TABLEAU FORMULAS FOR SKEW GROTHENDIECK POLYNOMIALS

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Abstract. An element of a Weyl group of classical type is skew if it is the left factor in a reduced factorization of a Grassmannian element. The skew Grothendieck polynomials are those which are indexed by skew elements of the Weyl group. We define set-valued tableaux which are fillings of the associated skew Young diagrams and use them to prove tableau formulas for the skew double Grothendieck polynomials in all four classical Lie types. We deduce tableau formulas for the Grassmannian Grothendieck polynomials and the K-theoretic analogues of the (double mixed) skew Stanley functions in the respective Lie types.

0. Introduction

The double Grothendieck polynomials of Lascoux and Schützenberger [LS] and Kirillov and Naruse [KN] represent the (stable) Schubert classes in the equivariant K-theory of complete flag manifolds, in each of the four classical Lie types. When the indexing Weyl group element is skew, in the sense of [T1, T3], we call these polynomials *skew Grothendieck polynomials*. The goal of this article is to prove tableau formulas for the skew Grothendieck polynomials, building on our earlier work [T6], which dealt with the skew Schubert polynomials.

Each skew signed permutation is associated with a pair of (typed, k-strict) partitions $\lambda \supset \mu$. We introduce *set-valued tableaux* on the skew Young diagram λ/μ by extending Buch's definition [B] in type A to types B, C, and D, in a way which is natural from a Lie-theoretic point of view. Our combinatorial formulas for Grothendieck polynomials are expressed as sums over set-valued tableaux on this skew shape. The main results are the first such theorems for symplectic and orthogonal Grothendieck polynomials, even in the single case.

The skew elements of the symmetric group S_n are the 321-avoiding or fully commutative permutations studied in [BJS, St]. A set-valued tableau formula for their double Grothendieck polynomials was proved by Matsumura [M2], following earlier results in [ACT] and [M1] for the single polynomials. As in [T6], the theorems of this paper are new even in type A, and provide an alternative to the main result of [M2], which has a direct analogue in types B, C, and D. We note that the (type A) skew stable Grothendieck polynomials introduced and studied in [FK1, B, LP, Y] do not represent Schubert classes and therefore are different than the ones found in [ACT, M1, M2] and the present work.

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The K-theoretic Stanley functions F_w^B, F_w^C, and F_w^D lie at the center of Kirillov and Naruse's approach to Grothendieck polynomials. When w is a skew element of the Weyl group, the theory developed here also produces set-valued tableau formulas for them. Since the lowest degree terms of these formal power series are the ordinary skew Stanley functions, we deduce new combinatorial formulas for the latter (Examples 10 and 14). These *barred k-tableau* and *barred typed k'-tableau* formulas refine the k-tableau and typed k'-tableau formulas of [T1, T3] by separating the powers of 2 which appear in the multiplicities there.

Our theorems specialize to obtain formulas for the *Grassmannian Grothendieck polynomials*, that is, the Grothendieck polynomials indexed by Grassmannian elements. In types B and C, this answers a question of Hudson, Ikeda, Matsumura, and Naruse [HIMN2, Sec. 1]. The maximal Grassmannian (and fully commutative) case of this problem was addressed earlier by Ikeda and Naruse in [IN, Sec. 9] (see also [GK]). Our formulas for maximal Grassmannian polynomials differ from loc. cit. just as our main result in type A differs from [M2, Thm. 3.1].

The straightforward proofs were found by modifying the arguments of [T6], this time employing the idCoxeter algebra, the Hecke product on the Weyl group, and the new definitions of set-valued tableaux in the orthogonal and symplectic Lie types. Other known approaches to tableau formulas in type A do not suffice for our purposes, since the Grassmannian elements in types B, C, and D are not fully commutative. This is reflected, e.g., in the difference between Proposition 1 and Propositions 3, 8, and their corollaries. Nevertheless, we arrive at a general theory of such formulas which is uniform across the four types, following [T1, T3, T6].

