) as Schur type determinants. We will not reproduce these formulas here, but we do need some more notation to recall the ones that we will require.

For any integer \( j \geq 0 \), define the elementary and complete symmetric functions \( e_j(Y) \) and \( h_j(Y) \) by the generating series

\[
\prod_i (1 + y_it) = \sum_j e_j(Y)t^j \quad \text{and} \quad \prod_i (1 - y_it)^{-1} = \sum_j h_j(Y)t^j.
\]
and the product of $u\lambda$ Stanley symmetric functions and splitting formulas.

1.3. mutations

Reduced factorization

Given any permutation $\sigma \in S_{\infty}$ is Grassmannian if there exists an $m \geq 1$ such that $\sigma_i < \sigma_{i+1}$ for all $i \neq m$. The shape of such a Grassmannian permutation $\sigma$ is the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_{m+1-j} = \sigma_j - j$ for $1 \leq j \leq m$. If $\sigma \in S_{\infty}$ is a Grassmannian permutation with a unique descent at $m$ and shape $\lambda$, then we have

$$\mathfrak{S}_\sigma(Y) = s_\lambda(y_1, \ldots, y_m).$$

A short proof of (5) starting from the formula (3) for $\mathfrak{S}_{\sigma_\alpha}(Y)$ is in [M2, (4.8)].
If \( A(Y) := A_1(y_1)A_1(y_2) \cdots \), then the function \( G_w(Y) \) defined for \( w \in S_n \) by

\[
G_w(Y) := \langle A(Y), w \rangle
\]
is symmetric in \( Y \). \( G_w \) is the type A Stanley symmetric function, which was introduced in [S].\(^1\) If \( \tilde{A}(Z) = \tilde{A}_1(z_1)\tilde{A}_1(z_2) \cdots \), then we define the double Stanley symmetric function \( G_w(Y/Z) \) by

\[
G_w(Y/Z) := \langle \tilde{A}(Z)A(Y), w \rangle = \sum_{uv = w} G_{u^{-1}}(-Z)G_v(Y)
\]
with the sum over all reduced factorizations \( uv = w \) in \( S_\infty \).

Given \( m \geq 1 \) and any \( w \in S_n \), the permutation \( 1_m \times w \in S_{m+n} \) is defined by \( (1_m \times w)(j) = j \) for \( 1 \leq j \leq m \) and \( (1_m \times w)(j) = w(j-m) \) for \( j > m \). We say that a permutation \( w \) is increasing up to \( m \) if \( w(1) < w(2) < \cdots < w(m) \). If \( w \) is increasing up to \( m \) and \( w^{-1} \) is increasing up to \( \ell \), then we have the following key identity:

\[
\mathfrak{S}_w(Y) = \sum_{v(1_m \times u) = w} G_v(y_1, \ldots, y_m)\mathfrak{S}_u(y_{m+1}, y_{m+2}, \ldots)
\]
where the sum is over all reduced factorizations \( v(1_m \times u) = w \) in \( S_\infty \). Equation (7) admits a double version: let \( G_v(Y_{(m)}/Z_{(\ell)}) \) denote the polynomial obtained from \( G_v(Y/Z) \) by setting \( y_i = z_j = 0 \) for all \( i > m \) and \( j > \ell \). Then if \( w \) is increasing up to \( m \) and \( w^{-1} \) is increasing up to \( \ell \), we have

\[
\mathfrak{S}_w(Y, Z) = \sum_{\ell > \ell} \mathfrak{S}_{u^{-1}}(-Z_{\ell})G_v(Y_{(m)}/Z_{(\ell)})\mathfrak{S}_u(Y_{>m}),
\]
where \( Y_{>m} := (y_{m+1}, y_{m+2}, \ldots) \), \( -Z_{>\ell} := (-z_{\ell+1}, -z_{\ell+2}, \ldots) \), and the sum is over all reduced factorizations \( (1_\ell \times u)v(1_m \times u') = w \) in \( S_\infty \). Equations (7) and (8) are easy to show directly from the definitions of \( \mathfrak{S}_w(Y, Z) \) and \( G_v(Y/Z) \) (see [T5, §1.4] for a detailed proof of (7); the proof of (8) is similar).

We say that a permutation \( w \in S_\infty \) is compatible with the sequence \( a : a_1 < \cdots < a_p \) of positive integers if all descent positions of \( w \) are contained in \( a \). Let \( b : b_1 < \cdots < b_q \) be a second sequence of positive integers and assume that \( w \) is compatible with \( a \) and \( w^{-1} \) is compatible with \( b \). We say that a reduced factorization \( u_1 \cdots u_{p+q-1} = w \) is compatible with \( a, b \) if \( u_j(i) = i \) whenever \( j < q \) and \( i \leq b_{q-j} \) or whenever \( j > q \) and \( i \leq a_{j-q} \) (and where we set \( u_j(0) = 0 \)). Set \( Y_i := \{y_{a_i-1+1}, \ldots, y_{b_i}\} \) for each \( i \geq 1 \) and \( Z_j := \{z_{b_{j-1+1}}, \ldots, z_{b_j}\} \) for each \( j \geq 1 \).

**Proposition 1** ([BKTY]). Suppose that \( w \) and \( w^{-1} \) are compatible with \( a \) and \( b \), respectively. Then the Schubert polynomial \( \mathfrak{S}_w(Y, Z) \) satisfies

\[
\mathfrak{S}_w = \sum G_{u_1}(0/Z_q) \cdots G_{u_{p-q}}(0/Z_p)G_{u_q}(Y_1/Z_1)G_{u_{q+1}}(Y_2) \cdots G_{u_{p+q-1}}(Y_p)
\]
summed over all reduced factorizations \( u_1 \cdots u_{p+q-1} = w \) compatible with \( a, b \).

**Proof.** The result follows easily by using (8) and iterating the identity (7). \( \square \)

When the Stanley symmetric function \( G_w \) is expanded in the basis of Schur functions, one obtains a formula

\[
G_w(Y) = \sum_{\lambda : |\lambda| = \ell(w)} a^{\lambda w}_\lambda s_\lambda(Y)
\]

\(^1\)In Stanley’s paper, the function \( G_{w^{-1}} \) is assigned to \( w \).
for some nonnegative integers $a^w_{u}$. There exist several different combinatorial interpretations of these coefficients, for instance using the transition trees of Lascoux and Schützenberger (see for example [M2, (4.37)]). One also knows that $a^w_{u} = a^{w^{-1}}_{u}$.

**Theorem 1** ([BKTY], Thm. 4). Suppose that $w$ is compatible with $a$ and $w^{-1}$ is compatible with $b$. Then we have

$$
\mathcal{G}_w = \sum_{\lambda} a^w_{\lambda} s_{\lambda_1}(0/Z_1) \cdots s_{\lambda_{p-1}}(0/Z_2) s_{\lambda_p}(Y_1/Z_1) s_{\lambda_{p+1}}(Y_2) \cdots s_{\lambda_{p+q-1}}(Y_p)
$$

summed over all sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^{p+q-1})$, where

$$
a^w_{\lambda} := \sum_{u_1 \cdots u_{p+q-1} = w} a^u_{\lambda} u_1 \cdots a^u_{\lambda_{p+q-1}},
$$

summed over all reduced factorizations $u_1 \cdots u_{p+q-1} = w$ compatible with $a$, $b$.

**Proof.** The result follows from Proposition 1 by using the equation (9). $\square$

2. **Flagged Schur polynomials and duality**

2.1. **Flagged Schur polynomials.** Let $\alpha = \{\alpha_j\}_{1 \leq j \leq \ell}$, $\beta = \{\beta_j\}_{1 \leq j \leq \ell}$, and $\rho = \{\rho_j\}_{1 \leq j \leq \ell}$ be three integer sequences, and let $t = (t_1, t_2, \ldots)$ be a sequence of independent variables. The Schur type determinant

$$
S^p_{\alpha/\beta}(h(t)) := \det \left( h_{\alpha_i - \beta_j + j - i}(t) \right)_{1 \leq i, j \leq \ell}
$$

is called a flagged Schur polynomial. We define the polynomial $S^p_{\alpha/\beta}(e(t))$ in a similar way. The method of Gessel and Viennot [GV] or Wachs [Wa] shows that for any partition $\lambda$ and increasing composition $\rho$, we have

$$
S^p_{\lambda}(h(t)) = \sum t^U
$$

where the sum is over all column strict Young tableaux $U$ of shape $\lambda$ whose entries in the $i$-th row are $\leq \rho_i$ for all $i \geq 1$.

**Proposition 2.** Let $\lambda$ and $\mu$ be two partitions of length at most $\ell$ with $\max(\lambda_1, \mu_1) \leq k$. Then we have

$$
\det \left( h_{\lambda_i - \mu_j + j - i}(t) \right)_{1 \leq i, j \leq \ell} = \det \left( e_{\lambda_i' - \mu_j' + j - i}(t) \right)_{1 \leq i, j \leq k}.
$$

**Proof.** The argument follows the one in [M1, I.2, eq. (2.9)]. Let $N := k + \ell$ and define the matrices

$$
A := \left( h_{i-j}(-t) \right)_{0 \leq i, j \leq N-1} \quad \text{and} \quad B := \left( h_{i-j}^{i+j+1}(t) \right)_{0 \leq i, j \leq N-1}.
$$

It is well known that $A$ and $B$ are inverse to each other (see for example [M1, I.3, Ex. 21]). Therefore each minor of $A$ is equal to the complementary cofactor of $B^t$, the transpose of $B$. For the minor of $A = (e_{i-j}(-t))$ with row indices $\lambda'_i + k - i$ $(1 \leq i \leq k)$ and column indices $\mu'_j + k - j$ $(1 \leq j \leq k)$, the complementary cofactor of $B^t = (h_{i-j}^{i+j+1}(t))$ has row indices $k - 1 + i - \lambda_i$ $(1 \leq i \leq \ell)$ and column indices $k - 1 + j - \mu_j$ $(1 \leq j \leq \ell)$. The equality (11) follows by taking determinants. $\square$
2.2. The duality involution. For any \( r \geq 1 \), let \( \delta_r \) denote the partition \((r, r-1, \ldots, 1)\). Let \( Y_n(\alpha) := \langle y_1, \ldots, y_n \rangle \) and \( I_n \subset \mathbb{Z}[Y_n(\alpha)] \) be the ideal generated by the elementary symmetric polynomials \( e_i(Y_n(\alpha)) \) for \( 1 \leq i \leq n \). Set \( H_n := \mathbb{Z}[Y_n(\alpha)]/I_n \) and let \( \mathcal{H}_n \) be the \( \mathbb{Z} \)-linear subspace of \( \mathbb{Z}[Y_n(\alpha)] \) spanned by the monomials \( y^\alpha := y_1^{\alpha_1} \cdots y_n^{\alpha_n} \) for all compositions \( \alpha \leq \delta_n \).

Let \( \mathcal{Y} = \text{GL}_n/B \) be the manifold which parametrizes complete flags 

\[
E_\bullet : \ 0 \subseteq E_1 \subseteq \cdots \subseteq E_n = \mathbb{C}^n
\]

of subspaces of \( \mathbb{C}^n \). It is well known that the inclusion \( \mathcal{H}_n \subset \mathbb{Z}[Y_n(\alpha)] \) induces an isomorphism of abelian groups \( \mathcal{H}_n \cong H_n \), and that \( H_n \) is isomorphic to the cohomology ring of the flag manifold \( \mathcal{Y} \). Moreover, the Schubert polynomials \( \mathcal{S}_\nu(Y) \) for \( \nu \in S_n \) form a \( \mathbb{Z} \)-basis of \( \mathcal{H}_n \), and represent the Schubert classes in \( \mathcal{Y} \) under the above isomorphism (see for example [Fu3, §10]).

Define an involution \( \varpi \mapsto \varpi^* \) of \( S_n \) by setting \( \varpi^* := \varpi_0 \varpi \varpi_0 \), and define the duality involution \( * : \mathcal{H}_n \to \mathcal{H}_n \) to be the \( \mathbb{Z} \)-linear map determined by \( * \mathcal{S}_\nu(Y) := \mathcal{S}_{\nu^*}(Y) \), for each \( \nu \in S_n \). We let \( \mathbb{Z}[Y_n(\alpha)] \to \mathbb{Z}[Y_n(\alpha)] \) be the ring involution given by \((y_1, \ldots, y_n) \mapsto (-y_1, \ldots, -y_n)\), which descends to a ring involution \( D : H_n \to H_n \).

The following result states that \( * \) and \( D \) are mutually compatible.

**Lemma 1.** For any permutation \( \varpi \in S_n \), we have

\[
D(\mathcal{S}_\nu(Y) + I_n) = \mathcal{S}_{\nu^*}(Y) + I_n
\]

in the quotient ring \( H_n \).

**Proof.** Modulo the defining relations for the ideal \( I_n \) of \( \mathbb{Z}[Y_n(\alpha)] \), we have

\[
A_1(y_1)A_1(y_2) \cdots A_1(y_n) \equiv 1.
\]

Set

\[
B_1(t) := (1 + tu_{n-1})(1 + tu_{n-2}t) \cdots (1 + tu_1)
\]

so that

\[
B_1(t)^{-1} = (1 - tu_1)(1 - tu_2) \cdots (1 - tu_{n-1}).
\]

Observe that (13) may be written as

\[
A_1(y_1)A_2(y_2) \cdots A_{n-1}(y_{n-1}) : B_{n-1}(y_2)B_{n-2}(y_3) \cdots B_1(y_n) \equiv 1
\]

or equivalently

\[
A_1(y_1)A_2(y_2) \cdots A_{n-1}(y_{n-1}) \equiv B_1(y_n)^{-1}B_2(y_{n-1})^{-1} \cdots B_{n-1}(y_{n-1})^{-1}.
\]

Notice that \( (s_i)^* = s_{n-i} \) for each simple transposition \( s_i \) in \( S_n \). It follows that for any \( \varpi \in S_n \), we have

\[
\mathcal{S}_\nu(Y) = \langle A_1(y_1)A_2(y_2) \cdots A_{n-1}(y_{n-1}), \varpi \rangle
\]

\[
= \langle B_1(y_n)^{-1}B_2(y_{n-1})^{-1} \cdots B_{n-1}(y_{n-1})^{-1}, \varpi \rangle = D \mathcal{S}_{\nu^*}(Y),
\]

as required. \( \Box \)

Geometrically, the above involutions correspond to the duality isomorphism \( \mathcal{Y} \to \mathcal{Y} \) which sends each complete flag \( E_\bullet \) in \( \mathbb{C}^n \) to the dual flag \( E^*_\bullet \) in \( (\mathbb{C}^n)^* \). Here \( E^*_i \) is defined as the kernel of the canonical linear map \( (\mathbb{C}^n)^* \to E^*_{n-i} \), for \( 1 \leq i \leq n \).

For more details on this, see [Fu3, §10, Exercise 13].
Example 1. One can prove the equality (12) directly in the case when $\mathfrak{S}_\omega(Y)$ is an elementary symmetric polynomial $e_i(y_1, \ldots, y_r)$. We have $\prod_{i=1}^n (1 + y_i t) = 1$ in $H_n$, and hence
\[
\prod_{i=1}^r (1 + y_i t) = \prod_{j=r+1}^n \frac{1}{1 + y_j t}
\]
which implies that $e_i(y_1, \ldots, y_r) = (-1)^i h_i(y_{r+1}, \ldots, y_n)$ in $H_n$, for any $i, r \in [1, n]$. Applying the automorphism $D$ gives
\[
D(e_i(y_1, \ldots, y_r) + I_n) = h_i(y_1, \ldots, y_{n-r}) + I_n.
\]

2.3. Reverse double Schubert polynomials.

Definition 1. For any $\varpi \in S_\infty$, define the reverse double Schubert polynomial
\[
\widehat{\mathfrak{S}}_{\varpi}(Y, Z) := \sum_{uv = \varpi} \mathfrak{S}_u(Y)\mathfrak{S}_v(-Z),
\]
where the sum is over all reduced factorizations $uv = \varpi$.

The adjective ‘reverse’ in Definition 1 is justified by comparing (15) with formula (6). Observe also that if $\varpi \in S_n$, then $\widehat{\mathfrak{S}}_{\varpi}(Y, Z) = \langle \widehat{\mathfrak{S}}(Y, Z), \varpi \rangle$, where
\[
\widehat{\mathfrak{S}}(Y, Z) := A_1(y_1) \cdots A_{n-1}(y_{n-1}) \bar{A}_{n-1}(z_{n-1}) \cdots \bar{A}_1(z_1)
\]
and, for each variable $t$, the factors $A_i(t)$ and $\bar{A}_i(t)$ are defined as in §1.1.

Choose any integer $m \geq 0$, let $\delta'_n(m) := (m + 1, \ldots, m + n - 1)$, and $\delta'_n := (1, 2, \ldots, n - 1)$. Let $\Omega := (\omega_1, \ldots, \omega_m)$ be an $m$-tuple of independent variables and $\Omega + Y = (\omega_1, \ldots, \omega_m, y_1, y_2, \ldots)$ denote the concatenation of the alphabets $\Omega$ and $Y$. For every $i \in [1, m + n - 1]$, define $A_i(t) := (1 + tu_{m+n-1}) \cdots (1 + tu_i)$ and $\bar{A}_i(t) := A_i(t)^{-1}$. Furthermore, set $A(\Omega) := A_1(\omega_1) \cdots A_m(\omega_m)$ and
\[
\widehat{\mathfrak{S}}(\Omega + Y, Z) := A(\Omega) A_{m+1}(y_1) \cdots A_{m+n-1}(y_{n-1}) \bar{A}_{m+1}(z_1) \cdots \bar{A}_{m+n-1}(z_{n-1}).
\]
Let $\varpi_0$ denote the longest element of $S_n$ and consider the polynomial
\[
\widehat{\mathfrak{S}}_{1_m \times \varpi_0}(\Omega + Y, Z) := \langle \widehat{\mathfrak{S}}(\Omega + Y, Z), 1_m \times \varpi_0 \rangle.
\]

Proposition 3. For every integer $m \geq 0$, we have
\[
\widehat{\mathfrak{S}}_{1_m \times \varpi_0}(\Omega + Y, Z) = S_{\delta'_n(m)}^{\delta'_n}(h(\Omega + Y, -Z)),
\]
where the superscript $(\delta'_n(m), \delta'_n)$ indicates the number of $\omega$, $y$, and $z$ variables used in each row of the flagged Schur polynomial.

Proof. Let $\Omega' := (\omega'_1, \ldots, \omega'_m)$. Then
\[
\mathfrak{S}_{1_m \times \varpi_0}(\Omega + Y, \Omega' + Z) = \sum_{v_2} \mathfrak{S}_{v_2}(-Z) G_u(\Omega/\Omega') \mathfrak{S}_{v_1}(Y),
\]
so setting $\omega'_j = 0$ for each $j \in [1, m]$ gives
\[
\mathfrak{S}_{1_m \times \varpi_0}(\Omega + Y, Z) = \sum_{u, v_1 : \ell(u) + \ell(v_1) = \ell(uv_1)} G_u(\Omega) \mathfrak{S}_{v_1}(Y) \mathfrak{S}_{uv_1 \varpi_0}(-Z).
\]
Moreover, it follows from equations (7) and (16) that
\[
\widehat{\mathfrak{S}}_{1_m \times \varpi_0}(\Omega + Y, Z) = \sum_{u, v_1 : \ell(u) + \ell(v_1) = \ell(uv_1)} G_u(\Omega) \mathfrak{S}_{v_1}(Y) \mathfrak{S}_{uv_1 \varpi_0}(-Z).
\]
Suppose that $Y_1, \ldots, Y_t$ and $Z_1, \ldots, Z_t$ denote finite sets of independent variables. For any integer vector $\alpha = (\alpha_1, \ldots, \alpha_t)$, we introduce the multi-Schur polynomial

$$S_{\alpha}(Y_1 - Z_1; \ldots; Y_t - Z_t) := \det(h_{\alpha_i+j-i}(Y_i/Z_i))_{1 \leq j \leq t}.$$ 

These generalize the supersymmetric Schur polynomials defined in §1.2.

A permutation $\pi$ is called vexillary if it is 2143-avoiding, that is, there is no sequence $i < j < k < r$ such that $\pi_j < \pi_i < \pi_r < \pi_k$. The Schubert polynomials which are indexed by vexillary permutations may be expressed as multi-Schur polynomials (see [M2, (6.16)]). Let $Y_r := (y_1, \ldots, y_r)$ and $Z_r := (z_1, \ldots, z_r)$ for each $r \geq 0$. Since the permutation $1_m \times \pi_0$ is vexillary, we deduce from loc. cit. that

$$\mathfrak{S}_{1_m \times \pi_0}(\Omega + Y, \Omega' + Z) = S_{\delta_{n-1}}(\Omega + Y_1 - \Omega' - Z_{n-1}; \ldots; \Omega + Y_{(n-1)} - \Omega' - Z_{(n-1)})$$

and therefore, setting

$$\mathfrak{S}_{1_m \times \pi_0}(\Omega + Y, Z) = S_{\delta_{n-1}}(\Omega + Y_1 - Z_{n-1}; \ldots; \Omega + Y_{(n-1)} - Z_{(n-1)})$$

which sends $(z_1, \ldots, z_n)$ to $(-z_n, \ldots, -z_1)$ and leaves all the remaining variables fixed. Let $I^n_\pi \subset \mathbb{Z}[\Omega, Y_{(n)}, Z_{(n)}]$ be the ideal generated by the elementary symmetric polynomials $e_i(Z_{(n)})$ for $1 \leq i \leq n$. It follows from (18), (19), and Lemma 1 that

$$D_z(\mathfrak{S}_{1_m \times \pi_0}(\Omega + Y, Z) + I^n_\pi) = \mathfrak{S}_{1_m \times \pi_0}(\Omega + Y, Z) + I^n_\pi.$$ 

Furthermore, for $r, s \in [1, n-1]$, equation (14) implies that

$$D_z(h_j(Y_r - Z_s)) \equiv h_j(Y_r, -Z_{n-s})$$

modulo the ideal $I^n_\pi$. It follows from (20) that

$$D_z(\mathfrak{S}_{1_m \times \pi_0}(\Omega + Y, Z)) \equiv S_{\delta_{n-1}}(h(\Omega + Y_1, -Z_1), \ldots, h(\Omega + Y_{n-1}, -Z_{n-1}))$$

$$= S_{\delta_{n-1}}(\mathfrak{S}_{\delta_{n-1}}^{(\delta_{n-1}^{\cdots (m)}\delta_{n-1})}(h(\Omega + Y, -Z))).$$

We deduce that equation (17) holds modulo $I^n_\pi$.

Let $\mathcal{H}^\pi_\pi$ be the $\mathbb{Z}[\Omega, Y_{(n)}]$-linear subspace of $\mathbb{Z}[\Omega, Y_{(n)}, Z_{(n)}]$ spanned by the monomials $s^\alpha$ for $0 \leq \alpha \leq \delta_{n-1}$. Then the monomial expression (10) for the flagged Schur polynomial $S_{\delta_{n-1}}^{(\delta_{n-1}^{\cdots (m)}\delta_{n-1})}(h(\Omega + Y, -Z))$ in (17) proves that the latter lies in $\mathcal{H}^\pi_\pi$.

Since the same is clearly true of $\mathfrak{S}_{1_m \times \pi_0}(\Omega + Y, Z)$, the proposition follows. □

The next result is obtained by setting $m = 0$ in Proposition 3.

**Corollary 1.** We have $\mathfrak{S}_{\pi_0}(Y, Z) = S_{\delta_{n-1}}^{(\delta_{n-1}^{\cdots (m)}\delta_{n-1})}(h(Y, -Z))$.

**3. The Type C Theory**

3.1. **Schubert polynomials and divided differences.** The Weyl group for the root system of type $B_n$ or $C_n$ is the hyperoctahedral group $W_n$, which consists of signed permutations on the set $\{1, \ldots, n\}$. The group $W_n$ is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$ and the sign change $s_0(1) = T$ (as is customary, we use a bar over an entry with a negative sign). There is a natural embedding $W_n \hookrightarrow W_{n+1}$ defined by adding the fixed point $n+1$, and we let $W_\infty := \cup_n W_n$. The notions of length, reduced words, and descents of elements
of $W_{\infty}$ are defined as in the case of the symmetric group $S_{\infty}$, only now the simple reflections are indexed by the integers in the set $\mathbb{N}_0 := \{0, 1, \ldots\}$.

The nilCoxeter algebra $W_n$ of the hyperoctahedral group $W_n$ is the free associative algebra with unit generated by the elements $u_0, u_1, \ldots, u_{n-1}$ modulo the relations

$$
u_i^2 = 0 \quad i \in \mathbb{N}_0;$$

$$u_iu_j = u_ju_i \quad |i - j| \geq 2;$$

$$u_iu_{i+1}u_i = u_{i+1}u_iu_{i+1} \quad i > 0;$$

$$u_0u_1u_0u_1 = u_1u_0u_1u_0.$$

For any $w \in W_n$, define $u_w := u_{a_1} \cdots u_{a_t}$, where $a_1 \cdots a_t$ is any reduced word for $w$. Then the $u_w$ for $w \in W_n$ form a free $\mathbb{Z}$-basis of $W_n$. As in §1.1, we denote the coefficient of $u_w \in W_n$ in the expansion of the element $\xi \in W_n$ by $\langle \xi, w \rangle$.

Let $t$ be a variable and define

$$C(t) := (1 + tu_{n-1}) \cdots (1 + tu_1)(1 + tu_0)(1 + tu_1) \cdots (1 + tu_{n-1}).$$

Suppose that $X = (x_1, x_2, \ldots)$ is another infinite sequence of commuting variables, let $C(X) := C(x_1)C(x_2) \cdots$, and for $w \in W_n$, define

$$(21) \quad \mathfrak{C}_w(X; Y, Z) := \langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1)C(X)A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \rangle.$$

Set $\mathfrak{C}_w(X; Y) := \mathfrak{C}_w(X; Y, 0)$. The polynomials $\mathfrak{C}_w(X; Y)$ are the type C Schubert polynomials of Billey and Haiman [BH] and the $\mathfrak{C}_w(X; Y, Z)$ are their double versions introduced by Ikeda, Mihalcea, and Naruse [IMN1].

Note that $\mathfrak{C}_w$ is really a polynomial in the $Y$ and $Z$ variables, with coefficients which are formal power series in $X$, with integer coefficients. These power series are symmetric in the $X$ variables, since $C(s)C(t) = C(t)C(s)$, for any two commuting variables $s$ and $t$ (see [FK, Prop. 4.2]). We set

$$(22) \quad F_w(X) := \mathfrak{C}_w(X; 0, 0) = \langle C(X), w \rangle$$

and call $F_w$ the type C Stanley symmetric function indexed by $w \in W_n$. We deduce from (22) that the coefficients of $F_w(X)$ have a combinatorial interpretation (compare with [BH, (3.5)]). Observe also that we have $F_w = F_{w^{-1}}$.

The above definition of $\mathfrak{C}_w(X; Y, Z)$ implies that it is stable under the natural inclusions $W_n \hookrightarrow W_{n+1}$ of the Weyl groups, and hence is well defined for $w \in W_{\infty}$. Equation (21) implies the relation

$$(23) \quad \mathfrak{C}_w(X; Y, Z) = \sum_{uv\varpi = w} \mathfrak{C}_{u^{-1}}(-Z)F_v(X)\mathfrak{C}_{\varpi}(Y)$$

summed over all reduced factorizations $uv\varpi = w$ with $u, \varpi \in S_{\infty}$.

The type C Stanley symmetric functions $F_w(X)$ lie in the ring $\Gamma$ of Schur $Q$-functions $Q_\lambda(X)$. For each $r \in \mathbb{Z}$, define the basic function $q_r(X)$ by the equation

$$\prod_{i=1}^\infty \frac{1 + x_i t}{1 - x_i t} = \sum_{r=0}^\infty q_r(X)t^r.$$ 

For any integer vector $\alpha$, let $q_\alpha := \prod \alpha_i$, and define $Q_\alpha := Q_\alpha(X)$ by

$$(24) \quad Q_\alpha := R^\infty q_\alpha$$

where the raising operator expression $R^\infty$ is given by

$$R^\infty := \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}.$$
Following [M1, III.8], we can equivalently write formula (24) using a Schur Pfaffian [Sc]. More precisely, for integer vectors $\alpha = (\alpha_1, \alpha_2)$ with only two parts, we have

$$Q_{(\alpha_1, \alpha_2)} = \frac{1 - R_{12}}{1 + R_{12}} q_{(\alpha_1, \alpha_2)} = q_{\alpha_1} q_{\alpha_2} + 2 \sum_{j \geq 1} (-1)^j q_{\alpha_1 + j} q_{\alpha_2 - j},$$

while for an integer vector $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with three or more components,

$$Q_\alpha = \text{Pfaffian}(Q_{(\alpha_i, \alpha_j)})_{1 \leq i < j \leq 2\ell'}$$

where $\ell'$ is the least positive integer such that $2\ell' \geq \ell$.

A partition $\lambda$ is strict if all its (non-zero) parts $\lambda_i$ are distinct. It is known that the $Q_\lambda(X)$ for $\lambda$ a strict partition form a free $\mathbb{Z}$-basis of the ring $\Gamma := \mathbb{Z}[q_1(X), q_2(X), \ldots]$. For any $w \in W_\infty$, we have an identity

$$F_w(X) = \sum_{\lambda: |\lambda| = \ell(w)} e_\lambda^w Q_\lambda(X),$$

summed over all strict partitions $\lambda$ with $|\lambda| = \ell(w)$. At this stage we only need to know that an equation (26) exists with $e_\lambda^w \in \mathbb{Q}$. This latter fact follows immediately from the cancellation rule $C(t)C(-t) = 1$ and a result of Pragacz [P, Thm. 2.11] (see also [FK, §4]). We refer to [B, BH, La] for three different combinatorial proofs that the coefficients $e_\lambda^w$, when non-zero, are positive integers. Equations (23) and (26) show that the Schubert polynomials $\mathcal{C}_w(X; Y, Z)$ lie in the ring $\Gamma[Y, Z]$.

We define an action of $W_\infty$ on $\Gamma[Y, Z]$ by ring automorphisms as follows. The simple reflections $s_i$ for $i > 0$ act by interchanging $y_i$ and $y_{i+1}$ and leaving all the remaining variables fixed, as in §1.1. The reflection $s_0$ maps $y_1$ to $-y_1$, fixes the $y_j$ for $j \geq 2$ and all the $z_j$, and satisfies

$$s_0(q_\ell(X)) := q_\ell(y_1, x_1, x_2, \ldots) = q_\ell(X) + 2 \sum_{j=1}^r y_1^j q_{\ell-j}(X).$$

For each $i \geq 0$, define the divided difference operator $\partial_i^w$ on $\Gamma[Y, Z]$ by

$$\partial_i^w f := \frac{f - s_i f}{y_i - y_{i+1}} \quad \text{for } i > 0.$$

Consider the ring involution $\omega: \Gamma[Y, Z] \to \Gamma[Y, Z]$ determined by

$$\omega(y_j) = -z_j, \quad \omega(z_j) = -y_j, \quad \omega(q_\ell(X)) = q_\ell(X)$$

and set $\partial_i^w := \omega \partial_i^w \omega$ for each $i \geq 0$.

The polynomials $\mathcal{C}_w(X; Y, Z)$ for $w \in W_\infty$ are the unique family of elements of $\Gamma[Y, Z]$ satisfying the equations

$$\partial_i^w \mathcal{C}_w = \begin{cases} \mathcal{C}_{w s_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \quad \partial_i^w \mathcal{C}_w = \begin{cases} \mathcal{C}_{s_i w} & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \geq 0$, together with the condition that the constant term of $\mathcal{C}_w$ is 1 if $w = 1$, and 0 otherwise. As a consequence, the the descents of $w$ and $w^{-1}$ determine the symmetries of the polynomial $\mathcal{C}_w(X; Y, Z)$. Note however the special role that descents at zero play here. Furthermore, as in type A, one can show that the double Schubert polynomials $\mathcal{C}_w(X; Y, Z)$ represent the universal Schubert classes in type C flag bundles, and therefore degeneracy loci of symplectic vector bundles, in the sense of [Fu2]. For a simple proof of these assertions, see [T6, §7.3]. The geometrization of the Schubert polynomials $\mathcal{C}_w$ will be discussed in §5.
3.2. The Schubert polynomial indexed by the longest element. Let \( w_0 \) denote the longest element in \( W_n \). A formula for the top single Schubert polynomial \( C_{w_0}(X; Y) \) was given by Billey and Haiman [BH, Prop. 4.15]. In this section, we derive the analogue of their result for the double Schubert polynomial \( C_{w_0}(X; Y, Z) \), and use it to give a new proof of a Pfaffian formula for \( C_{w_0} \) due to Ikeda, Mihaleca, and Naruse [IMN1, Thm. 1.2].

Observe first that \( w_0 u = uw_0 \) for any \( u \in S_n \). Using this in (23) gives

\[
C_{w_0}(X; Y, Z) = \sum_{u,v} F_u(X)\mathcal{S}_{u^{-1}}(-Z)\mathcal{S}_v(Y)
\]

where the \( u, v \) in the sum lie in \( S_n \), and the factorization \( wvu = w_0 \) is reduced. It follows that

\[
C_{w_0}(X; Y, Z) = \sum_{\sigma \in S_n} F_{w_0\sigma^{-1}}(X)\tilde{\mathcal{S}}_\sigma(Y, Z).
\]

**Proposition 4.** The equation

\[
C_{w_0}(X; Y, Z) = \sum_{\lambda \subseteq \delta_{n-1}} Q_{\delta_n + \lambda}(X)S_{\delta_{n-1}/\lambda'}^{(\delta_n^{m-1}, \delta_{n-1}^{m-1})}(h(Y, -Z))
\]

holds in \( \Gamma[Y, Z] \).

**Proof.** Recall that \( \varpi_0 \) denotes the longest permutation in \( S_n \). According to [BH, Thm. 3.16] and [La, Thm. 3.15], we have, for every \( u \in S_n \),

\[
F_{w_0 u}(X) = \sum_{\lambda} a^{u^{-1} \varpi_0} Q_{\delta_n + \lambda}(X),
\]

summed over partitions \( \lambda \). We deduce from (28) and (30) that

\[
C_{w_0}(X; Y, Z) = \sum_{\lambda} Q_{\delta_n + \lambda}(X) \sum_{\sigma \in S_n} a^{\varpi_0} \tilde{\mathcal{S}}_\sigma(Y, Z).
\]

It therefore suffices to show that for every partition \( \lambda \), we have

\[
\sum_{\sigma \in S_n} a^{\varpi_0} \tilde{\mathcal{S}}_\sigma(Y, Z) = S_{\delta_{n-1}/\lambda'}^{(\delta_n^{m-1}, \delta_{n-1}^{m-1})}(h(Y, -Z)).
\]

This is a generalization of [BH, Eqn. (4.63)], and its proof is similar.

Choose any integer \( m \geq 0 \) and define \( \Omega := (\omega_1, \ldots, \omega_m) \) and \( \delta_{n-1}^{(m)} := (m + 1, \ldots, m + n - 1) \) as in §2.3. For any integer vector \( \gamma = (\gamma_1, \ldots, \gamma_{n-1}) \), we have

\[
h_{(\delta_{n-1}^{(m)}, \delta_{n-1}^{(m)})}(\Omega + Y, -Z) = \sum_{\alpha \geq 0} h_{\gamma - \alpha}(\Omega)h_{\alpha}^{(\delta_{n-1}^{(m)}, \delta_{n-1}^{(m)})}(Y, -Z)
\]

summed over all compositions \( \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \). Moreover, for any such composition \( \alpha \), we have \( s_{\delta_{n-1} - \alpha}(\Omega) = 0 \) unless \( \delta_{n-1} - \alpha + \delta_{n-2} = \sigma(\lambda + \delta_{n-2}) \) for some partition \( \lambda \) and permutation \( \sigma \in S_{n-1} \), in which case \( s_{\delta_{n-1} - \alpha}(\Omega) = (-1)^\sigma s_\lambda(\Omega) \).
Using this and (32), we compute that
\[ S_{\delta_n-1}^{(\delta_{n-1}, \delta_{n-1})} (h(\Omega + Y, -Z)) = \prod_{i<j} (1 - R_{ij}) h_{\delta_n-1}^{(\delta_{n-1}, \delta_{n-1})} (\Omega + Y, Z) \]
\[ = \sum_{\alpha \geq 0} s_{\delta_n-1 - \alpha} (\Omega) h_\alpha^{(\delta_{n-1}, \delta_{n-1})} (Y, -Z) \]
\[ = \sum_{\lambda} s_\lambda (\Omega) \sum_{\sigma \in S_{\delta_n-1}} (-1)^\sigma h_{\delta_n-1 + \delta_{n-2} - \sigma(\lambda + \delta_{n-2})} (Y, -Z). \]

We deduce that
\[ S_{\delta_n-1}^{(\delta_{n-1}, \delta_{n-1})} (\Omega + Y, -Z) = \sum_{\lambda \subseteq S_{\delta_n-1}} s_\lambda (\Omega) S_{\delta_n-1 / \lambda}^{(\delta_{n-1}, \delta_{n-1})} (h(Y, -Z)). \]

It follows from (7), (9), and the definition (16) that
\[ \tilde{S}_{1_m \times \varpi_0} (\Omega + Y, Z) = \sum_{u \in \varpi_0} G_u (\Omega) \tilde{S}_\sigma (Y, Z) = \sum_{\lambda} \sum_{v \in S_n} a_{\lambda \varpi_0} s_\lambda (\Omega) \tilde{S}_\sigma (Y, Z) \]
and hence
\[ \tilde{S}_{1_m \times \varpi_0} (\Omega + Y, Z) = \sum_{\lambda} s_\lambda (\Omega) \sum_{\sigma \in S_n} a_{\lambda \varpi_0} \tilde{S}_\sigma (Y, Z). \]

By Proposition 3, the left hand sides of equations (33) and (34) coincide. Comparing the coefficients of \( s_\lambda (\Omega) \) on the right hand sides of the same equations completes the proof of (31), and hence of the proposition.