We remark that Schubert and Grothendieck polynomials do not give *intrinsic formulas* for the Schubert classes, that is, formulas which respect the symmetries of the underlying Weyl group elements. Intrinsic formulas for the equivariant Schubert classes in the equivariant cohomology ring of classical G/P spaces and corresponding ones in the theory of degeneracy loci of vector bundles were obtained in [T2] (in general) and [T5] (for amenable Weyl group elements). The problem of finding analogues of these results in K-theory remains open. For progress on this question, we refer the reader to [BKTY], which contains the general solution in type A, and [HIMN1], which studies the Grassmannian loci in types A, B, and C.

This article is organized as follows. Section 1 contains preliminary material on the relevant Weyl groups, Grothendieck polynomials, K-theoretic Stanley functions, partitions, and Grassmannian/skew (signed) permutations. The following Sections 2, 3, and 4 deal with set-valued tableaux and formulas for skew Grothendieck polynomials in the Lie types A, B/C, and D, respectively.

1. Preliminaries

This section recalls some essential background definitions and notation which will be used in this paper. For more details on the less standard among these, the reader may consult [KN] and [T1, T3, T6].

1.1. **Weyl groups and reduced words.** The Weyl group for the root system of type A_{n-1} is the *symmetric group* S_n of permutations of the set $\{1,\ldots,n\}$. The group S_n is generated by the simple transpositions $s_i=(i,i+1)$ for $1\leq i\leq n-1$. The Weyl group for the root system of type B_n or C_n is the *hyperoctahedral group* W_n of signed permutations on the set $\{1,\ldots,n\}$. The group W_n is generated by the transpositions s_i for $1\leq i\leq n-1$ and the sign change $s_0(1)=\overline{1}$ (here we set

$\bar{i} := -i$ for any $i \geq 1$). The elements of W_n are written in one line notation as n-tuples $(w_1, \ldots, w_n)$, where $w_i := w(i)$ for each $i \in [1, n]$.

There is a natural embedding of W_n in W_{n+1} defined by adding the fixed point $n+1$, and we let $W_\infty := \bigcup_n W_n$ and $S_\infty := \bigcup_n S_n$. The *length* of an element $w \in W_\infty$, denoted $\ell(w)$, is the least integer r such that we have an equation $w = s_{a_1} \cdots s_{a_r}$. In this case, the word $a_1 \cdots a_r$ is called a *reduced word* for w. We say that w is *decreasing down to* p if w has a reduced word $a_1 \cdots a_r$ such that $a_1 > \cdots > a_r \geq p$, and that w is *increasing up from* p if w has a reduced word $a_1 \cdots a_r$ such that $p \leq a_1 < \cdots < a_r$. Here p denotes a nonnegative integer.

The Weyl group $\widetilde{W}_n$ for the root system of type D_n is the subgroup of W_n consisting of all signed permutations with an even number of negative values. The group $\widetilde{W}_n$ is an extension of S_n by $s_\Box := s_0 s_1 s_0$, an element which sends $(1,2)$ to $(-2,-1)$ and fixes all integers $p \geq 3$. We define the natural embedding $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$ of Weyl groups as above and set $\widetilde{W}_\infty := \bigcup_n \widetilde{W}_n$.

The simple reflections in type D are indexed by the members of the set $\mathbb{N}_\Box := \{\Box, 1, 2, \ldots\}$, and the length and reduced words of the elements of $\widetilde{W}_\infty$ are defined using them. Given $p \geq 1$, we define $w \in \widetilde{W}_n$ to be decreasing down to p or increasing up from p in the same way as in types A, B, and C. However, when $p = \Box$, we say that $w \in \widetilde{W}_n$ is *decreasing down to* $\Box$ (resp. increasing up from $\Box$) if w has a reduced word $a_1 \cdots a_r$ which is a subword of $(n-1, \ldots, 2, \Box)$ (resp. a subword of $(\Box, 2, \ldots, n-1)$). In all of the classical Lie types, if w is decreasing down to p or increasing up from p, then the decreasing (resp. increasing) word $a_1 \cdots a_r$ for w is uniquely determined.