Our next goal is to express the top polynomial \( C_{w_0} (X; Y, Z) \) as a Schur multi-Pfaffian analogous to equations (24) and (25). For any \( k, r \in \mathbb{Z} \), we define the polynomial \( k e_p^r = k e_p^r (X; Y, Z) \) by
\[ k e_p^r := \sum_{i=0}^p \sum_{j=0}^p q_{p-j-i} (X) h_i^r (Y) h_j^r (Z). \]
The polynomials \( k e_p^r \) were first studied by Wilson in [W, Def. 6 and Prop. 6]. For any integer sequences \( \alpha, \beta, \rho \), define \( \rho e_\alpha^\beta := \prod_i \rho_i e_\alpha^{\beta_i} \) and \( \rho h_\alpha^\beta := \prod_i \rho_i h_\alpha^{\beta_i} \). Given any raising operator \( R \), let \( R^\rho e_\alpha^\beta := e^{\rho R} c^{\beta^R} \). Finally, define the Schur multi-Pfaffian \( R^\rho Q_\alpha^\beta (c) := R^\rho c^{\beta^R} e_\alpha^\beta \).

**Proposition 5.** The equation
\[ C_{w_0} (X; Y, Z) = \delta_{n-1} Q_{\delta_n + \delta_{n-1}} (c) \]
holds in \( \Gamma [Y, Z] \).

**Proof.** If we set \( k h_m^r (Y, -Z) := \sum_{j=0}^m h_j^r (Y) h_{m-j}^r (Z) \), then we have
\[ k e_p^r = \sum_{j=0}^p q_{p-j} (X) k h_j^r (Y, -Z). \]
In particular, if \( k, r \geq 0 \), then we have
\[ k h_m^r (Y, -Z) = e_m (y_1, \ldots, y_k, -z_1, \ldots, -z_r) = e_m (Y (k), -Z (r)) \]
so that \( k h_m^r (Y, -Z) = 0 \) whenever \( m > k + r \).
Notice, using (37), that for any integer vector $\gamma = (\gamma_1, \ldots, \gamma_n)$, we have
\[
\delta_{n-1}c_{\gamma}^{-\delta_{n-1}} = \sum_{0 \leq \alpha \leq 2\delta_{n-1}} q_{\gamma - \alpha}(X)^{\delta_{n-1}}h_{\alpha}^{-\delta_{n-1}}(Y, -Z),
\]
and hence, by the definition (24),
\[
\delta_{n-1}Q_{\delta_{n-1} + \delta_{n-1}}^{-\delta_{n-1}}(c) = \sum_{0 \leq \alpha \leq 2\delta_{n-1}} Q_{\delta_n + \delta_{n-1} - \alpha}(X)^{\delta_{n-1}}h_{\alpha}^{-\delta_{n-1}}(Y, -Z).
\]
Recall that the Schur $Q$-functions $Q_{\gamma}(X)$ for $\gamma \geq 0$ are alternating in the components of $\gamma$. Therefore, we have $Q_{\delta_n + \delta_{n-1} - \alpha}(X) = 0$ in the above sum unless
\[
\delta_n + \delta_{n-1} - \alpha = 1^n + \sigma(\delta_{n-1} + \lambda)
\]
for some partition $\lambda \subset \delta_{n-1}$ and permutation $\sigma \in S_{n-1}$. Observe that $\alpha = 2\delta_{n-1} - \sigma(\delta_{n-1} + \lambda)$ is uniquely determined from $\lambda$ and $\sigma$. It follows that
\[
\delta_{n-1}Q_{\delta_{n-1} + \delta_{n-1}}^{-\delta_{n-1}}(c) = \sum_{\lambda \subset \delta_{n-1}} \sum_{\sigma \in S_{n-1}} Q_{1^n + \sigma(\delta_{n-1} + \lambda)}(X)^{\delta_{n-1}}h_{2\delta_{n-1} - \sigma(\delta_{n-1} + \lambda)}(Y, -Z)
\]
\[
= \sum_{\lambda \subset \delta_{n-1}} Q_{\delta_n + \lambda}(X) \sum_{\sigma \in S_{n-1}} (-1)^\sigma \delta_{n-1}h_{2\delta_{n-1} - \sigma(\delta_{n-1} + \lambda)}(Y, -Z)
\]
\[
= \sum_{\lambda \subset \delta_{n-1}} Q_{\delta_n + \lambda}(X) S_{\delta_{n-1} + \lambda}^{-\delta_{n-1}}(e(Y, -Z)).
\]
Taking $\lambda = \delta_{n-1}$, $\mu = \lambda^\vee$, $k = n$, $\ell = n - 1$, and $t = (y_1, -z_1, \ldots, y_{n-1}, -z_{n-1})$ in Proposition 2 gives
\[
S_{\delta_{n-1} + \lambda}^{-\delta_{n-1}}(e(Y, -Z)) = S_{\delta_{n-1} + \lambda^\vee}^{-\delta_{n-1}}(h(Y, -Z)).
\]
The result now follows by using equations (29) and (38).

3.3. The Schubert polynomials indexed by maximal elements. Consider a sequence $a : a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$. The sequence $a$ parameterizes a parabolic subgroup $W_a$ of $W_n$, generated by the simple reflections $s_i$ for $i \notin \{a_1, \ldots, a_p\}$. We let $W_n^a$ denote the set of minimal length left $W_a$-coset representatives. Recall that
\[
W_n^a = \{w \in W_n \mid \ell(ws_i) = \ell(w) + 1, \forall i \notin \{a_1, \ldots, a_p\}\}.
\]
Let $w_0(a)$ denote the longest element in $W_n^a$; we have
\[
w_0(a) = \left\{ \begin{array}{ll}
\sigma_2 \cdots \sigma_3 a_2 + 1 \cdots \sigma a_p + 1 & \text{if } a_1 = 0, \\
1 \cdots a_1 \sigma_2 \cdots a_1 + 1 \cdots \sigma a_p + 1 & \text{if } a_1 > 0.
\end{array} \right.
\]
It is known (see for example [St, §2]) that $W_n^a$ is an order ideal of the left weak Bruhat order of $W_n$, and that $w_0(a)$ is the unique maximal element of $W_n^a$ under this ordering.

Fix an integer $k$ with $0 \leq k < n$. The elements of the set $W_n^{(k)}$ are the $k$-Grassmannian elements of $W_n$. Let $w^{(k,n)} = 1 \cdots k \pi \cdots k + 1$ denote the longest element of $W_n^{(k)}$. Following [TW, §6.2], we will require a formula analogous to (36) for the Schubert polynomial $C_{w^{(k,n)}}(X; Y, Z)$, which maps to Kazarian’s multi-Pfaffian formula from [Ka, Thm. 1.1]. Equivalent Pfaffian formulas for the Schubert
polynomials $c_{w_0(a)}(X; Y, Z)$ were obtained by Anderson and Fulton in [AF1]. Ikeda and Matsumura [IM, §8.2] gave proofs of these formulas by applying left difference operators to the top Schubert polynomial $c_{w_0}$, and we will follow that approach here.

For every $i \geq 0$, the operator $\partial_i := \partial_i^+$ on $\Gamma[Y, Z]$ satisfies the Leibnitz rule

\begin{equation}
\partial_i(fg) = (\partial_i f)g + (s_i f)\partial_i g.
\end{equation}

The argument depends on the following basic lemmas.

**Lemma 2** ([IM], Lemma 5.4). Suppose that $p, r \in \mathbb{Z}$ and let $k \geq 0$. For all $i \geq 0$, we have

\[\partial_i(k^r c_p) = \begin{cases} k^{r+1} c_{p-1} & \text{if } r = \pm i, \\ 0 & \text{otherwise.} \end{cases}\]

**Lemma 3** ([IM], Lemma 8.2). Suppose that $i \geq 0$ and $k > 0$. Then we have

\[k^i c_p = k^{-i} c_p + (z_{i+1} + y_k)^k c_p - i .\]

**Lemma 4** ([IM], Prop. 5.4). Suppose that $k, r \geq 0$ and $p > k + r$. Then we have

\[\partial_i(c_{k,r} Q_{(p_1, \ldots, p_r)}) = 0.\]

**Example 2.** For any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have

\[\partial_0(c_{\alpha,n} c_{1,0}^n c_{\alpha,2}^n \cdots c_{\alpha,n}^n) = c_{\alpha,1} c_{\alpha,2}^n \cdots c_{\alpha,n}^n.\]

If $\delta_{n} := (n, n - 1, \ldots, 2) \in \mathbb{Z}^{n-1}$, it follows from (27) and (36) that

\[c_{s_0 w_0}(X; Y, Z) = \partial_0 c_{w_0}(X; Y, Z) = \delta_{n-1} Q_{\delta_{n-1}}(c).\]

Arguing as in §3.2, we can show that

\begin{equation}
c_{s_0 w_0}(X; Y, Z) = \sum_{\lambda \in \mathbb{R}_{n}} Q_{\delta_{n-1} + \lambda}(X) S_{\delta_{n-1} + \lambda}(e(Y, -Z)).
\end{equation}

**Proposition 6.** We have

\begin{equation}
c_{w(k,n)}(X; Y, Z) = (k, \ldots, k) Q_{(n+k, n+k-1, \ldots, 2k+1)}(c).
\end{equation}

in $\Gamma[Y, Z]$.

**Proof.** If $v^{(k,n)} = \overline{k} \cdots \overline{1} \overline{1} \cdots \overline{k+1}$ is the longest element in $W_{n}^{(0,k)}$, then we have a reduced factorization $w_0 = v_1 v_2 v^{(k,n)}$, where

\begin{align}
v_1 &:= (s_{k-1} \cdots s_1) (s_{k-1} \cdots s_2) \cdots (s_{k-1} s_{k-2}) s_{k-1} \\
v_2 &:= (s_{n-1} \cdots s_{k+2}) (s_{n-1} \cdots s_{k+2}) \cdots (s_{n-1} s_{n-2}) s_{n-1}.
\end{align}

Using (27), this implies the equation

\[c_{w(k,n)} = \partial_{n-1} c_{n-2} \partial_{n-1} \cdots (\partial_{k+1} \cdots \partial_{n-1}) \partial_{k-1} c_{k-2} \partial_{k-1} \cdots (\partial_1 \cdots \partial_{k-1}) c_{w_0}.\]

Assume that $k \geq 2$, as the proof when $k \in \{0, 1\}$ is easier. Using Lemmas 2 and 3, for any $p \in \mathbb{Z}$ we have

\begin{equation}
\partial_{k-1} c_p = k \partial_{k-1} c_{p-1} = k^2 c_{p-1} + (t_{k-1} + y_k)^k c_{p-2}.
\end{equation}
Let $\epsilon_j$ denote the $j$-th standard basis vector in $\mathbb{Z}^n$. The Leibniz rule and (44) imply that for any integer vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have
\[
\partial_{k-1}^{-\delta_n-1} e_\alpha^{-\delta_n-1} = \delta_{n-1}^{-1-\epsilon_{n+1-k}} e_\alpha^{-\delta_n-1} + (t_{k-1} + y_{k-1}) \delta_{n-1}^{-1-\epsilon_{n+1-k}} e_\alpha^{-\delta_n-1} + \epsilon_{n+1-k}.
\]
We deduce from this and Lemma 4 that
\[
\partial_{k-1}^{-\delta_n-1} Q_{\delta_n+\delta_n-1}^{-\delta_n-1}(c) = \delta_{n-1}^{-1-\epsilon_{n+1-k}} Q_{\delta_n+\delta_n-1}^{-\epsilon_{n+1-k}}(c) = (n-1,k-2,2-2,\ldots,1) Q^{(1-n,\ldots,-1)}(2n-1,2k+1,2k-3,\ldots,1)(c).
\]
Iterating this calculation gives
\[
(\partial_1 \cdots \partial_{k-1}) C_{w_0} = (n-1,k-2,2-2,\ldots,1) Q^{(1-n,\ldots,-1)}(2n-1,2k+1,2k-3,\ldots,2,1)(c)
\]
and furthermore
\[
\partial_{k-1}^{-\delta_n-1} Q_{\delta_n+\delta_n-1}^{-\delta_n-1}(c) = (n-1,k,0,0,\ldots,0) Q^{(1-n,\ldots,-1)}(2n+1,2k+1,2k-3,\ldots,1)(c).
\]
Applying the operator $\partial_{k-1}^{-\delta_n-1} Q_{\delta_n+\delta_n-1}^{-\delta_n-1}$ to both sides of the above equation, we similarly obtain
\[
C_{w(k,n)} = (k,\ldots,k,0,0,\ldots,0) Q^{(1-n,\ldots,-1)}(n+k+1,k,k,k)(c).
\]
Since $w(k,n) = (s_0 \cdots s_{k-1}) \cdots (s_0 s_1) s_0 u(k,n)$, equation (27) gives
\[
C_{w(k,n)} = \partial_0(\partial_1 \cdots \partial_{k-1}) C_{w(k,n)},
\]
Finally, since $(\rho,\beta,\gamma)(c) = \rho Q_\alpha(\gamma)(c)$ for any integers $r$ and $b$, it follows that
\[
C_{w(k,n)} = \partial_0(\partial_1 \cdots \partial_{k-1}) C_{w(k,n)}
\]
and furthermore
\[
\partial_{k-1}^{-\delta_n-1} Q_{\delta_n+\delta_n-1}^{-\delta_n-1}(c) = (k,\ldots,k) Q^{(1-n,\ldots,-k)}(n+k+1,2k+1)(c).
\]

Since $w(k,n) = (w(k,n))^{-1}$, the polynomial $C_{w(k,n)}(X;Y,Z)$ should be symmetric in the $Z$ variables as well as in the $Y$ variables, however this is not reflected in equation (41). The next proposition makes this symmetry apparent. Recall (for example from [Fu3, §6.2]) that for any three partitions $\lambda$, $\mu$, and $\nu$, the Littlewood-Richardson number $N^\lambda_{\mu\nu}$ is the nonnegative integer defined by the equation of Schur $S$-functions
\[
s_\mu(t)s_\nu(t) = \sum_\lambda N^\lambda_{\mu\nu}s_\lambda(t).
\]
Let $\mu_0 := (2k)^{n-k} = (2k,\ldots,2k)$, and for every $\mu \subseteq \mu_0$, define $\mu^\vee := (2k - \mu_{n-k},\ldots,2k - \mu_1)$. Note that $(n+k, n+k-1, \ldots, 2k+1) = \delta_{n-k} + \mu_0$.

**Proposition 7.** We have
\[
C_{w(k,n)}(X;Y,Z) = (k,\ldots,k) Q_{\delta_n+\delta_n-1}^{(1-n,\ldots,-k)}(c)
\]
and furthermore
\[
(\nu_1,\nu_2) \subseteq \mu \subseteq \mu_0
\]
\[
N^\mu_{\nu_1\nu_2} Q_{\delta_n+\delta_n-1+\mu^{\vee}}(X) s_{\nu_1}(Y(k)) s_{\nu_2}(-Z(k)),
\]
in $\Gamma[Y,Z]$, where $N^\mu_{\nu_1\nu_2}$ denotes a Littlewood-Richardson number.
Proof. For any integer vector \( \gamma = (\gamma_1, \ldots, \gamma_{n-k}) \), we have

\[
\left( b_{\gamma_1, \ldots, \gamma_{n-k}} \right) = \sum_{\alpha \leq \delta_{n-k-1} + \mu_0} q_{\gamma - \alpha} \alpha (X) e^{(n+k-1, \ldots, 2k)} (Y_k, -Z)
\]

where the alphabet \( Y_k \) in the factor \( e^{(n+k-1, \ldots, 2k)} (Y_k, -Z) \) is constant, while \( Z \) varies down from \( Z^{(n-k)} \) to \( Z_k \). Now using (41) and (45) while applying the raising operator \( R^\infty \), along with the alternating property of Schur \( Q \)-functions, gives

\[
\mathcal{C}_{\omega(k,n)} (X; Y, Z) = \sum_{\mu \leq \mu_0} Q_{\delta_{n-k-1} + \mu_0 - \alpha} (X) e^{(n+k-1, \ldots, 2k)} (Y_k, -Z)
\]

\[
= \sum_{\mu \leq \mu_0} Q_{\delta_{n-k} + \mu} (X) \det (e^{(2k, \ldots, 2k)} (Y_k, -Z))_{1 \leq i,j \leq n-k}
\]

\[
= \sum_{\mu \leq \mu_0} Q_{\delta_{n-k} + \mu} (X) S_{\mu_0 / \mu} (e (Y_k, -Z)).
\]

We claim that for each partition \( \mu \subseteq \mu_0 \), we have

\[
\det (e^{(2k, \ldots, 2k)} (Y_k, -Z)) = \det (e^{(2k, \ldots, 2k)} (Y_k, -Z)) = S_{\mu_0 / \mu} (e (Y_k, -Z)).
\]

The proof of this follows [M2, (3.4)]. For each \( i, j \) with \( 1 \leq i, j \leq n-k \),

\[
e^{(n+k-1, \ldots, 2k)} (Y_k, -Z) = e^{(n+k-1, \ldots, 2k)} (Y_k, -Z_{n-k})
\]

\[
= \sum_{p=1}^{n-k} e_{p-i}(B_i) e^{(n+k-1, \ldots, 2k)} (Y_k, -Z_{n-k}),
\]

where \( B_i = (z_{k+1}, \ldots, z_{n-k}) \) (in particular, \( B_{n-k} = \emptyset \)). Therefore the matrix

\[
\{e^{(n+k-1, \ldots, 2k)} (Y_k, -Z)\}_{1 \leq i,j \leq n-k}
\]

is the product of the matrix

\[
\{e_{p-i}(B_i)\}_{1 \leq i,p \leq n-k},
\]

which is unitriangular, and the matrix

\[
\{e^{(n+k-1, \ldots, 2k)} (Y_k, -Z)\}_{1 \leq p,j \leq n-k}.
\]

Taking determinants completes the proof of the claim.

Since \( S_{\mu_0 / \mu} (e (Y_k, -Z)) = S_{\mu / \mu} (e (Y_k, -Z)) \), we deduce that

\[
\mathcal{C}_{\omega(k,n)} (X; Y, Z) = \sum_{\mu \leq \mu_0} Q_{\delta_{n-k} + \mu} (X) S_{\mu / \mu} (e (Y_k, -Z))
\]

\[
= (k, \ldots, k) Q_{\delta_{n-k} + \mu_0} (e).
\]
Furthermore, using [M1, I.(5.9)], we compute that
\[
\begin{align*}
\mathfrak{c}_{w(k,n)}(X; Y, Z) &= \sum_{\mu \subseteq \mu_0} Q_{\delta_{n-k} + \mu'}(X)S_\mu(e(Y)_k, -Z_k)) \\
&= \sum_{\mu \subseteq \mu_0} Q_{\delta_{n-k} + \mu'}(X)s_\mu'(Y)_k, -Z_k) \\
&= \sum_{\nu \subseteq \mu \subseteq \mu_0} Q_{\delta_{n-k} + \mu'}(X)s_{\mu'/\nu'}(Y)_k)s_\nu(-Z_k) \\
&= \sum_{\nu_1, \nu_2 \subseteq \mu \subseteq \mu_0} N_{\nu_1' \nu_2'}\delta_{n-k} + \mu'(X)s_{\nu_1'}(Y)_k)s_{\nu_2'}(-Z_k).
\end{align*}
\]
Since we have \(N_{\nu_1' \nu_2'} = N_{\nu_1' \nu_2}\), the result follows. \(\square\)

Although we only require the formula for \(\mathfrak{c}_{w(k,n)}(X; Y, Z)\), we will record the general result here for comparison with the orthogonal case, which is discussed in §4.4. According to [AF1] and [IM, Thm. 8.2], we have
\[
\mathfrak{c}_{w_0(a)}(X; Y, Z) = \rho(a)Q_{\lambda(a)}(c),
\]
where \(\lambda(a)\), \(\beta(a)\), and \(\rho(a)\) denote the sequences
\[
\lambda(a) = (n + a_p, \ldots, 2a_p + 1, \ldots, a_i + a_{i+1}, \ldots, 2a_i + 1, \ldots, a_1 + 1, \ldots, 2a_1 + 1);
\]
\[
\beta(a) = (1 - n, \ldots, -a_p, \ldots, 1 - a_{i+1}, \ldots, -a_i, \ldots, 1 - a_2, \ldots, -a_1);
\]
and
\[
\rho(a) = (a_p^{a_p - a_{p-1}}, \ldots, a_i^{a_i + 1 - a_{i+1}}, \ldots, a_1^{a_1 - a_2}).
\]

### 3.4. Theta polynomials.

The next step in this program is to prove formulas for the Schubert polynomials indexed by the \(k\)-Grassmannian elements of \(W_\infty\). We will see that they may be expressed using theta polynomials.

We say that a partition \(\lambda\) is \(k\)-strict if no part greater than \(k\) is repeated, that is, \(\lambda_j > k\) implies \(\lambda_{j+1} < \lambda_j\), for each \(j \geq 1\). There is an explicit bijection between \(k\)-Grassmannian elements \(w\) of \(W_\infty\) and \(k\)-strict partitions \(\lambda\), such that the elements in \(W_n\) correspond to those partitions whose diagram fits inside an \((n-k) \times (n+k)\) rectangle. According to [BKT1, §4.1 and §4.4], if the element \(w\) corresponds to the \(k\)-strict partition \(\lambda\), then the bijection is given by the equations
\[
\lambda_i = \begin{cases} 
|w_{k+i}| + k & \text{if } w_{k+i} < 0, \\
\# \{p \leq k : w_p > w_{k+i} \} & \text{if } w_{k+i} > 0.
\end{cases}
\]
Using the above bijection, we attach to any \(k\)-strict partition \(\lambda\) a finite set of pairs
\[
C(\lambda) := \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j \text{ and } w_{k+i} + w_{k+j} < 0\}
\]
and a sequence \(\beta(\lambda) = \{\beta_j(\lambda)\}_{j \geq 1}\) defined by
\[
\beta_j(\lambda) := \begin{cases} 
w_{k+j} + 1 & \text{if } w_{k+j} < 0, \\
w_{k+j} & \text{if } w_{k+j} > 0.
\end{cases}
\]

Following [BKT2], let \(\lambda\) be any \(k\)-strict partition, and consider the raising operator expression \(R^\lambda\) given by
\[
R^\lambda := \prod_{i < j} (1 - R_{ij}) \prod_{(i,j) \in C(\lambda)} (1 + R_{ij})^{-1}.
\]
For any integer sequences $\alpha$ and $\beta$, define $c^\beta_\alpha := \prod_i k c^\beta_\alpha$. According to [TW, W], the double theta polynomial $\Theta_\lambda(X;Y(k), Z)$ is defined by

$$\Theta_\lambda(X;Y(k), Z) := R^\lambda \cdot c^\beta_\lambda.$$  

The single theta polynomial $\Theta_\lambda(X;Y(k))$ of [BKT2] is given by

$$\Theta_\lambda(X;Y(k)) := \Theta_\lambda(X | Y(k), 0).$$

Note that we are working here with the images of the theta polynomials $\Theta$ and $\lambda$ in $\Gamma[Y,Z]$ of double Schubert polynomials, following [T5, T6].

Fix a rank $n$ and let

$$\lambda_0 := (n + k, n + k - 1, \ldots, 2k + 1)$$

be the $k$-strict partition associated to the $k$-Grassmannian element $w^{(k,n)}$ of maximal length in $W_n$. We deduce from Proposition 6 and the definition (49) that

$$\Theta_{\lambda_0}(X;Y(k), Z)$$

in $\Gamma[Y,Z]$. It follows from [IM, Lemma 5.5] that for all $i \geq 1$ and indices $p$ and $q$, we have

$$\partial_i c^{(i)}_{(p,q)} = c^{(i+1)}_{(p+1,q)} + c^{(i+1)}_{(p,q+1)} = (1 + R_{12}) c^{(i+1)}_{(p+1,q+1)}.$$  

Using this identity and Lemma 2, it is shown in [TW, Prop. 5] that if $\lambda$ and $\mu$ are $k$-strict partitions such that $|\lambda| = |\mu| + 1$ and $w_\lambda = w_{\mu}$ for some simple reflection $s_i \in W_\infty$, then we have

$$\partial_i \Theta_{\lambda}(X;Y(k), Z) = \Theta_{\mu}(X;Y(k), Z)$$

in $\Gamma[Y,Z]$. Now (50) and (51) imply that for any $k$-strict partition $\lambda$ with associated $k$-Grassmannian element $w_\lambda$, we have

$$\Theta_{w_\lambda}(X;Y, Z) = \Theta_\lambda(X;Y(k), Z)$$

in $\Gamma[Y,Z]$. In particular, we recover the equality

$$\Theta_{w_\lambda}(X;Y) = \Theta_{\lambda}(X;Y(k))$$

in $\Gamma[Y]$ from [BKT2, Prop. 6.2] for the single polynomials.

### 3.5. Mixed Stanley functions and splitting formulas

Following [T5, §2], for any $w \in W_\infty$, the double mixed Stanley function $J_w(X;Y)$ is defined by the equation

$$J_w(X;Y/Z) := \langle \hat{A}(Z)C(X)A(Y), w \rangle = \sum_{uvw = w} G_{u^{-1}}(-Z)F_w(X)G_w(Y),$$

where the sum is over all reduced factorizations $uvw = w$ with $u,v \in S_\infty$. The single mixed Stanley function $J_w(X;Y)$ is given by setting $Z = 0$ in $J_w(X;Y/Z)$. Observe that $J_w(X;Y/Z)$ is separately symmetric in the three sets of variables $X$, $Y$, and $Z$, and that we have $J_w(X;0) = F_w(X)$.

Fix an integer $k \geq 0$. We say that an element $w \in W_\infty$ is increasing up to $k$ if $0 < w_1 < w_2 < \cdots < w_k$ (this condition is automatically true if $k = 0$). If $w$ is increasing up to $k$, then [BH, Eqn. (2.5)] and equation (7) have a natural analogue for the restricted mixed Stanley function $J_w(X;Y(k))$, which is obtained
from \( J_w(X;Y) \) after setting \( y_i = 0 \) for \( i > k \). In this case, according to [T5, Prop. 5], we have

\[
\mathcal{C}_w(X;Y) = \sum_{v(1_k \times w) = w} J_v(X;Y_{(k)}) \mathcal{S}_w(y_{k+1}, y_{k+2}, \ldots),
\]

where the sum is over all reduced factorizations \( v(1_k \times w) = w \) in \( W_\infty \) with \( w \in S_\infty \). Moreover, there is a double version of equation (53) which is parallel to (8).

Let \( J_v(X;Y_{(k)}/Z_{(\ell)}) \) denote the power series obtained from \( J_v(X;Y/Z) \) by setting \( y_i = z_j = 0 \) for all \( i > k \) and \( j > \ell \). Then if \( w \) is increasing up to \( k \) and \( w^{-1} \) is increasing up to \( \ell \), we have

\[
\mathcal{C}_w(X;Y,Z) = \sum \mathcal{S}_{u-1}(-Z_{>\ell}) J_v(X;Y_{(k)}/Z_{(\ell)}) \mathcal{S}_w(Y_{>k}),
\]

where \( Y_{>k} := (yk+1, yk+2, \ldots) \), \( -Z_{>\ell} := (-z^{\ell+1}, -z^{\ell+2}, \ldots) \), and the sum is over all reduced factorizations \( (1_\ell \times u)v(1_k \times w) = w \) in \( W_\infty \) with \( u, w \in S_\infty \).

We say that an element \( w \in W_\infty \) is compatible with the sequence \( a : a_1 < \cdots < a_q \) of elements of \( N_0 \) if all descent positions of \( w \) are contained in \( a \). Let \( b : b_1 < \cdots < b_q \) be a second sequence of elements of \( N_0 \) and assume that \( w \) is compatible with \( a \) and \( w^{-1} \) is compatible with \( b \). We say that a reduced factorization \( u_1 \cdots u_{p+q-1} = w \) is compatible with \( a, b \) if \( u_i \in S_\infty \) for all \( i \not= q \), \( u_j(i) = i \) whenever \( j < q \) and \( i \leq b_{q-j} \) or whenever \( j > q \) and \( i \leq a_{j-q} \). Set \( Y_i := \{y_{a_{i-1}+1}, \ldots, y_{a_i}\} \) for each \( i \geq 1 \) and \( Z_j := \{z_{b_{j-1}+1}, \ldots, z_{b_j}\} \) for each \( j \geq 1 \).

**Proposition 8.** Suppose that \( w \) and \( w^{-1} \) are compatible with \( a \) and \( b \), respectively. Then the Schubert polynomial \( \mathcal{C}_w(X;Y,Z) \) satisfies

\[
\mathcal{C}_w = \sum G_{u_1}(0/Z_q) \cdots G_{u_{q-1}}(0/Z_2) J_{u_q}(X;Y_{1}/Z_1) G_{u_{q+1}}(Y_2) \cdots G_{u_{p+q-1}}(Y_p)
\]

summed over all reduced factorizations \( u_1 \cdots u_{p+q-1} = w \) compatible with \( a, b \).

**Proof.** The result is established by combining the identity (7) with (54). \( \square \)

If \( w \) is increasing up to \( k \), then the following generalization of equation (26) holds (see [T5, Thm. 1]):

\[
J_w(X;Y_{(k)}) = \sum_{\lambda: |\lambda| = \ell(w)} e^w_\lambda \Theta_\lambda(X;Y_{(k)}),
\]

where the sum is over \( k \)-strict partitions \( \lambda \) with \( |\lambda| = \ell(w) \). The *mixed Stanley coefficients* \( e^w_\lambda \) in (55) are nonnegative integers. In fact, to any \( w \in W_\infty \) increasing up to \( k \) we associate a \( k \)-transition tree \( T^k(w) \) whose leaves are \( k \)-Grassmannian elements, and \( e^w_\lambda \) is equal to the number of leaves of the tree \( T^k(w) \) which have shape \( \lambda \). The proof of (55) in [T5] is a straightforward application of Billey’s transition equations for symplectic flag manifolds [B] combined with equation (52).

Assume that \( w \) is increasing up to \( k \) and \( w^{-1} \) is increasing up to \( \ell \). At present there is no clear analogue of equation (55) for the (restricted) double mixed Stanley function \( J_w(X;Y_{(k)}/Z_{(\ell)}) \). However, we have

\[
J_w(X;Y_{(k)}/Z_{(\ell)}) = \sum_{u,v \in \mu} G_{u^{-1}}(-Z_{(\ell)}) J_v(X;Y_{(k)})
\]

\[
= \sum_{u,v \in \mu} G_{u^{-1}}(Y_{(k)}) J_v(X; -Z_{(\ell)}),
\]
where the factorizations under the sum signs are reduced with \( u \in S_\infty \). We can now use equations (9) and (55) in (56) and (57) to obtain two dual expansions of \( J_w(X; Y_{(k)}/Z_{(\ell)}) \) as a positive sum of products of Schur S-polynomials with theta polynomials.

**Example 3.** Let \( w = 231 = s_1s_2 \in W_3 \) and take \( k = \ell = 1 \). We have

\[
s_231(X; Y, Z) = q_2(X) + q_1(X)(y_1 + y_2 - z_1) + (y_1 - z_1)(y_2 - z_1)
\]

and hence \( J_{231}(X; Y_{(1)}/Z_{(1)}) = q_2(X) + q_1(X)(y_1 - z_1) - (y_1 - z_1)z_1 \). Equality (56) gives

\[
J_{231}(X; Y_{(1)}/Z_{(1)}) = J_{231}(X; Y_{(1)}) + G_{213}(-Z_{(1)})J_{132}(X; Y_{(1)}) + G_{312}(-Z_{(1)})
\]

\[
= \Theta_2(X; y_1) + s_1(-z_1)\Theta_1(X; y_1) + s_2(-z_1)
\]

\[
= (q_2(X) + q_1(X)y_1) + (-z_1)(q_1(X) + y_1) + z_1^2,
\]

while equality (57) gives

\[
J_{231}(X; Y_{(1)}/Z_{(1)}) = J_{312}(X; -Z_{(1)}) + G_{132}(Y_{(1)})J_{213}(X; -Z_{(1)}) + G_{231}(Y_{(1)})
\]

\[
= \Theta_{(1,1)}(X; -z_1) + s_1(y_1)\Theta_1(X; -z_1) + s_{(1,1)}(y_1)
\]

\[
= (q_2(X) - q_1(X)z_1 + z_1^2) + y_1(q_1(X) - z_1).
\]

**Theorem 2** ([T5], Cor. 1). Suppose that \( w \) is compatible with \( a \) and \( w^{-1} \) is compatible with \( b \), where \( b_1 = 0 \). Then we have

\[
\mathcal{C}_w = \sum_{\lambda} f^w_\lambda s_{\lambda^1}(0/Z_q) \cdots s_{\lambda^{p-1}}(0/Z_2)\Theta_{\lambda^p}(X; Y_1)s_{\lambda^{q+1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)
\]

summed over all sequences of partitions \( \lambda = (\lambda^1, \ldots, \lambda^{p+q-1}) \) with \( \lambda^p \) \( a_1 \)-strict, where

\[
f^w_\lambda := \sum_{u_1 \cdots u_p = w} a_{\lambda^1}^{u_1} \cdots a_{\lambda^{p-1}}^{u_{p-1}} e_{\lambda^p}^{u_p+1} a_{\lambda^{p+1}}^{u_p+2} \cdots a_{\lambda^{p+q-1}}^{u_{p+q-1}}
\]

summed over all reduced factorizations \( u_1 \cdots u_{p+q-1} = w \) compatible with \( a, b \).

**Proof.** The result follows from Proposition 8 by using equations (9) and (55). \( \square \)

4. The Type D Theory

For the orthogonal Lie types B and D we work with coefficients in the ring \( \mathbb{Z}[\frac{1}{2}] \). For \( w \in W_\infty \), the type B double Schubert polynomial \( \mathfrak{B}_w \) of [IMN1] is related to the type C Schubert polynomial by the equation \( \mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w \), where \( s(w) \) denotes the number of \( i \) such that \( w_i < 0 \). We will therefore omit any further discussion of type B, and concentrate on the even orthogonal case. The exposition is parallel to that of §3, but there are some variations in the results and in their proofs.

### 4.1. Schubert Polynomials and Divided Differences

The Weyl group \( \widetilde{W}_n \) for the root system \( D_n \) is the subgroup of \( W_n \) consisting of all signed permutations with an even number of sign changes. The group \( \widetilde{W}_n \) is an extension of \( S_n \) by the element \( s_{\square} = s_0s_1s_0 \), which acts on the right by

\[
(s_1, s_2, \ldots, s_{n-1})s_{\square} = (s_2, s_1, s_3, \ldots, s_n).
\]

There is a natural embedding \( \widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1} \) of Weyl groups defined by adjoining the fixed point \( n+1 \), and we let \( \widetilde{W}_\infty := \cup_n \widetilde{W}_n \). The elements of the set \( \tilde{N}_{\square} := \{ \square, 1, \ldots \} \)
index the simple reflections in \( \widetilde{W}_\infty \); these are used to define the reduced words and descents of elements in \( \widetilde{W}_\infty \) as in the previous sections.

The nilCoxeter algebra \( \widetilde{W}_n \) of \( \widetilde{W}_n \) is the free associative algebra with unit generated by the elements \( u_0, u_1, \ldots, u_{n-1} \) modulo the relations

\[
\begin{align*}
    u_i^2 &= 0 & i & \in \mathbb{N}_0 ; \\
    u_0 u_1 &= u_1 u_0 \\
    u_0 u_2 u_0 &= u_2 u_0 u_2 \\
    u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} & i & > 0 ; \\
    u_i u_j &= u_j u_i & j > i + 1, & \text{and } (i, j) \neq (0, 2).
\end{align*}
\]

As in §3.1, for any \( w \in \widetilde{W} \), choose a reduced word \( a_1 \cdots a_d \) for \( w \), and define \( u_w := u_{a_1} \cdots u_{a_d} \). Denote the coefficient of \( u_w \in \widetilde{W}_n \) in the expansion of the element \( \xi \in \widetilde{W}_n \) in the \( u_w \) basis by \( \langle \xi, w \rangle \). Let \( t \) be a variable and, following Lam [La], define

\[
D(t) := (1 + tu_{n-1}) \cdots (1 + tu_2)(1 + tu_1)(1 + tu_0)(1 + tu_2) \cdots (1 + tu_{n-1}).
\]

Let \( D(X) := D(x_1)D(x_2) \cdots \), and for \( w \in \widetilde{W}_n \), define

\[
(60) \quad \mathfrak{D}_w(X; Y, Z) := \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1)D(X)A_1(y_1) \cdots A_{n-1}(y_{n-1}, w) \right\rangle.
\]

The power series \( \mathfrak{D}_w(X; Y) := \mathfrak{D}_w(X; Y, 0) \) are the type D Billey-Haiman Schubert polynomials, and the \( \mathfrak{D}_w(X; Y, Z) \) are their double versions from [IMN1].