An element of W_n is called *unimodal* if it has a reduced word $a_1 \cdots a_r$ which is a subword of $\Omega_n^B := (n-1, \ldots, 1, 0, 1, \ldots, n-1)$. For a unimodal $w \in W_n$, the number of reduced words of w which are subwords of $\Omega_n^C := (n-1, \ldots, 1, 0, 0, 1, \ldots, n-1)$ is equal to $2^{n(w)}$ for a nonnegative integer $n(w)$. An element of $\widetilde{W}_n$ is called *unimodal* if it has a reduced word $a_1 \cdots a_r$ which is a subword of $\Omega_n^D := (n-1, \ldots, 2, 1, \Box, 2, \ldots, n-1)$. For a unimodal $w \in \widetilde{W}_n$, the number of reduced words of w which are subwords of Ω_n^D is equal to $2^{n'(w)}$ for a nonnegative integer $n'(w)$.

1.2. **Grothendieck polynomials and K-theoretic Stanley functions.** The double Grothendieck polynomials for the classical Lie groups studied here are due to Lascoux and Schützenberger [LS] and Fomin and Kirillov [FK1, FK2] (in type A) and Kirillov and Naruse [KN] (in types B, C, and D). Note that these objects depend on a formal variable β, and in the latter three Lie types they are not polynomials but power series of unbounded degree in the Z variables. We refer the reader to [KN, Sec. 6] for the precise way in which Grothendieck polynomials represent the stable equivariant Schubert classes in the equivariant connective K-theory of complete flag manifolds. When the parameter β is set equal to zero, we obtain the double Schubert polynomials defined by Ikeda, Mihalcea, and Naruse [IMN], up to a change of sign in the Y variables.

The idCoxeter algebra $\mathbb{W}_n^\beta$ of W_n is the free unital associative $\mathbb{Z}[\beta]$-algebra generated by the elements $\pi_0, \pi_1, \ldots, \pi_{n-1}$ modulo the relations

$$\begin{array}{rclr}
\pi_i^2 & = & \beta\pi_i & i \geq 0\,; \\
\pi_i\pi_j & = & \pi_j\pi_i & |i-j| \geq 2\,; \\
\pi_i\pi_{i+1}\pi_i & = & \pi_{i+1}\pi_i\pi_{i+1} & i > 0\,; \\
\pi_0\pi_1\pi_0\pi_1 & = & \pi_1\pi_0\pi_1\pi_0. &
\end{array}$$

For every $w \in W_n$, define $\pi_w := \pi_{a_1} \ldots \pi_{a_r}$, where $a_1 \cdots a_r$ is any reduced word for w. The elements π_w for $w \in W_n$ form a free $\mathbb{Z}[\beta]$-basis of $\mathbb{W}_n^\beta$. We denote the coefficient of $\pi_w \in \mathbb{W}_n^\beta$ in the expansion of the element $\alpha \in \mathbb{W}_n^\beta$ by $\langle \alpha, w\rangle$.

Let t be an indeterminate and define

$$\begin{aligned} A_i(t) &:= (1+t\pi_{n-1})(1+t\pi_{n-2})\cdots(1+t\pi_i)\,; \\ A_i'(t) &:= (1+t\pi_i)(1+t\pi_{i+1})\cdots(1+t\pi_{n-1})\,; \\ B(t) &:= (1+t\pi_{n-1})\cdots(1+t\pi_1)(1+t\pi_0)(1+t\pi_1)\cdots(1+t\pi_{n-1})\,; \\ C(t) &:= (1+t\pi_{n-1})\cdots(1+t\pi_1)(1+t\pi_0)(1+t\pi_0)(1+t\pi_1)\cdots(1+t\pi_{n-1}). \end{aligned}$$