The double Schubert polynomial \( \mathfrak{D}_w(X; Y, Z) \) is stable under the natural inclusions \( \widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1} \), and hence is well defined for \( w \in \widetilde{W}_\infty \). We set

\[
E_w(X) := \mathfrak{D}_w(X; 0, 0) = \langle D(X), w \rangle
\]

call \( E_w \) the type D Stanley symmetric function indexed by \( w \in \widetilde{W}_n \). Observe that we have \( E_w = E_{w^{-1}} \). Equation (60) implies the relation

\[
(61) \quad \mathfrak{D}_w(X; Y, Z) = \sum_{uv=\varnothing = w} \mathfrak{S}_{u^{-1}}(-Z)E_v(X)\mathfrak{S}_{\varnothing}(Y)
\]

summed over all reduced factorizations \( uw\varnothing = w \) with \( u, \varnothing \in S_{\infty} \).

For each strict partition \( \lambda \), the Schur \( P \)-function \( P_\lambda(X) \) is defined by the equation

\[
P_\lambda(X) := 2^{-\ell(\lambda)}Q_\lambda(X), \quad \text{where } \ell(\lambda) \text{ denotes the length of } \lambda.
\]

The type D Stanley symmetric functions \( E_w(X) \) lie in the ring \( \Gamma' := \mathbb{Z}[P_1, P_2, \ldots] \) of Schur \( P \)-functions.

In fact, for any \( w \in \widetilde{W}_\infty \), we have an equation

\[
(62) \quad E_w(X) = \sum_{\lambda: |\lambda| = \ell(w)} d_\lambda^w P_\lambda(X)
\]

summed over all strict partitions \( \lambda \) with \( |\lambda| = \ell(w) \). Since \( D(t)D(-t) = 1 \), it follows from [P, Thm. 2.11] that an identity (62) exists with coefficients \( d_\lambda^w \in \mathbb{Z} \). Given equation (61), this implies that \( \mathfrak{D}_w(X; Y, Z) \) is an element of \( \Gamma'[X; Y, Z] \), for any \( w \in \widetilde{W}_\infty \). For three different proofs that \( d_\lambda^w \geq 0 \), see [B, BH, La].

We define an action of \( \widetilde{W}_\infty \) on \( \Gamma'[Y, Z] \) by ring automorphisms as follows. The simple reflections \( s_i \) for \( i > 0 \) act by interchanging \( y_i \) and \( y_{i+1} \) and leaving all the remaining variables fixed, as in §1.1. The reflection \( s_0 \) maps \( (y_1, y_2) \) to \( (-y_2, -y_1), \)
fixes the \( y_j \) for \( j \geq 3 \) and all the \( z_j \), and satisfies, for any \( r \geq 1 \),
\[
s_{\square}(P_r(X)) := P_r(y_1, y_2, x_1, x_2, \ldots )
= P_r(X) + (y_1 + y_2) \sum_{j=0}^{r-1} \left( \sum_{a+b=j} y_1^a y_2^b \right) Q_{r-1-j}(X).
\]

For each \( i \in \mathbb{N}_0 \), define the divided difference operator \( \partial_i^w \) on \( \Gamma'[Y,Z] \) by
\[
\partial_i^w := \frac{f - s_{\square} f}{y_i - y_i} \quad \text{for } i > 0.
\]

Consider the ring involution \( \omega : \Gamma'[Y,Z] \to \Gamma'[Y,Z] \) determined by
\[
\omega(y_j) = -z_j, \quad \omega(z_j) = \omega(y_j) = -y_j,
\omega(P_r(X)) = P_r(X)
\]
and set \( \partial_i^w := \omega \partial_i^w \omega \) for each \( i \in \mathbb{N}_0 \).

The polynomials \( \mathcal{D}_w(X;Y,Z) \) for \( w \in \tilde{W}_\infty \) are the unique family of elements of \( \Gamma'[Y,Z] \) satisfying the equations
\[
(63) \quad \partial_i^w \mathcal{D}_w = \begin{cases} \mathcal{D}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \quad \partial_i^w \mathcal{D}_w = \begin{cases} \mathcal{D}_{s_i w} & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{otherwise,} \end{cases}
\]
for all \( i \in \mathbb{N}_0 \), together with the condition that the constant term of \( \mathcal{D}_w \) is 1 if \( w = 1 \), and 0 otherwise. As in §1.1 and §3.1, it follows that descents of \( w \) and \( w^{-1} \) determine the symmetries of the double Schubert polynomial \( \mathcal{D}_w(X;Y,Z) \), and that the polynomials \( \mathcal{D}_w \) represent degeneracy loci of even orthogonal vector bundles, in the sense of [Fu2].

4.2. Schur \( P \)-functions and their double analogues. Let \( n \geq 1 \) be an integer and \( \ell \in [1,n] \). Let \( \alpha = (\alpha_1, \ldots, \alpha_\ell) \) be a composition, and define a polynomial \( P_\alpha^{(\ell)}(x_1, \ldots, x_n) \) by the equation
\[
(64) \quad P_\alpha^{(\ell)}(x_1, \ldots, x_n) := \frac{1}{(n-\ell)!} \sum_{w \in S_n} \varpi \left( x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell} \prod_{i \leq \ell, i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \right).
\]

Let \( S_{n-\ell} \) denote the subgroup of \( S_n \) consisting of permutations of \( \{ \ell + 1, \ldots, n \} \).

Since the expression \( x^{\alpha} \prod_{i \leq \ell, i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \) is symmetric in \( (x_{\ell+1}, \ldots, x_n) \), we deduce that
\[
(65) \quad P_\alpha^{(\ell)}(x_1, \ldots, x_n) = \sum_{\sigma \in S_n/S_{n-\ell}} \sigma \left( x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell} \prod_{i \leq \ell, i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \right).
\]

It follows from [Iv1, Prop. 1.1(c)] that \( P_\alpha^{(\ell)}(x_1, \ldots, x_n) = 0 \) if \( \alpha_i = \alpha_j \) for some \( i \neq j \).

Hence, the polynomial \( P_\alpha^{(\ell)}(x_1, \ldots, x_n) \) is alternating in the indices \( (\alpha_1, \ldots, \alpha_\ell) \).

**Lemma 5.** Assume that \( n \) is even. If \( \alpha_\ell = 0 \), then we have
\[
P_\alpha^{(\ell)}(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } \ell \text{ is odd,} \\ P_\alpha^{(\ell-1)}(x_1, \ldots, x_n) & \text{if } \ell \text{ is even.} \end{cases}
\]
Proof. According to [Iv2, Prop. 2.4], for any \( m \geq 1 \), we have

\[
\sum_{\varpi \in \mathcal{S}_m} \varpi \left( \prod_{j=2}^{m} \frac{x_1 + x_j}{x_1 - x_j} \right) = \begin{cases} 
0 & \text{if } m \text{ is even,} \\
(m - 1)! & \text{if } m \text{ is odd.}
\end{cases}
\]

Let \( H \cong S_{n+1-\ell} \) denote the subgroup of \( S_n \) consisting of permutations of \( \{\ell, \ldots, n\} \), and set \( P^{(\ell,n)}_\alpha := P^{(\ell,n)}_\alpha(x_1, \ldots, x_n) \). Using (65) and equation (66), we compute that

\[
(n-\ell)! P^{(\ell,n)}_\alpha = \sum_{\sigma \in \mathcal{S}_n/H} \sum_{\varpi \in \mathcal{H}} \sigma \varpi \left( \prod_{i<\ell,i<j} x_i + x_j x_i - x_j \right)
= \sum_{\varpi \in \mathcal{H}} \sum_{\sigma \in \mathcal{S}_n/H} \sigma \left( \prod_{i<\ell,i<j} x_i + x_j x_i - x_j \right) \prod_{j=\ell+1}^{n} x_{\varpi(j)} x_{\varpi(\ell)} - x_{\varpi(j)} x_{\varpi(\ell)}
= \sum_{\sigma \in \mathcal{S}_n/H} \sigma \left( \prod_{i<\ell,i<j} x_i + x_j x_i - x_j \right) \prod_{j=\ell+1}^{n} x_{\varpi(j)} x_{\varpi(\ell)} - x_{\varpi(j)} x_{\varpi(\ell)}
= \begin{cases} 
0 & \text{if } \ell \text{ is odd,} \\
(n-\ell)! P^{(\ell-1,n)}_\alpha & \text{if } \ell \text{ is even.}
\end{cases}
\]

Let \( t = (t_1, t_2 \ldots) \) be a sequence of independent variables, as in §2.1, and define \((x \mid t)^r := (x - t_1) \cdots (x - t_r)\). Given a strict partition \( \lambda \) of length \( \ell \) and \( n \geq \ell \), Ivanov’s double Schur \( P \)-function \( P_\lambda(x_1, \ldots, x_n \mid t) \) is defined by

\[
P_\lambda(x_1, \ldots, x_n \mid t) := \frac{1}{(n-\ell)!} \sum_{\varpi \in \mathcal{S}_n} \varpi \left( \prod_{i=1}^{\ell} (x_1 \mid t)^{\lambda_i} \prod_{i<\ell,i<j} x_i + x_j x_i - x_j \right) .
\]

Following [IMN1, §4.2], we let \( P_\lambda(X \mid t) \) denote the (even) projective limit of the functions \( P_\lambda(x_1, \ldots, x_{2m} \mid t) \) as \( m \to \infty \). We have that \( P_\lambda(X \mid 0) = P_\lambda(X) \) is the Schur \( P \)-function indexed by the partition \( \lambda \).

**Proposition 9.** Let \( \lambda \) be a strict partition of length \( \ell \).

(a) Suppose that \( \ell \) is even and \( \lambda = \delta_{\ell-1} + \mu \) for some partition \( \mu \). Then

\[
P_\lambda(X \mid t) = \sum_{\nu \subseteq \mu} P_{\delta_{\ell-1}+\nu}(X) S^\lambda_{\mu/\nu} (e(-t)).
\]

(b) Suppose that \( \ell \) is odd and \( \lambda = \delta_{\ell} + \mu \) for some partition \( \mu \). Then

\[
P_\lambda(X \mid t) = \sum_{\nu \subseteq \mu} P_{\delta_{\ell}+\nu}(X) S^\lambda_{\mu/\nu} (e(-t)).
\]

Proof. For any \( r \geq 1 \), we have \((x \mid t)^r = \sum_{p=0}^{r} x^p e^{r-p}(-t)\). Therefore for any partition \( \lambda \) of length \( \ell \), we have

\[
\prod_{i=1}^{\ell} (x_i \mid t)^{\lambda_i} = \sum_{\alpha \geq 0} x^{\alpha} e^{\lambda_{\alpha}(-t)} .
\]
It follows from (64), (67), and (68) that for any \( n \geq \ell \), we have
\[
P_\lambda(x_1, \ldots, x_n \mid t) = \sum_{0 \leq \alpha \leq \lambda} P_\alpha^{(\ell)}(x_1, \ldots, x_n) e_{\lambda-\alpha}^\lambda(-t).
\]

Taking the even projective limit as \( n \to \infty \) gives
\[
(69) \quad P_\lambda(X \mid t) = \sum_{0 \leq \alpha \leq \lambda} P_\alpha^{(\ell)}(X) e_{\lambda-\alpha}^\lambda(-t).
\]

For part (a), using (69) and Lemma 5 gives
\[
P_\lambda(X \mid t) = \sum_{\mu \subset \nu} (-1)^\sigma P_{\delta_\nu-1+\nu}(X) e_{\delta_\nu-1+\mu-\sigma(\delta_\nu+1)}(-t)
\]
\[
= \sum_{\mu \subset \nu} P_{\delta_\nu+\nu}(X) S_{\mu/\nu}^\lambda(e(-t)).
\]

For part (b), we similarly obtain
\[
P_\lambda(X \mid t) = \sum_{\mu \subset \nu} (-1)^\sigma P_{\delta_\nu+\nu}(X) e_{\delta_\nu+\mu-\sigma(\delta_\nu+\nu)}(-t)
\]
\[
= \sum_{\mu \subset \nu} P_{\delta_\nu+\nu}(X) S_{\mu/\nu}^\lambda(e(-t)).
\]

\( \square \)

**Corollary 2.** (a) If \( n \) is odd, then
\[
P_{2\delta_{n-1}}(X \mid t) = \sum_{\nu \subset \delta_n} P_{\delta_{n-2}+\nu}(X) S_{\delta_{n-1}/\nu}^{2\delta_{n-1}}(e(-t)).
\]
(b) If \( n \) is even, then
\[
P_{2\delta_{n-1}}(X \mid t) = \sum_{\nu \subset \delta_{n-1}} P_{\delta_{n-1}+\nu}(X) S_{\delta_{n-1}/\nu}^{2\delta_{n-1}}(e(-t)).
\]

According to [Iv3, §9] and [IN, §8.3], we have a Pfaffian formula
\[
(70) \quad P_\lambda(X \mid t) = \text{Pfaffian}(P_\mu(X \mid t))_{1 \leq i < j \leq 2\ell},
\]
where \( 2\ell' \) is the least even integer which is greater than or equal to \( \ell(\lambda) \). In equation (70), we use the conventions that \( P_{a,b}(X \mid t) := -P_{b,a}(X \mid t) \) whenever \( 0 \leq a \leq b \), and \( P_{n,0}(X \mid t) := P_n(X \mid t) \). We will require a raising operator expression analogous to (24) for the functions \( P_\lambda(X \mid t) \). This uses a more involved Pfaffian formalism which stems from the work of Knuth [Kn, §4] and Kazarian [Ka, App. C and D].

For any \( r \in \mathbb{Z} \), we define the polynomial \( c_r^p = c_r^p(X \mid t) \) by
\[
c_r^p := \sum_{j=0}^{\mu} q_{p-j}(X)e_j^\lambda(-t).
\]

For any integer sequences \( \alpha, \beta \), let
\[
c_\alpha^\beta := c_\alpha^\beta c_\alpha^\beta \cdots
\]
where, for each \( i \geq 1 \),
\[
c_\alpha^\beta := c_\alpha^\beta + \begin{cases} (-1)^j e_\alpha^\lambda(-t) & \text{if } \beta_i = \alpha_i > 0, \\ 0 & \text{otherwise.} \end{cases}
\]
If $R := \prod_{i<j} R_{ij}^{n_{ij}}$ is any raising operator, denote by $\text{supp}(R)$ the set of all indices $i$ and $j$ such that $n_{ij} > 0$. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \ldots, \beta_\ell)$ be integer vectors, set $\nu := R\alpha$, and define

$$R \ast \tilde{c}_{\alpha} = \tilde{c}_{\nu} := \tilde{c}_{\nu_1} \cdots \tilde{c}_{\nu_\ell}$$

where for each $i \geq 1$,

$$\tilde{c}_{\nu_i} := \begin{cases} c_{\nu_i} & \text{if } i \in \text{supp}(R), \\ \tilde{c}_{\nu_i} & \text{otherwise}. \end{cases}$$

**Proposition 10.** For any strict partition $\lambda$, we have

$$P_\lambda(X \mid t) = 2^{-\ell(\lambda)} R^\infty \ast \tilde{c}_{\lambda}.$$ 

**Proof.** It follows from [Kn, Ka] that the equation

$$R^\infty \ast \tilde{c}_{\lambda} = 2^{\ell(\lambda)} \text{Pfaffian}(P_{\lambda_i, \lambda_j}(X \mid t))_{i<j}$$

holds if and only if it holds for all strict partitions $\lambda$ of length $\ell$ at most 3. The latter is a formal identity which is straightforward to check from the definitions; compare with [AF2, App. A] and [IMN2, §2.3]. We conclude from (70) and (72) that (71) is also true. \hfill \qed

### 4.3. The Schubert polynomial indexed by the longest element.

Let $\tilde{w}_0$ denote the longest element in $W_n$. We have

$$\tilde{w}_0 = \begin{cases} (\bar{1}, \ldots, \bar{n}) & \text{if } n \text{ is even}, \\ (1, 2, \ldots, n) & \text{if } n \text{ is odd}. \end{cases}$$

A formula for the top single Schubert polynomial $\mathcal{D}_{\tilde{w}_0}(X ; Y)$ was given by Billey and Haiman [BH, Prop. 4.15]. In this section, we derive the analogue of their result for the double Schubert polynomial $\mathcal{D}_{\tilde{w}_0}(X ; Y, Z)$, and use it to give a combinatorial proof of the Pfaffian formula for $\mathcal{D}_{\tilde{w}_0}$ from [IMN1, Thm. 1.2].

**Proposition 11.** If $n$ is even, then we have

$$\mathcal{D}_{\tilde{w}_0}(X ; Y, Z) = \sum_{\lambda \subseteq \delta_{n-1}} P_{\delta_{n-1}+\lambda}(X) S_{\delta_{n-1}/\lambda}^{(\delta_{n-1}, \delta_{n-1})}(e(Y, -Z))$$

in $\Gamma'[Y, Z]$.

**Proof.** Using equation (61), we have

$$\mathcal{D}_{\tilde{w}_0}(X ; Y, Z) = \sum_{uvw = \tilde{w}_0} E_u(X) S_{u^{-1}}(-Z) S_v(Y).$$

summed over all reduced factorizations $uvw = \tilde{w}_0$ in $\tilde{W}_\infty$ with $u, v \in S_\infty$. If $n$ is even, then every permutation in $S_n$ commutes with $\tilde{w}_0$, and it follows that

$$\mathcal{D}_{\tilde{w}_0}(X ; Y, Z) = \sum_{\sigma \in S_n} E_{\tilde{w}_0 \sigma^{-1}}(X) S_\sigma(Y, Z).$$

According to [BH, Thm. 3.16] and [La, Thm. 5.14], we have, for every $u \in S_n$,

$$E_{\tilde{w}_0 u}(X) = \sum_\lambda a^{\tilde{w}_0 \sigma^{-1}} \lambda P_{\delta_{n-1}+\lambda}(X).$$

We deduce from (73) and (74) that

$$\mathcal{D}_{\tilde{w}_0}(X ; Y, Z) = \sum_\lambda P_{\delta_{n-1}+\lambda}(X) \sum_{\sigma \in S_n} a^{\tilde{w}_0 \sigma} \tilde{S}_\sigma(Y, Z).$$
Let $f_k$ be defined depending on the context. We also let $f_{k}^{r}$.

The result now follows by combining equations (31) and (38).