Suppose that $X = (x_1, x_2, \ldots)$, $Y = (y_1, y_2, \ldots)$, and $Z = (z_1, z_2, \ldots)$ are three infinite sequences of commuting independent variables. For any $\varpi \in S_n$, the type A Grothendieck polynomial $\mathfrak{G}_\varpi$ is given by

$$\mathfrak{G}_\varpi(X,Y) := \left\langle A_{n-1}'(y_{n-1})\cdots A_1'(y_1)A_1(x_1)\cdots A_{n-1}(x_{n-1}), \varpi\right\rangle, \tag{1}$$

while its single version is defined by $\mathfrak{G}_\varpi(X) := \mathfrak{G}_\varpi(X,0)$. If $A(X) := A_1(x_1)A_1(x_2)\cdots$ and $A'(Y) := \cdots A_1'(y_2)A_1'(y_1)$, then the stable Grothendieck polynomial G_ϖ of Fomin and Kirillov [FK1] is given by

$$G_\varpi(X,Y) := \lim_{m\to\infty} \mathfrak{G}_{1^m\times\varpi}(X,Y) = \langle A'(Y)A(X), \varpi\rangle\,,$$

where $1^m \times \varpi \in S_{m+n}$ is the permutation defined by $1^m\times\varpi(i) = i$ for $i \in [1,m]$ and $1^m\times\varpi(i) = m+\varpi(i-m)$ if $i \in [m+1, m+n]$. Following Fomin and Kirillov (see [FK1] and [FK2, Cor. 6.5]), the formal power series G_ϖ is a K-theoretic analogue of the (type A) double Stanley symmetric function.

Let $B(Z) := B(z_1)B(z_2)\cdots$, $C(Z) := C(z_1)C(z_2)\cdots$, and for $w \in W_n$, define the type B and type C Grothendieck polynomials $\mathfrak{G}_w^B$ and $\mathfrak{G}_w^C$ by

$$\mathfrak{G}_w^B(Z;X,Y) := \left\langle A_{n-1}'(y_{n-1})\cdots A_1'(y_1)B(Z)A_1(x_1)\cdots A_{n-1}(x_{n-1}), w\right\rangle \tag{2}$$

and

$$\mathfrak{G}_w^C(Z;X,Y) := \left\langle A_{n-1}'(y_{n-1})\cdots A_1'(y_1)C(Z)A_1(x_1)\cdots A_{n-1}(x_{n-1}), w\right\rangle, \tag{3}$$

while their single versions are given by $\mathfrak{G}_w^B(Z;X) := \mathfrak{G}_w^B(Z;X,0)$ and $\mathfrak{G}_w^C(Z;X) := \mathfrak{G}_w^C(Z;X,0)$, respectively. To compare with [KN], note that the polynomial called $\mathcal{G}_w^B(a,b;x)$ in op. cit. would be the polynomial denoted by $\mathfrak{G}_w^B(x;a,b)$ here. The polynomials $\mathfrak{G}_w^B$ and $\mathfrak{G}_w^C$ are stable under the inclusion of W_n in W_{n+1}; it follows that $\mathfrak{G}_w^B$, $\mathfrak{G}_w^C$, and $\mathfrak{G}_\varpi$ are well defined for $w \in W_\infty$ and $\varpi \in S_\infty$, respectively. The type B and type C K-theoretic Stanley functions F_w^B and F_w^C of [KN] are given by $F_w^B(Z) := \langle B(Z), w\rangle$ and $F_w^C(Z) := \langle C(Z), w\rangle$.

The idCoxeter algebra $\widetilde{\mathbb{W}}_n^\beta$ of the group $\widetilde{W}_n$ is the free unital associative $\mathbb{Z}[\beta]$-algebra generated by the elements $\pi_\square, \pi_1, \ldots, \pi_{n-1}$ modulo the relations

$$\begin{array}{rcll} \pi_i^2 &=& \beta\pi_i & i \in \mathbb{N}_\square\,; \\ \pi_\square\pi_1 &=& \pi_1\pi_\square\,; & \\ \pi_\square\pi_2\pi_\square &=& \pi_2\pi_\square\pi_2\,; & \\ \pi_i\pi_{i+1}\pi_i &=& \pi_{i+1}\pi_i\pi_{i+1} & i > 0\,; \\ \pi_i\pi_j &=& \pi_j\pi_i & j > i+1, \text{ and } (i,j) \neq (\square, 2). \end{array}$$