Fix $k \geq 0$, and set $k c_p(X; Y) := \sum_{i=0}^{p} g_{p-i}(X) h_{i}^{-k}(Y)$, so that we have

$$k c_{p}^{r}(X; Y, Z) = \sum_{j=0}^{p} k c_{p-j}(X; Y) h_{j}^{-r}(Z).$$

Define $k b_r := k c_r$ for $r < k$, $k b_r := \frac{1}{2} k c_r$ for $r > k$, and set

$$k b_k := \frac{1}{2} k c_k + \frac{1}{2} e_k(Y) \quad \text{and} \quad k b_k := \frac{1}{2} k c_k - \frac{1}{2} e_k(Y).$$

Let $f_k$ be an indeterminate of degree $k$, which will be equal to $k b_k$, $k b_k$, or $\frac{1}{2} k c_k$, depending on the context. We also let $f_0 \in \{0, 1\}$. For any $p, r \in \mathbb{Z}$, define $k c_p^r$ by

$$k c_p^r := k c_r + \begin{cases} (2 f_k - k c_k) e_{p-k}^{r}(-Z) & \text{if } r = k - p < 0, \\ 0 & \text{otherwise}. \end{cases}$$

In particular, we have

$$0 c_p^r = 0 c_p^r \pm \begin{cases} e_p^{r}(-Z) & \text{if } r = -p < 0, \\ 0 & \text{otherwise}, \end{cases}$$

and thus $0 c_p^{p} = 0 c_p^{p} \pm e_p^{r}(-Z)$ when $p > 0$, while $0 c_0^{0} = 1$. It follows from [IMN2, Eq. (2.14)] that

$$P_r(X \mid Z) = \frac{1}{2} 0 c_r^{r} - e_r^{r}(-Z),$$

that is, $P_r(X \mid Z) = \frac{1}{2} 0 c_r^{r}$ with the choice of $f_0 = 0$.

Recall that we have defined left divided differences $\partial_i = \partial_i^r$ for each $i \in \mathbb{N}$. These operators satisfy the same Leibnitz rule (39) as in the type C case. We now have the following even orthogonal analogues of Lemmas 2 and 3.

**Lemma 6** ([T7], Prop. 2). Suppose that $p, r \in \mathbb{Z}$ and let $k \geq 0$ and $i \geq 1$.

a) We have

$$\partial_i^{(k c_p^r)} = \begin{cases} k c_{p-i}^{r+1} & \text{if } r = \pm i, \\ 0 & \text{otherwise}. \end{cases}$$

b) If $p > k$, we have

$$\partial_i^{(k c_p^{p-k})} = \begin{cases} k c_{p-i-1}^{k-p+1} & \text{if } i = p - k \geq 2, \\ 2 f_k & \text{if } i = p - k = 1, \\ 0 & \text{otherwise}. \end{cases}$$

**Lemma 7.** Suppose that $i \geq 0$ and $k > 0$. Then we have

$$k c_p^{-i} = k c_p^{-i} + (z_{i+1} + y_k) k c_p^{-i-1}.$$

**Proof.** We know from Lemma 3 that

$$(75) \quad k c_p^{-i} = k c_p^{-i} + (z_{i+1} + y_k) k c_p^{-i-1}.$$

If $i \neq p - k$ or $i = 0$ there is nothing more to prove. If $i = p - k > 0$ the result follows from (75) and the fact that

$$e_k^{r}(Y) e_{p-k}^{r}(-Z) = e_{k-i}^{r+1}(Y) e_{p-k+1}^{r+1}(-Z) + (z_{p-k+1} + y_k) e_{k-i}^{r+1}(Y) e_{p-k}^{r}(-Z).$$
For any integer sequences $\alpha$, $\beta$ and composition $\rho$, let
\[ \rho^\beta_{\alpha} := \rho_1^\beta_1 \rho_2^\beta_2 \ldots \]
where, for each $i \geq 1$,
\[ \rho_i^\beta_{\alpha_i} := \rho_i^\beta_1 \left\{ \frac{(-1)^i e^{\rho_i(Y)} e^{\alpha_i - \rho_i}(-Z)}{0} \right\} \text{ if } \beta_i = \rho_i - \alpha_i < 0, \]
\[ \text{otherwise.} \]

**Definition 2.** Let $\rho$ be a composition and $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, $\beta = (\beta_1, \ldots, \beta_\ell)$ be two integer vectors. Let $R$ be any raising operator, set $\nu := R\alpha$, and define
\[ R \ast \rho^\beta_{\alpha} = \rho^\beta_{\nu} := \rho_1^\beta_1 \rho_2^\beta_2 \ldots \rho_\ell^\beta_\ell \]
where for each $i \geq 1$,
\[ \rho_i^\beta_{\nu_i} := \begin{cases} \rho_i^\beta_1 & \text{if } i \in \text{supp}(R), \\ \rho_i^\beta_{\alpha_i} & \text{otherwise.} \end{cases} \]

Set
\[ (76) \]
\[ \rho \widehat{P}^\beta(\xi) := 2^{-\ell} R^\infty \ast \rho^\beta_{\alpha}. \]

**Lemma 8.** Suppose that $\beta_i = \rho_i - \alpha_i < 0$ for every $i \in [1, \ell]$. Then we have
\[ (77) \]
\[ \rho \widehat{P}^\beta(\xi) = 2^{-\ell} \text{ Pfaffian}(\rho_i^\beta_{\alpha_i}, \rho_i^\beta_{\nu_i})_{i < j}. \]

In addition, if $\alpha_j = \alpha_{j+1}$ and $\beta_j = \beta_{j+1}$ for some $j \in [1, \ell - 1]$, then
\[ (78) \]
\[ \rho \widehat{P}^\beta(\xi) = 0. \]

**Proof.** Arguing as in §4.2 for the double Schur $P$-functions, one shows that the raising operator expression $R^\infty \ast \rho^\beta_{\alpha}$ in (76) may be written formally as the Schur-type Pfaffian in (77). The proof of the vanishing statement (78) is similar to [IM, Prop. 5.4]. Suppose that $k, r \geq 0$, let $\xi$ be a formal variable, and
\[ F(\xi) := \sum_{p=0}^k c_p^r \xi p = \prod_{i=1}^k \frac{1 + x_i \xi}{1 - x_i \xi} \prod_{j=1}^k (1 + y_j \xi) \prod_{m=1}^r (1 - z_m \xi) \]
be the generating function for the sequence $\{c_p^r\}_{p \geq 0}$. Then we clearly have
\[ (79) \]
\[ F(\xi) F(-\xi) = \prod_{j=1}^k (1 - y_j^2 \xi^2) \prod_{m=1}^r (1 - z_m^2 \xi^2). \]

Equating the like even powers of $\xi$ on both sides of (79) gives
\[ \frac{1 - R_{12}^k}{1 + R_{12}^k} c_p^r, \quad c_p^r = \begin{cases} e_k(y_1^2, \ldots, y_k^2) c_r(z_1^2, \ldots, z_r^2) & \text{if } p = k + r, \\ 0 & \text{if } p > k + r. \end{cases} \]

We deduce that if $p \geq k + r$, then
\[ \frac{1 - R_{12}^k}{1 + R_{12}^k} c_p^r = 0, \]
and therefore that $k, k \widehat{P}^r_{p, p}(\xi) = 0$. Equation (78) now follows using (77) and the alternating properties of Pfaffians, as in [Ka, §1] and [IM, §4]. \qed

The next result is equivalent to Ikeda, Mihalcea, and Naruse’s Pfaffian formula for $D_{\alpha_0}$ from [IMN1, Thm. 1.2].
Proposition 12. For any integer \( n \geq 1 \), we have
\[
(80) \quad \mathcal{D}_{\tilde{w}_0}(X;Y,Z) = \delta_{n-1} \tilde{P}_{2\delta_{n-1}}(c)
\]
in \( \Gamma'[Y,Z] \).

Proof. It follows from Propositions 10 and 11 and Corollary 2(b) that (80) holds when \( n \) is even. Assume that \( n \) is even for the rest of this proof. Since we have
\[
s_{n-1} \cdots s_1 \circ s_{n-1} \tilde{w}_0^{(n)} = \tilde{w}_0^{(n-1)}
\]
in \( \tilde{W}_n \), we get using (63) a corresponding equation of divided differences
\[
\mathcal{D}_{\tilde{w}_0^{(n-1)}} = (\partial_{n-1} \cdots \partial_2)(\partial \circ \partial_1)(\partial_2 \cdots \partial_{n-1})\mathcal{D}_{\tilde{w}_0^{(n)}}
\]
in \( \Gamma'[Y,Z] \). We therefore obtain the equality
\[
(81) \quad \mathcal{D}_{\tilde{w}_0^{(n-1)}} = 2^{1-n}(\partial_{n-1} \cdots \partial_2)(\partial \circ \partial_1)(\partial_2 \cdots \partial_{n-1}) \left(R^\infty * \delta_{n-1}C_{\delta_{n-1}} \right).
\]
We next compute the action of the divided differences on the right hand side of (81). By using Lemmas 7 and 8 and arguing as in §3.3, we see that
\[
(\partial_2 \cdots \partial_{n-1}) \left(R^\infty * \delta_{n-1}C_{\delta_{n-1}} \right) = R^\infty * (n-2,n-3,\ldots,1)C_{(2n-3,n-5,\ldots,3)}.
\]
For any integer vector \( \alpha := (\alpha_1, \ldots, \alpha_{n-1}) \), we have
\[
\partial_t (n-2,n-3,\ldots,1)C_{\alpha} = \delta_{n-2} \frac{1}{C_{\alpha}} \left((1-n,2-n,\ldots,-2)1\right) g_0^{(n-1)} - 1,
\]
where
\[
g_0 = \begin{cases} 2^{1/2} f_1 & \text{if } p = 1, \\ 1 & \text{otherwise}. \end{cases}
\]
According to [T7, §1], for \( k \geq 1 \), we have
\[
(82) \quad \partial \circ (kC_p) = 2^k C_p \quad \text{and} \quad \partial \circ (kB_k) = \partial \circ (kB_k) = \delta_{2k-1}.
\]
We therefore also have \( \partial \circ (1f_1) = 1 \) by (82) and \( \partial \circ (1g_0) = 0 \) for \( p \leq 0 \). It follows that
\[
(83) \quad \partial_t (n-2,n-3,\ldots,1)C_{\alpha} = \begin{cases} 2^k \delta_{n-2} C_{(\alpha_1,\ldots,\alpha_{n-2})} & \text{if } \alpha_{n-1} = 2, \\ 0 & \text{if } \alpha_{n-1} < 2. \end{cases}
\]
We deduce from (83) that \( \partial \circ \partial_t \) commutes with the action of the raising operators \( R \) in the expansion of \( R^\infty \) in its \( \star \)-action on \( (n-2,n-3,\ldots,1)C_{(2n-3,2n-5,\ldots,3)} \), and hence that
\[
(84) \quad (\partial \circ \partial_1 \cdots \partial_{n-1}) \left(R^\infty * \delta_{n-2}C_{(2n-3,2n-5,\ldots,3)} \right) = 2 R^\infty * \delta_{n-2}C_{(2n-3,2n-5,\ldots,3)}.
\]
We continue applying Lemma 6(b) to compute the action of \( \partial_{n-1} \cdots \partial_2 \) on \( R^\infty * \delta_{n-2}C_{(2n-3,2n-5,\ldots,3)} \), to conclude that
\[
\mathcal{D}_{\tilde{w}_0^{(n-1)}} = 2^{2-n} (\partial_{n-1} \cdots \partial_2) \left(R^\infty * \delta_{n-2}C_{(2n-3,2n-5,\ldots,3)} \right) = 2^{2-n} R^\infty * \delta_{n-2}C_{2\delta_{n-2}} \quad \text{and hence that (80) holds for all } n \geq 1, \text{ as required.} \]
Corollary 3. If \( n \) is odd, then we have

\[
\mathfrak{D}_{\bar{w}_0}(X; Y, Z) = \sum_{\lambda \subseteq \delta_n} P_{\delta_n - 2 + \lambda}(X) S^{(\delta_n - 1, \delta_n - 1)}_{\delta_n / \lambda}(e(Y, -Z))
\]

in \( \Gamma'[Y, Z] \).

Proof. This follows by combining Proposition 12 with (71) and Corollary 2(a). \( \square \)

Note the similarity between formulas (40) and (84). It would be interesting to expose a more direct argument connecting the two to each other.

4.4. The Schubert polynomials indexed by maximal elements. Consider a sequence \( a : a_1 < \cdots < a_p \) of elements of \( \mathbb{N}_{\square} \) with \( a_p < n \). The sequence \( a \) parametrizes a parabolic subgroup \( \tilde{W}_a \) of \( \tilde{W}_n \), which is generated by the simple reflections \( s_i \) for \( i \notin \{a_1, \ldots, a_p\} \). In type D, we will only consider sequences \( a \) with \( a_1 \neq 1 \), since these suffice to parametrize all the relevant homogeneous spaces and degeneracy loci, up to isomorphism.\(^2\) This claim is due to the natural involution of the Dynkin diagram of type \( D_n \). Geometrically, it is explained by the fact that any isotropic subspace \( E_{n-1} \) of \( \mathbb{C}^{2n} \) (equipped with an orthogonal form) with \( \dim(E_{n-1}) = n - 1 \) can be uniquely extended to a two-step flag \( E_{n-1} \subset E_n \) with \( E_n \) maximal isotropic and in a given family (compare with [T6, §6.3.2]).

Define the set \( \tilde{W}_n^a \) by

\[ \tilde{W}_n^a := \{ w \in \tilde{W}_n \mid \ell(ws_i) = \ell(w) + 1, \forall i \notin \{a_1, \ldots, a_p\} \} \]

and let \( \tilde{w}_0(a) \) denote the longest element in \( \tilde{W}_n^a \). We have

\[ \tilde{w}_0(a) = \begin{cases} \underbar{a}_{2} \cdots \underbar{2} \overline{\overline{a}}_{3} \cdots a_{2} + 1 \cdots \overline{\overline{a}}_{p} + 1 \cdots \overline{\overline{a}}_{p} + 1 & \text{if } a_1 = \square, \\ \overline{\overline{1}} \cdots a_1 \underbar{a}_{2} \cdots a_{1} + 1 \cdots a_{p} + 1 \cdots a_{p} + 1 & \text{if } a_1 \neq \square, \end{cases} \]

where \( \overline{\overline{1}} \) is equal to either 1 or \( \overline{1} \), specified so that \( \tilde{w}_0(a) \) contains an even number of barred integers.

Fix an element \( k \in \mathbb{N}_{\square} \) with \( \square \leq k < n \), and set \( \tilde{W}_n^{(k)} := \tilde{W}_n^{(\square, 1)} \). The elements of the set \( \tilde{W}_n^{(k)} \) are the \( k \)-Grassmannian elements of \( \tilde{W}_n \). Let \( \tilde{w}^{(k, n)} := \overline{\overline{1}} \cdots k \overline{\overline{1}} \cdots k + 1 \overline{\overline{1}} \overline{\overline{1}} \cdots k \) denote the longest element of \( \tilde{W}_n^{(k)} \). Following [T7, §3.2], we will require a formula analogous to (80) for the Schubert polynomial \( \mathfrak{D}_{\tilde{w}^{(k, n)}}(X; Y, Z) \), which maps to Kazarian’s multi-Pfaffian formula from [Ka, Thm. 1.1]. Corresponding Pfaffian formulas for the Schubert polynomials \( \mathfrak{D}_{\tilde{w}_0(n)}(X; Y, Z) \) were obtained in [AF1].

Proposition 13. We have

\[
\mathfrak{D}_{\tilde{w}^{(k, n)}}(X; Y, Z) = (k, \ldots, k) \mathfrak{P}_{(n+k-1, \ldots, n+k)}^{1-n-k, \ldots, -k}(c).
\]

in \( \Gamma'[Y, Z] \).

Proof. Let \( \overline{\overline{1}} \cdots k \overline{\overline{1}} \cdots k + 1 \overline{\overline{1}} \overline{\overline{1}} \cdots k \) be the longest element in \( \tilde{W}_n^{(\square, k)} \). Then we have a reduced factorization \( \tilde{w}_0 = v_1 v_2 \tilde{w}^{(k, n)} \), where \( v_1 \) and \( v_2 \) are defined by (42) and (43), as in the type C case. Using the equations (63), we obtain the relation

\[
\mathfrak{D}_{\tilde{w}^{(k, n)}} = \partial_{n-1} (\partial_{n-2} \partial_{n-1}) \cdots (\partial_{k+1} \cdots \partial_{n-1}) \cdot \partial_{k-1} (\partial_{k-2} \partial_{k-1}) \cdots (\partial_{1} \cdots \partial_{k-1}) \mathfrak{D}_{\tilde{w}_0}.
\]

\(^2\)This convention is simpler than the one used in [T5, §6] and [T6, §5.3].
Assume that \( k \geq 2 \), as the proof is easier if \( k \in \{\Box, 1\} \). First, using Lemmas 6 and 7, for any \( p \in \mathbb{Z} \) we have
\[
\partial_{k-1}^{-1} c_p^{-1} = k^{-1} c_p^{-1} = k^{-2} c_p^{-1} = (t_{k-1} + y_{k-1}) \partial_{k-1}^{-1} c_p^{-1} + (t_{k-1} + y_{k-1})^{k-1} c_p^{-1}.
\]
The Leibnitz rule and (85) imply that for any integer vector \( \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \), we have
\[
\partial_{k-1} \delta_{\alpha} \delta_{\alpha-1} = \delta_{\alpha-1}^{-1} \epsilon_{\alpha+1-k} \delta_{\alpha} \delta_{\alpha-1} + (t_{k-1} + y_{k-1})^{\delta_{\alpha-1}^{-1} \epsilon_{\alpha+1-k}} \delta_{\alpha} \delta_{\alpha-1} + \epsilon_{\alpha+1-k}.
\]
The last equation implies that \( \partial_{k-1} \) commutes with \( \star \) action of the raising operators \( \hat{R} \) in the expansion of \( \hat{R}^{\infty} \) on \( \delta_{\alpha} \delta_{\alpha-1} \). We deduce from this and Lemma 8 that
\[
\partial_{k-1} \delta_{\alpha} \delta_{\alpha-1} (c) = \delta_{\alpha-1}^{-1} \epsilon_{\alpha+1-k} \hat{P}_{2k-1} (c)
\]
Iterating this calculation gives
\[
(\partial_1 \cdots \partial_{k-1}) \hat{D} \hat{w}_0 = (n-1, k, k-2, k-3, \ldots, 0) \hat{P}_{2k-1} (c)
\]
and furthermore
\[
\partial_{k-1} (\partial_{k-2} \partial_{k-1}) \cdots (\partial_1 \partial_{k-2} \partial_{k-1}) \hat{w}_0 = (n-1, k, k, k-2, k-3, k-4, \ldots, 1) \hat{P}_{2k-1} (c).
\]
Applying the operator \( \partial_{n-1} \partial_{n-2} \partial_{n-1} \cdots \partial_{k+1} \cdots \partial_{n-1} \) to the latter, we similarly get
\[
\hat{D}_{\hat{w}_{(k,n)}} = (k, k, k, k, 0, 0, 0, 0) \hat{P}_{2n-1} (c)
\]
We also have \( \tilde{w}_{(k,n)} = (s \Box s_2 \cdots s_{k-1}) \cdots (s \Box s_2) \Box \tilde{w}_{(k,n)} \), and hence (63) gives
\[
\hat{D}_{\tilde{w}_{(k,n)}} = (k, k, k, 0, 0, 0) \hat{P}_{2n-1} (c).
\]
Finally, using the fact that
\[
(\rho, \tau) \hat{P}_{(\alpha, 0)} (c) = \rho \hat{P}_{\alpha} (c),
\]
we compute that
\[
\hat{D}_{\tilde{w}_{(k,n)}} = \partial_1 (\partial_2 \partial_{\Box}) \cdots (\partial_{k-1} \cdots \partial_2 \partial_{\Box}) \hat{D}_{\tilde{w}_{(k,n)}}
\]
More generally, using similar arguments to those above, we can prove that
\[
\hat{D}_{\tilde{w}_{(\lambda, \alpha)}} (X; Y, Z) = \rho (\lambda) \hat{P}_{\lambda (\alpha)} (c),
\]
where \( \lambda (\alpha) \), \( \beta (\alpha) \), and \( \rho (\alpha) \) denote the sequences
\[
\lambda (\alpha) = (n + a_p - 1, \ldots, 2a_p, \ldots, a_i + a_{i+1} - 1, \ldots, 2a_i, \ldots, a_1 + a_2 - 1, \ldots, 2a_1);\]  
\[
\beta (\alpha) = (1 - n, \ldots, -a_p, \ldots, 1 - a_i + 1, \ldots, -a_i, \ldots, 1 - a_2, \ldots, -a_1); \]
and \[
\rho (\alpha) = (a_p^{n-a_p}, \ldots, a_i^{a_i+1-a_i}, \ldots, a_1^{a_2-a_1}).
\]
4.5. Eta polynomials. A typed $k$-strict partition is a pair consisting of a $k$-strict partition $\lambda$ together with an integer $\text{type}(\lambda) \in \{0, 1, 2\}$, which is positive if and only if $\lambda_j = k$ for some index $j$. We assume that $k > 0$ here, although it is straightforward to include the case $k = 0$, where a typed 0-strict partition is simply a strict partition (of type zero). There is a bijection between the $k$-Grassmannian elements $w$ of $\tilde{W}_\infty$ and typed $k$-strict partitions $\lambda$, under which the elements in $\tilde{W}_n$ correspond to typed partitions whose diagram fits inside an $(n - k) \times (n + k - 1)$ rectangle, obtained as follows. If the element $w$ corresponds to the typed partition $\lambda$, then for each $j \geq 1$,

$$\lambda_j = \begin{cases} |w_{k+j}| + k - 1 & \text{if } w_{k+j} < 0, \\ \# \{p \leq k : |w_p| > w_{k+j} \} & \text{if } w_{k+j} > 0 \end{cases}$$

while $\text{type}(\lambda) > 0$ if and only if $|w_1| > 1$, and in this case $\text{type}(\lambda)$ is equal to 1 or 2 depending on whether $w_1 > 0$ or $w_1 < 0$, respectively. To any typed $k$-strict partition $\lambda$, we associate a finite set of pairs $C(\lambda)$ and a sequence $\beta(\lambda) = \{\beta_j(\lambda)\}_{j \geq 1}$ using this bijection and the same equations (46) and (47) as in the type C case.

The Schubert polynomials indexed by $k$-Grassmannian elements are given by eta polynomials. For any typed $k$-strict partition $\lambda$, the raising operator expression $R^\lambda$ is defined by equation (48), as before. Let $\ell$ denote the length of $\lambda$, let $\ell_k(\lambda)$ denote the number of parts $\lambda_i$ which are strictly greater than $k$, let $m := \ell_k(\lambda) + 1$ and $\beta := \beta(\lambda)$. If $R := \prod_{1 \leq j} R_{ij}^{\ell_j}$ is any raising operator, denote by $\text{supp}(R)$ the set of all indices $i$ and $j$ such that $n_{ij} > 0$ and $j < m$, and set $\nu := R\lambda$. If $\text{type}(\lambda) = 0$, then define

$$R \ast e^\beta_\lambda = \nu \ast e^\beta_\nu := \nu_i^{\beta_1} \cdots \nu^\beta_\nu$$

where for each $i \geq 1$,

$$e^\beta_\nu := \begin{cases} k_{\nu_i} & \text{if } i \in \text{supp}(R), \\ k_{\nu_i} & \text{otherwise.} \end{cases}$$

For any $p, r \in \mathbb{Z}$ and $s \in \{0, 1\}$, define

$$a^s_p := \frac{1}{2} k c_p + \omega^s_p, \quad b^s_k := k b_k + \omega^s_k, \quad \text{and} \quad \tilde{b}^s_k := k \tilde{b}_k + \omega^s_k,$$

where $\omega^s_p(X; Y, Z) := \sum_{j=1}^p k c_{p-j} h^s_j(-Z)$.

If $\text{type}(\lambda) > 0$ and $R$ involves any factors $R_{ij}$ with $i = m$ or $j = m$, then define

$$R \ast e^\beta_\lambda := \nu_i^{\beta_1} \cdots \nu_{m-1}^{\beta_{m-1}} a^{\beta_m}_{\nu_{m+1}} c_{\nu_{m+1}}^{\beta_m} \cdots c_{\nu_j}^{\beta_j},$$

where $c_{\nu_i}^{\beta_i} := k e_{\nu_i}^{\beta_i}$ for each $i$. If $R$ has no such factors, then define

$$R \ast e^\beta_\lambda := \begin{cases} \nu_i^{\beta_1} \cdots \nu_{m-1}^{\beta_{m-1}} b^\beta_{\nu_{m+1}} c_{\nu_{m+1}}^{\beta_m} \cdots c_{\nu_j}^{\beta_j} & \text{if } \text{type}(\lambda) = 1, \\ \nu_i^{\beta_1} \cdots \nu_{m-1}^{\beta_{m-1}} \tilde{b}^\beta_{\nu_{m+1}} c_{\nu_{m+1}}^{\beta_m} \cdots c_{\nu_j}^{\beta_j} & \text{if } \text{type}(\lambda) = 2. \end{cases}$$

Following [T7], define the double eta polynomial $H_{\lambda}(X; Y(k), Z)$ by

$$(86) \quad H_{\lambda}(X; Y(k), Z) := 2^{-\ell_k(\lambda)} R^\lambda \ast e^\beta_\lambda.$$

The single eta polynomial $H_{\lambda}(X; Y(k))$ of [BKT2] is given by

$$H_{\lambda}(X; Y(k)) := H_{\lambda}(X; Y(k), 0).$$
As in §3.4, we note that we are working here with the images in the ring $\Gamma[Y, Z]$ of the eta polynomials $H_\lambda(c)$ and $H_\lambda(c | t)$ from [T6, T7].

Fix a rank $n$ and let

$$\tilde{\lambda}_0 := (n + k - 1, n + k - 2, \ldots, 2k)$$

be the typed $k$-strict partition associated to the $k$-Grassmannian element $\tilde{w}^{(k,n)}$ of maximal length in $\tilde{W}_n$. We deduce from Proposition 13 and (86) that

\begin{equation}
\mathfrak{D}_{\tilde{w}^{(k,n)}}(X; Y, Z) = H_{\tilde{\lambda}}(X; Y_{(k)}, Z).
\end{equation}

Using raising operators, it is shown in [T7, Prop. 5] that if $\lambda$ and $\mu$ are typed $k$-strict partitions such that $|\lambda| = |\mu| + 1$ and $w_\lambda = s_i w_\mu$ for some simple reflection $s_i \in \tilde{W}_\infty$, then we have

\begin{equation}
\partial_i H_\lambda(X; Y_{(k)}, Z) = H_\mu(X; Y_{(k)}, Z)
\end{equation}

in $\Gamma'[Y, Z]$. It follows easily from (87) and (88) that for any typed $k$-strict partition $\lambda$ with associated $k$-Grassmannian element $w_\lambda$, we have

$$\mathfrak{D}_{w_\lambda}(X; Y, Z) = H_\lambda(X; Y_{(k)}, Z)$$

in $\Gamma'[Y, Z]$. In particular, we recover the equality

\begin{equation}
\mathfrak{D}_{w_\lambda}(X; Y) = H_\lambda(X; Y_{(k)})
\end{equation}

in $\Gamma'[Y]$ from [BKT3, Prop. 6.3] for the single Schubert and eta polynomials.

4.6. Mixed Stanley functions and splitting formulas. For any $w \in \tilde{W}_\infty$, the double mixed Stanley function $I_w(X; Y)$ is defined by the equation

$$I_w(X; Y/Z) := \langle A(Z)D(X)A(Y), w \rangle = \sum_{uv \varpi = w} G_{u^{-1}}(-Z)E_v(X)G_{\varpi}(Y),$$

where the sum is over all reduced factorizations $uv \varpi = w$ with $u, \varpi \in S_\infty$. The single mixed Stanley function $I_w(X; Y)$ from [T5, §6] is given by setting $Z = 0$ in $I_w(X; Y/Z)$.

Fix an element $k \in \mathbb{N}$. If $k \geq 2$, we say that an element $w \in \tilde{W}_\infty$ is increasing up to $k$ if $|w_1| < |w_2| < \cdots < |w_k$. Furthermore, we adopt the convention that every element of $\tilde{W}_\infty$ is increasing up to $\varnothing$ and increasing up to $1$. If $w$ is increasing up to $k$, then there is an analogue of (53) for the restricted mixed Stanley function $I_w(X; Y_{(k)})$, which is obtained from $I_w(X; Y)$ after setting $y_i = 0$ for $i > k$. In this case, according to [T5, Eqn. (33)], we have

\begin{equation}
\mathfrak{D}_w(X; Y) = \sum_{v(1_k \times \varpi) = w} I_v(X; Y_{(k)})G_{\varpi}(y_{k+1}, y_{k+2}, \ldots),
\end{equation}

where the sum is over all reduced factorizations $v(1_k \times \varpi) = w$ in $\tilde{W}_\infty$ with $\varpi \in S_\infty$. Akin to §1.3 and §3.5, equation (90) has a double version: let $I_v(X; Y_{(k)} / Z_{(\ell)})$ be the power series obtained from $I_v(X; Y/Z)$ by setting $y_i = z_j = 0$ for all $i > k$ and $j > \ell$. Then if $w$ is increasing up to $k$ and $w^{-1}$ is increasing up to $\ell$, we have

\begin{equation}
\mathfrak{D}_w(X; Y, Z) = \sum \mathfrak{S}_{w^{-1}}(-Z_{>\ell})I_v(X; Y_{(k)}/Z_{(\ell)})G_{\varpi}(Y_{>k}),
\end{equation}

where the sum is over all reduced factorizations $(1_\ell \times u)v(1_k \times \varpi) = w$ in $\tilde{W}_\infty$ with $u, \varpi \in S_\infty$.\footnote{The paper [T7] assumes that $k > 0$, but the proofs also work (and are simpler) when $k = 0.$}
We say that an element \( w \in \tilde{W}_\infty \) is compatible with the sequence \( a : a_1 < \cdots < a_p \) of elements of \( \mathbb{N}_0 \) if all descent positions of \( w \) are contained in \( a \) (following §4.4, we assume that \( a_1 \neq 1 \)). Let \( b : b_1 < \cdots < b_q \) be a second sequence of elements of \( \mathbb{N}_0 \), and suppose that \( w \) is compatible with \( a \) and \( w^{-1} \) is compatible with \( b \). The notion of a reduced factorization \( u_1 \cdots u_{p+q-1} = w \) compatible with \( a, b \) and the sets of variables \( Y_i \) and \( Z_j \) for \( i, j \geq 1 \) are defined exactly as in §3.5.

**Proposition 14.** Suppose that \( w \) and \( w^{-1} \) are compatible with \( a \) and \( b \), respectively. Then the Schubert polynomial \( D_w(X; Y, Z) \) satisfies

\[
D_w = \sum G_{u_1}(0/Z_1) \cdots G_{u_{q-1}}(0/Z_2) I_u(X; Y_1/Z_1) G_{u_{q-1}}(Y_2) \cdots G_{u_{p+q-1}}(Y_p)
\]

summed over all reduced factorizations \( u_1 \cdots u_{p+q-1} = w \) compatible with \( a, b \).

**Proof.** The result is shown by using (91) and iterating the identity (7). \( \square \)

If \( w \) is increasing up to \( k \), then the following generalization of equation (62) holds (see [T5, Eqn. (32)]):

\[
I_w(X; Y(k)) = \sum_{\lambda : \ell = \ell(w)} d^w_\lambda H_\lambda(X; Y(k)),
\]

where the sum is over typed \( k \)-strict partitions \( \lambda \) with \( |\lambda| = \ell(w) \). The mixed Stanley coefficients \( d^w_\lambda \) in (92) are nonnegative integers which have a combinatorial interpretation, as in §3.5. The proof of (92) in [T5] uses (89), and is similar to the type C case.

Assume that \( w \) is increasing up to \( k \) and \( w^{-1} \) is increasing up to \( \ell \). Then the (restricted) double mixed Stanley function \( I_w(X; Y(k)/Z(\ell)) \) satisfies

\[
I_w(X; Y(k)/Z(\ell)) = \sum_{u \in w^{-1} = w} G_{u_1}(-Z(\ell)) I_u(X; Y(k))
\]

\[
= \sum_{u \in w^{-1} = w} G_{u_1}((Y_k) I_u(X; -Z(\ell))
\]

where the sums are over reduced factorizations as shown, with \( u \in S_\infty \). We can now use equations (9) and (92) in (93) and (94) to obtain two expansions of \( I_w(X; Y(k)/Z(\ell)) \) as a positive sum of products of Schur \( S \)-polynomials with eta polynomials, as in Example 3.

**Theorem 3** ([T5], Cor. 3). Suppose that \( w \) is compatible with \( a \) and \( w^{-1} \) is compatible with \( b \), where \( b_1 = \square \). Then we have

\[
D_w = \sum_{\lambda} g^w_\lambda s_{\lambda^1}(0/Z_1) \cdots s_{\lambda^{p+q-1}}(0/Z_2) H_{\lambda^1}(X; Y_1) s_{\lambda^{p+q-1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)
\]

summed over all sequences of partitions \( \lambda = (\lambda^1, \ldots, \lambda^{p+q-1}) \) with \( \lambda^1 \) \( a_1 \)-strict and typed, where

\[
g^w_\lambda := \sum a^{u_1}_{\lambda^1} \cdots a^{u_{q-1}}_{\lambda^{q-1}} d^{u_q}_{\lambda^q} a^{u_{q+1}}_{\lambda^{q+1}} \cdots a^{u_{p+q-1}}_{\lambda^{p+q-1}}
\]

summed over all reduced factorizations \( u_1 \cdots u_{p+q-1} = w \) compatible with \( a, b \).

**Proof.** This follows from Proposition 14 by using equations (9) and (92). \( \square \)
In this section, we discuss the precise way in which the Schubert polynomials given in the previous sections represent degeneracy loci of vector bundles in the sense of [Fu2]. This has been addressed in earlier work (see [T5, §4] and [T6, §6, 7]), but our aim here is to provide some more detailed historical comments, which include the author’s papers [T2, T3]. We restrict attention to the symplectic case, as the orthogonal types are analogous to type C, and the situation in type A has been understood since [Fu1]. Throughout the section $X := (x_1, x_2, \ldots)$ and $Y := (y_1, y_2, \ldots)$ will denote two sequences of commuting independent variables, and for every integer $n \geq 1$, we set $X_n := (x_1, \ldots, x_n)$ and $Y_n := (y_1, \ldots, y_n)$.

Consider a vector bundle $E \to M$ of rank $2n$ over a smooth algebraic variety $M$, equipped with an everywhere nondegenerate skew-symmetric form $E \otimes E \to \mathbb{C}$. Assume that we are given two complete flags of subbundles of $E$

\[
E_0 : 0 \subset E_1 \subset \cdots \subset E_{2n} = E \quad \text{and} \quad F_0 : 0 \subset F_1 \subset \cdots \subset F_{2n} = E
\]

with rank $E_i = \text{rank } F_i = i$ for each $i$, while $E_{n+i} = E_{n-i}^\perp$ and $F_{n+i} = F_{n-i}^\perp$ for $0 \leq i < n$. For any $w$ in the Weyl group $W_n$, we have the degeneracy locus

\[
\mathfrak{X}_w := \{ x \in M \mid \dim(E_i(x) \cap F_j(x)) \geq d_w(i, j) \quad \forall i \in [1, n], j \in [1, 2n] \},
\]

where $d_w(i, j) \in \mathbb{Z}$ and the inequalities in (96) are exactly those which define the Schubert variety $X_w(F_i)$ in the flag variety $\text{IF}_n := \text{Sp}_{2n} / B$ (the precise values $d_w(i, j)$ are given in [T6, §6.2]). We assume that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in $M$, and seek a formula which expresses the cohomology class $[\mathfrak{X}_w] \in \text{H}^*(M)$ as a universal polynomial in the Chern classes of the vector bundles $E_i$ and $F_j$.

When the vector bundles $F_j$ are trivial, the answer to the above degeneracy locus problem coincides with the answer to the Giambelli problem for $\text{H}^*(\text{IF}_n)$, which amounts to a theory of (single) symplectic Schubert polynomials. In this setting, the $E_i$ are the universal (or tautological) vector bundles over $M = \text{IF}_n$. The cohomology ring of $\text{IF}_n$ has a standard Borel presentation [Bo] as a quotient ring

\[
\text{H}^*(\text{IF}_n, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n] / J_n,
\]

where the variables $x_i$ represent the characters of the Borel subgroup $B$, or the Chern roots of the dual of the Lagrangian subbundle $E_n$, and $J_n$ denotes the ideal generated by the $W_n$-invariant polynomials of positive degree. The aim of a theory of Schubert polynomials is to provide a combinatorially explicit and natural set of polynomial representatives $\{ \mathfrak{C}_w(X_n) \}_{w \in W_n}$ for the Schubert classes $\{ [X_w] \}_{w \in W_n}$ in the presentation (97) of $\text{H}^*(\text{IF}_n, \mathbb{Z})$.

Among the many desirable attributes of the Schubert polynomials $\mathfrak{C}_w(X_n)$, the stability property is perhaps the most important. This states that we have

\[
\mathfrak{C}_{j_n(w)}(x_1, \ldots, x_n, 0) = \mathfrak{C}_w(x_1, \ldots, x_n), \quad \forall w \in W_n,
\]

where $j_n : W_n \to W_{n+1}$ is the natural inclusion map of Weyl groups. The significance of this property was already recognized in the work of Lascoux and Schützenberger [LS] on type A Schubert polynomials, where – together with the fact that they represent Schubert classes – it completely characterizes them.