For any element $w \in \widetilde{W}_n$, choose a reduced word $a_1\cdots a_r$ for w, and define $\pi_w := \pi_{a_1}\ldots\pi_{a_r}$. Denote the coefficient of $\pi_w \in \widetilde{\mathbb{W}}_n^\beta$ in the expansion of the element $\alpha \in \widetilde{\mathbb{W}}_n^\beta$ in the π_w basis by $\langle \alpha, w\rangle$, and define

$$D(t) := (1+t\pi_{n-1})\cdots(1+t\pi_2)(1+t\pi_1)(1+t\pi_\square)(1+t\pi_2)\cdots(1+t\pi_{n-1}).$$

Let $D(Z) := D(z_1)D(z_2)\cdots$, and for $w \in \widetilde{W}_n$, define the type D Grothendieck polynomial $\mathfrak{G}_w^D$ by

$$\mathfrak{G}_w^D(Z;X,Y) := \left\langle A'_{n-1}(y_{n-1})\cdots A'_1(y_1)D(Z)A_1(x_1)\cdots A_{n-1}(x_{n-1}), w\right\rangle, \tag{4}$$

and set $\mathfrak{G}_w^D(Z;X) := \mathfrak{G}_w^D(Z;X,0)$. The Grothendieck polynomial $\mathfrak{G}_w^D(Z;X,Y)$ is stable under the natural inclusions $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$, and hence is well defined for $w \in \widetilde{W}_\infty$. Following [KN], the type D K-theoretic Stanley function F_w^D is defined by $F_w^D(Z) := \langle D(Z), w\rangle$.

Given any Weyl group elements u, v, w, we say that w is the *Hecke product* $u \circ v$ of u and v if $\pi_u\pi_v = \beta^{\ell(u)+\ell(v)-\ell(w)}\pi_w$ in the corresponding idCoxeter algebra. The Hecke product $\circ$ is characterized by the relations $s_i \circ w = s_i w$, if $\ell(s_i w) > \ell(w)$, and $s_i \circ w = w$, if $\ell(s_i w) < \ell(w)$, for each simple reflection s_i. This defines an associative product on the Weyl group such that $u \circ v = uv$ if and only if $\ell(uv) = \ell(u)+\ell(v)$. In the latter case, we say that the product uv is *reduced*. If $u_1 \circ \cdots \circ u_r = w$, then $u_1 \circ \cdots \circ u_r$ is called a *Hecke factorization* of w.

1.3. Partitions and Grassmannian/skew Weyl group elements.

1.3.1. *Type A.* A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a finite weakly decreasing sequence of nonnegative integers, which we identify with its Young diagram of boxes. The *length* of λ is the number of non-zero parts λ_i. The containment relation $\mu \subset \lambda$ of partitions is defined by using their diagrams, and the set-theoretic difference $\lambda \smallsetminus \mu$ is the *skew diagram* λ/μ. A skew diagram is a *horizontal strip* (resp. *vertical strip*) if it does not contain two boxes in the same column (resp. row), and, following [B, Sec. 4], a *rook strip* if it contains no two boxes in the same row or column.

Fix a nonnegative integer m. An element ϖ in S_∞ is m-*Grassmannian* if $\ell(\varpi s_i) > \ell(\varpi)$ for all $i \neq m$. Every m-Grassmannian permutation ϖ corresponds to a unique partition λ of length at most m, called the *shape* of ϖ, satisfying $\lambda = (\varpi_m - m, \ldots, \varpi_1 - 1)$. When the shape λ and m are given, we denote the corresponding permutation by ϖ_λ.

A permutation $\varpi \in S_\infty$ is called *skew* if there exists an m-Grassmannian permutation ϖ_λ (for some m) and a reduced factorization $\varpi_\lambda = \varpi\varpi'$ in S_∞. In this case, the right factor ϖ' equals ϖ_μ for some m-Grassmannian permutation ϖ_μ, and we have $\mu \subset \lambda$. We say that ϖ is associated to the pair (λ, μ) and write $\varpi = \varpi_{\lambda/\mu}$. Moreover, we have $\ell(\varpi) = |\lambda/\mu| := \sum_i(\lambda_i - \mu_i)$, and there is a 1-1 correspondence between reduced factorizations uv of ϖ and partitions ν with $\mu \subset \nu \subset \lambda$.

1.3.2. *Types B and C.* Fix a nonnegative integer k. We denote the box in row r and column c of a Young diagram by $[r, c]$, and call the box $[r, c]$ a *left box* if $c \leq k$ and a *right box* if $c > k$. We say that the boxes $[r, c]$ and $[r', c']$ are k-*related* if $|c-k-1|+r = |c'-k-1|+r'$ and k'-*related* if $|c-k-\frac{1}{2}|+r = |c'-k-\frac{1}{2}|+r'$.

A partition λ is called k-*strict* if no part greater than k is repeated. The number of parts λ_i of λ which are greater than k is the k-*length* of λ, denoted by $\ell_k(\lambda)$. We define $\lambda_0 := \infty$ and agree that the diagram of λ includes all boxes $[0, c]$ in row zero. The *rim* of λ is the set of boxes $[r, c]$ of its Young diagram such that box $[r+1, c+1]$ lies outside of the diagram.

If $\mu \subset \lambda$ are two k-strict partitions, we let R (resp. $\mathbb{A}$) denote the set of right boxes of μ (including boxes in row zero) which are bottom boxes of λ in their column and are (resp. are not) k'-related to a left box of λ/μ. The pair $\mu \subset \lambda$ forms a *k-horizontal strip* λ/μ if (a) λ/μ is contained in the rim of λ, and the right boxes of λ/μ form a horizontal strip; (b) no two boxes in R are k'-related; and (c) if two boxes of λ/μ lie in the same column, then they are k'-related to exactly two boxes of R, which both lie in the same row. We let $n(\lambda/\mu)$ denote the number of connected components of $\mathbb{A}$ which do not have a box in column $k+1$. Here two boxes in $\mathbb{A}$ are connected if they share a vertex or an edge.

An element w of W_∞ is *k-Grassmannian* if it satisfies $\ell(ws_i) > \ell(w)$ for all $i \neq k$. Each k-Grassmannian element $w \in W_\infty$ corresponds to a unique k-strict partition λ, called the *shape* of w. We have

$$\lambda_i = \begin{cases} k + |w_{k+i}| & \text{if } w_{k+i} < 0, \\ \#\{j \in [1,k] \,:\, w_j > w_{k+i}\} & \text{if } w_{k+i} > 0. \end{cases}$$

A formula for the inverse of this map is found in [T6, Sec. 3.1]. Note that if w lies in S_∞, then its shape in types B and C is the partition whose diagram is the transpose of its shape in type A. If the shape λ and k are given, then we denote the corresponding Weyl group element by w_λ.

1.3.3. *Type D.* Fix a positive integer k. A *typed k-strict partition* is a pair consisting of a k-strict partition λ together with an integer $\mathrm{type}(\lambda) \in \{0,1,2\}$, which is positive if and only if $\lambda_i = k$ for some index i.

If $\mu \subset \lambda$ are two typed k-strict partitions, the sets R and $\mathbb{A}$ are defined as in types B and C, replacing 'k'-related' by '$(k-1)$-related'. A pair $\mu \subset \lambda$ of typed k-strict partitions forms a *typed k'-horizontal strip* λ/μ if $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$ and conditions (a), (b), and (c) in the definition of a k-horizontal strip hold, again replacing 'k'-related' by '$(k-1)$-related'. We define $n'(\lambda/\mu)$ to be one less than the number of connected components of $\mathbb{A}$.

An element $w \in \widetilde{W}_\infty$ has type 0 if $|w_1| = 1$, type 1 if $w_1 > 1$, and type 2 if $w_1 < -1$. We say that $w \in \widetilde{W}_\infty$ is *k-Grassmannian* if $\ell(ws_i) > \ell(w)$ for all $i \neq k$, if $k > 1$, and for all $i \notin \{\Box, 1\}$, if $k = 1$. There is a type-preserving bijection between the k-Grassmannian elements w of $\widetilde{W}_\infty$ and typed k-strict partitions λ. We have

$$\lambda_i = \begin{cases} k - 1 + |w_{k+i}| & \text{if } w_{k+i} < 0, \\ \#\{j \in [1,k] \,:\, |w_j| > w_{k+i}\} & \text{if } w_{k+i} > 0. \end{cases}$$

See [T6, Sec. 4.1] for a description of inverse of this map. If the Weyl group element w corresponds to the typed k-strict partition λ, then we denote $w = w(\lambda, k)$ by w_λ.