The inclusions $j_n$ induce surjections

\[
\cdots \to \text{H}^*(\text{IF}_{n+1}, \mathbb{Z}) \to \text{H}^*(\text{IF}_n, \mathbb{Z}) \to \cdots
\]
and the inverse limit of the system (98) in the category of graded rings is the stable cohomology ring $\mathbb{H}(\text{IF})$. The stability property implies that the symplectic Schubert polynomials lift to give representatives $e_w(X)$ of the stable Schubert classes $\sigma_w := \lim_{\to} [X_w]$, one for every $w \in W_\infty$. Unlike the situation in type $A$, the $e_w(X)$ will no longer be polynomials in $X$, but formal power series (see Example 4 below). Moreover, a special role is played by the subring of $\mathbb{H}(\text{IF})$ invariant under the action of the symmetric group $S_\infty$, whose classes are represented by symmetric power series, and which is isomorphic to the stable cohomology ring $\mathbb{H}(\text{LG})$ of the Lagrangian Grassmannian $\text{LG}(n, 2n)$.

The Giambelli problem for the cohomology ring of $\text{LG}(n, 2n)$ was solved by Pragacz [P] by using the theory of Schur’s $Q$-functions, and the resulting isomorphism between $H(\text{LG}(n, 2n), \mathbb{Z})$ and a certain quotient of the ring $\Gamma$ of Schur $Q$-functions was further studied by Józefiak [Jo]. Let $\Lambda$ denote the ring of symmetric functions in the variables $X$, so that $\Lambda = \mathbb{Z}[e_1(X), e_2(X), \ldots]$, and let $I$ be the ideal of $\Lambda$ generated by the symmetric functions in $X^2 := (x_1^2, x_2^2, \ldots)$ of positive degree. According to [Jo, Cor. 2.3], the surjective map $\eta : \Lambda \to \Gamma$ with $\eta(e_i(X)) := q_i(X)$ for all $i \geq 1$ induces an isomorphism $\Lambda/I \cong \Gamma$.

Define a map $\phi_n : \mathbb{Z}[X] \to \mathbb{Z}[X_n]$ by $x_i \mapsto x_i$ for $i \leq n$, while $x_i \mapsto 0$ for $i > n$. If $\Lambda_n := \mathbb{Z}[e_1(X_n), \ldots, e_n(X_n)] = \mathbb{Z}[X_n]^{S_n}$ is the ring of symmetric polynomials in $X_n$, then $\phi_n$ induces an homomymous map $\Lambda \to \Lambda_n$. Setting $I_n := \phi_n(I) = \Lambda_n(e_1(X_n^2), \ldots, e_n(X_n^2))$, we then have a commutative diagram of rings

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\pi} & \Lambda/I \\
\downarrow{\phi_n} & & \downarrow{\psi_0} \\
\Lambda_n/I_n & \xrightarrow{\phi_0} & H^*(\text{LG}(n, 2n))
\end{array}
$$

where the horizontal arrows are isomorphisms. The map $\psi_0$ sends $e_i(X_n)$ to the $i$-th Chern class $c_i(E/E_n)$ of the universal quotient bundle over $\text{LG}(n, 2n)$. The resulting surjection $\Gamma \to H^*(\text{LG}(n, 2n))$ maps $Q_\lambda(X)$ to the Schubert class $[X_\lambda]$, for any strict partition $\lambda$ with $|\lambda| \leq n$, and to zero, otherwise.

Since the combinatorial theory of Schur $Q$-functions was well understood and analogous to the type $A$ theory of Schur $S$-functions, the above picture provided a satisfactory way to do classical Schubert calculus on LG. The study of related problems in the theory of degeneracy loci [PR], Arakelov theory [T1], and quantum cohomology [KT], however, required representatives for the Schubert classes in the Borel presentation of $H^*(\text{LG}(n, 2n))$, and hence in the ring $\Lambda_n$. The answer was provided by Pragacz and Ratajski’s theory [PR] of $Q$-polynomials $\tilde{Q}_\lambda(X_n)$, which were extended to the $Q$-functions $\tilde{Q}_\lambda(X)$ in [T2, §1.1]. For each strict partition $\lambda$, $\tilde{Q}_\lambda(X)$ and $\tilde{Q}_\lambda(X_n)$ are defined by the raising operator expressions

$$
\tilde{Q}_\lambda(X) := R^\infty e_\lambda(X) \text{ and } \tilde{Q}_\lambda(X_n) := \phi_n(\tilde{Q}_\lambda(X)) = R^\infty e_\lambda(X_n),
$$
where as usual \( e_\lambda := \prod_i e_{\lambda_i} \). The geometrization of the Schur Q-functions \( Q_\lambda(X) \) is then displayed in the diagram

\[
\begin{array}{ccc}
Q_\lambda(X) & \xrightarrow{\pi} & \tilde{Q}_\lambda(X) \\
\downarrow{\phi_n} & & \downarrow{\psi_n} \\
\tilde{Q}_\lambda(X_n) & \xrightarrow{\psi_0} & [X_\lambda].
\end{array}
\]

In other words, by geometrization we mean the choice of substitution \( Q_\lambda(X) \mapsto \tilde{Q}_\lambda(X) \) shown above, which lifts the ring homomorphisms

\[
\Gamma \xrightarrow{\pi} \Lambda/I \xrightarrow{\phi_n} \Lambda_n/I_n
\]

to maps of abelian groups

\[
\Gamma \xrightarrow{\pi} \Lambda \xrightarrow{\phi_n} \Lambda_n.
\]

The next step in the story was to extend the above picture to the entire Weyl group \( W_n \), and thus obtain a theory of symplectic Schubert polynomials \( C_w(X_n) \) for \( H^*(IF_n, \mathbb{Z}) \). In a fundamental paper which built on the work of Lascoux-Schützenberger [LS] and Pragacz [P], Billey and Haiman [BH] found the combinatorially explicit family of type C Schubert polynomials \( C_w(X; Y) \) of §3.1. These objects are actually formal power series, realized as nonnegative integer linear combinations of products \( Q_\lambda(X)S_\varpi(Y) \) of Schur Q-functions and type A Schubert polynomials. The \( C_w(X; Y) \) for \( w \in W_\infty \) form a \( \mathbb{Z} \)-basis of a ring \( \Gamma[Y] \) isomorphic to \( H^*(IF) \), and map to the stable Schubert classes \( \sigma_w \) under this isomorphism.

The problem with the Billey-Haiman power series \( C_w(X; Y) \) was that they were not related in [BH] to the Borel presentation (97) in a way that retained combinatorial control over their coefficients. The \( \lambda \)-ring substitution used in [BH, §2] involved the odd power sums, and led to Schubert polynomials in the root variables \( x_i \) which were quite complicated (see [FK, §7] for a discussion of this). In 2006, motivated in part by an application to arithmetic intersection theory, the author resolved this issue by constructing a family of symplectic Schubert polynomials \( C_w(X_n) \) which are a geometrization of the \( C_w(X; Y) \), in the same sense that the polynomials \( \tilde{Q}_\lambda(X_n) \) are a geometrization of the power series \( Q_\lambda(Y) \).

Here are the details of this work, which eventually appeared in [T2]. Let \( J \) be the ideal of \( \Lambda[X] \) generated by the elementary symmetric functions \( e_p(X^2) \) for \( p \geq 1 \). The map \( \phi_n : \Lambda[X] \to \mathbb{Z}[X_n] \) sends \( J \) to the ideal \( J_n = (e_1(X_n^2), \ldots, e_n(X_n^2)) \). Define an isomorphism \( \pi : \Gamma[Y] \to \Lambda[X]/J \) by setting \( \pi(q_i(X)) := e_i(X) \) and \( \pi(y_i) := -x_i \) for all \( i \geq 1 \). We then have a diagram

\[
\begin{array}{ccc}
\Gamma[Y] & \xrightarrow{\pi} & \Lambda[X]/J \\
\downarrow{\phi_n} & & \downarrow{\psi} \\
\mathbb{Z}[X_n]/J_n & \xrightarrow{\psi} & H^*(IF_n)
\end{array}
\]

where the horizontal arrows are again ring isomorphisms. The map \( \psi \) is determined by \( \psi(x_i) = -c_1(E_{n+1-i}/E_{n-i}) \) for \( 1 \leq i \leq n \). Given any \( w \in W_n \), apply equations
(23) and (26) to write

\[ C_w(X; Y) = \sum_{v, w, \lambda} e^v Q_\lambda(X) S_\lambda(Y) \]

where the sum is over all reduced factorizations \( v w = w \) and strict partitions \( \lambda \) such that \( \pi \in S_n \) and \( |\lambda| = \ell(v) \). Following [T2, §2.2], define the symplectic Schubert polynomials \( C_w(X_n) \) and power series \( C_w(X) \) by the equations

\[ C_w(X) := \sum_{v, \pi, \lambda} e^v \tilde{Q}_\lambda(X) S_\lambda(-X_n) \quad \text{and} \quad C_w(X_n) := \phi_n(C_w(X)). \]

The geometrization of the Schubert polynomials \( C_w(X; Y) \) is then displayed in the diagram

\[
\begin{array}{ccc}
C_w(X; Y) & \xrightarrow{\pi} & C_w(X) \\
\downarrow \phi_n & & \downarrow \\
C_w(X_n) & \xrightarrow{\psi} & [X_w].
\end{array}
\]

**Example 4.** The linear Schubert polynomials \( C_{s_i} \) are indexed by the simple reflections \( s_i \) in \( W_{\infty} \). For each \( i \in \mathbb{N}_0 \), we have

\[ C_{s_i}(X; Y) = q_i(X) + \mathcal{S}_{s_i}(Y) = 2 \left( \sum_{j=1}^{\infty} x_j \right) \quad \text{for the Billey-Haiman polynomials, while} \]

\[ C_{s_i}(X) = \sum_{j=i+1}^{\infty} x_j \quad \text{and} \quad C_{s_i}(X_n) = \begin{cases} x_{i+1} + \cdots + x_n & \text{if } i < n, \\ 0 & \text{otherwise.} \end{cases} \]

There remained one missing ingredient to fully solve the problem of Schubert polynomials in the classical Lie types: define double versions of the Billey-Haiman polynomials, and extend the above picture to that setting. Fortunately, there was progress in this direction, as in 2005 Ikeda [I] had shown how Ivanov’s factorial Schur \( Q \)-functions [Iv3] may be used to represent the torus-equivariant Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian. At the March 2007 workshop on Schubert calculus in Banff, the author suggested to Ikeda that he should work on creating a double version of the Billey-Haiman theory. Ikeda was also informed about the author’s theory of symplectic Schubert polynomials \( C_w(X_n) \), and asked to include an extension of the substitution \( C_w(X; Y) \rightarrow C_w(X_n) \) to the double case, as it was important for geometric applications. In 2008, the work [IMN1] was announced, which used localization techniques as in [I, IN] to construct the required theory of polynomials \( C_w(X; Y) \). Moreover, the information necessary to extend the author’s geometrization of the \( C_w(X; Y) \) to equivariant cohomology was provided in [IMN1, §10]. This was done explicitly in 2009, with the announcement of our paper [T5], and simultaneously generalized, to deal with the degeneracy loci coming from any isotropic partial flag variety.

Following [T5, §4], to describe how the \( C_w(X; Y, Z) \) represent degeneracy loci of vector bundles, recall from [Gr] and [T6] that these loci and their cohomology
classes pull back from the Borel mixing space $BM_n := BB \times_{BSp_{2n}} BB$, which is the universal case of (96) and the degeneracy locus problem. Let

$$\mathbb{H}(BM) := \lim_{\leftarrow} H^*(BM_n, \mathbb{Z})$$

be the stable cohomology ring of $BM_n$. Define the supersymmetric functions $e_p(X/Y)$ for $p \geq 0$ by

$$e_p(X/Y) := \sum_{j=0}^{p} e_j(X)h_{p-j}(Y),$$

set $\tilde{\Lambda} := \mathbb{Z}[e_1(X/Y), e_2(X/Y), \ldots]$, and let $\tilde{J}$ be the ideal of $\tilde{\Lambda}[X,Y]$ generated by the fundamental relations

$$e_p^2(X/Y) + 2 \sum_{j=1}^{p} (-1)^j e_{p+j}(X/Y)e_{p-j}(X/Y)$$

for all $p \geq 1$. Define an isomorphism

$$\tilde{\pi} : \Gamma[Y,Z] \to \tilde{\Lambda}[X,Y]/\tilde{J}$$

by setting $\tilde{\pi}(q_i(X)) := e_i(X/Y)$, $\tilde{\pi}(y_i) := -x_i$, and $\tilde{\pi}(z_i) := y_i$, for each $i \geq 1$. Moreover, let $\tilde{\phi}_n : \tilde{\Lambda}[X,Y] \to \mathbb{Z}[X_n, Y_n]$ be the map $x_i \mapsto x_i$ and $y_i \mapsto y_i$ for $i \leq n$, while $x_i \mapsto 0$ and $y_i \mapsto 0$ for $i > n$, and set $\tilde{J}_n := \tilde{\phi}_n(\tilde{J}) \subset \mathbb{Z}[X_n, Y_n]$. (It is not hard to show that $\tilde{J}_n$ is equal to the ideal of $\mathbb{Z}[X_n, Y_n]$ generated by the differences $e_p(X_n^2) - e_p(Y_n^2)$ for $1 \leq p \leq n$.)

We then have a commutative diagram of rings

$$\begin{array}{ccc}
\Gamma[Y,Z] & \xrightarrow{\tilde{\pi}} & \tilde{\Lambda}[X,Y]/\tilde{J} \\
\downarrow{\tilde{\phi}_n} & & \downarrow{\tilde{\psi}} \\
\mathbb{Z}[X_n, Y_n]/\tilde{J}_n & \xrightarrow{\tilde{\psi}} & H^*(BM_n, \mathbb{Z})
\end{array}$$

extending the previous ones, where again the horizontal maps are isomorphisms. The map $\tilde{\psi}$ satisfies $\tilde{\psi}(x_i) = -c_1(E_{n+1-i}/E_{n-i})$ and $\tilde{\psi}(y_i) = -c_1(F_{n+1-i}/F_{n-i})$ for each $i$ with $1 \leq i \leq n$.

Given $w \in W_n$, apply equations (23) and (26) to write

$$\mathcal{C}_w(X;Y,Z) = \sum_{u,v,\varpi,\lambda} e_v^w \Theta_{u^{-1}}(-Z)Q_{\lambda}(X)\Theta_{\varpi}(Y)$$

where the sum is over all reduced factorizations $uw\varpi = w$ and strict partitions $\lambda$ such that $u, \varpi \in S_n$ and $|\lambda| = \ell(v)$. Define the supersymmetric $Q$-function $\tilde{Q}_\lambda(X/Y)$ by the equation

$$\tilde{Q}_\lambda(X/Y) := R^\infty e_\lambda(X/Y).$$

and the double symplectic Schubert polynomials $\mathcal{C}_w(X_n, Y_n)$ and power series $\mathcal{C}_w(X, Y)$ by the equations

$$\mathcal{C}_w(X, Y) := \sum_{u,v,\varpi,\lambda} e_v^w \Theta_{u^{-1}}(-Y_n)\tilde{Q}_\lambda(X/Y)\Theta_{\varpi}(-X_n)$$
and $\mathcal{C}_w(X_n, Y_n) := \tilde{\phi}_n(\mathcal{C}_w(X/Y))$. The geometrization of the double Schubert polynomials $\mathcal{C}_w(X; Y, Z)$ is exhibited in the diagram

$$\begin{array}{ccc}
\mathcal{C}_w(X; Y, Z) & \xrightarrow{\pi} & [X_w] \\
\downarrow \tilde{\phi}_n & & \downarrow \\
\mathcal{C}_w(X_n, Y_n) & \xrightarrow{\tilde{\psi}} & [X_w].
\end{array}$$

As before, we observe that the substitutions $Q_\lambda(X) \mapsto \tilde{Q}_\lambda(X/Y)$, $y_i \mapsto -x_i$, and $z_i \mapsto y_i$ lift the ring homomorphisms

$$\Gamma[Y, Z] \xrightarrow{\pi} \tilde{\Lambda}[X, Y] \xrightarrow{\tilde{\phi}_n} Z[X_n, Y_n] \xrightarrow{\tilde{\psi}} [X_w].$$

to maps of abelian groups

$$\Gamma[Y, Z] \xrightarrow{\pi} \tilde{\Lambda}[X, Y] \xrightarrow{\tilde{\phi}_n} Z[X_n, Y_n].$$

Since the variables $\{-x_i\}$ and $\{-y_i\}$ for $1 \leq i \leq n$ represent the Chern roots of the isotropic vector bundles in (95), it is now a simple matter to translate the formulas in this paper into Chern class formulas for degeneracy loci. For each $r \geq 0$, define $c_r(E - E_i - F_j)$ by the equation of total Chern classes

$$c(E - E_i - F_j) := c(E)c(E_i)^{-1}c(F_j)^{-1}.$$ 

Then the geometrization map $\omega_n := \tilde{\psi}\tilde{\phi}_n\pi$ sends $q_r(X)$ to $c_r(E - E_n - F_n)$ for each $r \geq 0$, and $Q_\lambda(X)$ to $Q_\lambda(E - E_n - F_n)$ for every strict partition $\lambda$.

More generally, following [T5, Thm. 3], the substitution which maps the theta polynomial $\Theta_\lambda(X; Y(\ell))$ to $\Theta_\lambda(E - E_{n-k} - F_n)$ for each $k$-strict partition $\lambda$ is applied to treat the degeneracy loci which come from any symplectic partial flag variety (see also [T6, Remark 4]). Computing the image of formula (58), we thus obtain that the class of the degeneracy locus $X_w$ in $H^*(M)$ is equal to

$$\sum_{\lambda} f_w^s \chi^t(F_n + b_{q-1} - F_n + b_q) \cdots \Theta_\lambda^t(E - E_{n-a_1} - F_n) \cdots s_{\lambda^{n-1}}(E_{n-a_{p-1}} - E_{n-a_p})$$

with the coefficients $f_w^s$ given by equation (59).

The polynomials $k^p c_p^r$ defined in (35) are particularly useful to work with, as we have the Chern class equation

$$\omega_n^t(k^p c_p^r) = c_p(E - E_{n-k} - F_{n+r})$$

(compare with [TW, Eqn. (31)]). For the top Schubert polynomial $\mathcal{C}_{w_0}(X; Y, Z)$, equation (36) maps to the multi-Pfaffian formula

$$(99) \quad [X_{w_0}] = Q_{\delta_{n+\delta_{n-1}}}(E - E_{1,2,...,n-1} - F_{1,2,...,n-1})$$

in $H^*(M)$. Following [TW, Cor. 1], the Chern polynomial in (99) is defined as the image of the polynomial $R^1 c_\alpha^t + \delta_{n-1}$ under the $\mathbb{Z}$-linear map which sends the noncommutative monomial $c_\alpha$ to $\prod_i c_{\alpha_i}(E - E_i - F_i)$, for every integer sequence $\alpha$. The geometric substitutions of this section can thus be used to relate some of the equations for Schubert polynomials found in the present paper to the Chern class formulas in [Ka, AF1, AF2] and elsewhere.
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