A Weyl group element w in Lie type B or C (resp. D) is called *skew* if there exists a k-Grassmannian element w_λ for some $k \geq 0$ (resp. for some $k \geq 1$) and a reduced factorization $w_\lambda = ww'$ in W_∞ (resp. $\widetilde{W}_\infty$). In this case, the right factor w' equals w_μ for some k-Grassmannian element w_μ, and we have $\mu \subset \lambda$. We say that (λ, μ) is a *compatible pair* of (typed) k-strict partitions and that w is associated to the pair (λ, μ). We write $w = w_{\lambda/\mu}$ and note that $\ell(w) = |\lambda/\mu|$. For example, any k-horizontal strip (resp. typed k'-horizontal strip) λ/μ is a compatible pair (λ, μ) of (typed) k-strict partitions.

We say that a sequence of (typed) k-strict partitions $\lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^p$ is *compatible* if $(\lambda^i, \lambda^{i-1})$ is a compatible pair for each $i \in [1,p]$. If $p = 2$ we say that

$\lambda^0 \subset \lambda^1 \subset \lambda^2$ is a *compatible triple* of such partitions. There is a 1-1 correspondence between reduced factorizations uv of $w_{\lambda/\mu}$ and (typed) k-strict partitions ν such that $\mu \subset \nu \subset \lambda$ is a compatible triple, with $u = w_{\lambda/\nu}$ and $v = w_{\nu/\mu}$.

2. Tableau formula for type A skew Grothendieck polynomials

In this section, we assume that all Grassmannian and skew permutations are taken with respect to a fixed positive integer m. An m-Grassmannian element $\varpi \in S_n$ satisfies $\ell(s_i\varpi) < \ell(\varpi)$ if and only if $\varpi = (\cdots i+1 \cdots | \cdots i \cdots)$, where the vertical line $|$ lies between ϖ_m and ϖ_{m+1}. Using this, it is easy to see that a skew permutation $\varpi_{\lambda/\mu}$ is decreasing down to 1 (resp. increasing up from 1) if and only if λ/μ is a horizontal (resp. vertical) strip.

Proposition 1. *Suppose that $\varpi_{\lambda/\mu} = u{\circ}v$ for some skew permutation $\varpi_{\lambda/\mu}$. Then there exist partitions ν and ρ with $\mu \subset \rho \subset \nu \subset \lambda$ such that ν/ρ is a rook strip, $u = \varpi_{\lambda/\rho}$, and $v = \varpi_{\nu/\mu}$. Moreover, the converse statement also holds.*

Proof. Suppose that $u = s_{a_1}\cdots s_{a_r}$ with $r = \ell(u)$, so that $\varpi_{\lambda/\mu} = s_{a_1}{\circ}\cdots{\circ}s_{a_r}{\circ}v$. Since $s_i{\circ}v$ is equal to v or to the reduced product s_iv for any i, it is clear that there exists a reduced factorization $\varpi_{\lambda/\mu} = \varpi'v$ for some ϖ', and hence that $v = \varpi_{\nu/\mu}$ for some partition ν with $\mu \subset \nu \subset \lambda$.

The assertions regarding u and ρ are proved using induction on $\ell(u)$. If $\ell(u) = 1$ then $u = s_i$ for some i, so $\varpi_{\lambda/\mu} = s_i{\circ}\varpi_{\nu/\mu}$. If $s_i{\circ}\varpi_{\nu/\mu} = s_i\varpi_{\nu/\mu}$, then we set $\rho := \nu$. If $s_i{\circ}\varpi_{\nu/\mu} = \varpi_{\nu/\mu}$, then $\nu = \lambda$ and $s_i\varpi_{\nu/\mu} = \varpi_{\rho/\mu}$ for some partition ρ with $\mu \subset \rho \subset \nu$ and $|\rho| = |\nu| - 1$, which satisfies the required property.

Assume next that $u = s_iu'$ with $\ell(u) = \ell(u') + 1$ and $\varpi_{\lambda/\mu} = u{\circ}\varpi_{\nu/\mu} = s_i{\circ}u'{\circ}\varpi_{\nu/\mu}$. It follows as above that $u'{\circ}\varpi_{\nu/\mu} = \varpi_{\zeta/\mu}$ for some partition $\zeta \subset \lambda$, so $s_i{\circ}\varpi_\zeta = \varpi_\lambda$. By the inductive hypothesis, we have $u' = \varpi_{\zeta/\rho}$ for some partition $\rho \subset \nu$ such that ν/ρ is a rook strip. If $|\zeta| = |\lambda| - 1$, then $s_i\varpi_\zeta = \varpi_\lambda$ is a reduced factorization, hence $u = \varpi_{\lambda/\rho}$ and we are done. If $\zeta = \lambda$, then $\ell(s_i\varpi_\lambda) = \ell(\varpi_\lambda) - 1$ implies that $\varpi_\lambda = (\cdots i{+}1 \cdots | \cdots i \cdots)$, that is, $\varpi_\lambda(e) = i{+}1$ and $\varpi_\lambda(f) = i$ where $e \leq m < f = m + 1 + i - e$, and $s_i\varpi_\lambda = \varpi_{\lambda \smallsetminus b}$ for some box b in row $m + 1 - e$. Now $\ell(s_i\varpi_{\lambda/\rho}) = \ell(s_iu') = \ell(u') + 1$ implies that b must lie in ρ. (Indeed, if not then we have $\rho \subset (\lambda \smallsetminus b)$, hence a reduced factorization $w_{\lambda \smallsetminus b} = vw_\rho$ for some v, therefore $u' = s_iv$ is also reduced and so $\ell(s_iu') = \ell(u') - 1$.) Since both $\lambda \smallsetminus b$ and $\widetilde{\rho} := \rho \smallsetminus b$ are partitions, we must also have $\varpi_\rho(e) = i + 1$ and $\varpi_\rho(f) = i$.

If $s_{a_1}\cdots s_{a_r}$ is a reduced word of $u' = \varpi_{\lambda/\rho}$, then we deduce that $\{a_1, \ldots, a_r\} \cap \{i{-}1, i, i{+}1\} = \emptyset$, and hence that $s_i\varpi_{\lambda/\rho} = \varpi_{\lambda/\rho}s_i$. Now $s_i\varpi_{\widetilde{\rho}} = \varpi_\rho$ and therefore $u\varpi_{\widetilde{\rho}} = s_i\varpi_{\lambda/\rho}\varpi_{\widetilde{\rho}} = \varpi_{\lambda/\rho}\varpi_\rho = \varpi_\lambda$. We conclude that $u = \varpi_{\lambda/\widetilde{\rho}}$. Finally, since $\lambda \smallsetminus b$ is a partition and $\rho \subset \nu \subset \lambda$, it follows that b is the only box in its row and column in $\nu/\widetilde{\rho}$, and therefore that $\nu/\widetilde{\rho} = (\nu/\rho) \cup b$ is a rook strip.

To prove the converse, suppose that $\mu \subset \rho \subset \nu \subset \lambda$ are such that ν/ρ is a rook strip. Then it is easy to see by induction on $|\nu/\rho|$ that $\varpi_{\nu/\rho} \circ \varpi_{\nu/\mu} = \varpi_{\nu/\mu}$. Therefore, we have $\varpi_{\lambda/\rho} \circ \varpi_{\nu/\mu} = \varpi_{\lambda/\nu} \circ \varpi_{\nu/\rho} \circ \varpi_{\nu/\mu} = \varpi_{\lambda/\nu} \circ \varpi_{\nu/\mu} = \varpi_{\lambda/\mu}$. □

Corollary 1. *For the skew element $\varpi_{\lambda/\mu} \in S_\infty$, we have*

$$\mathfrak{G}_{\varpi_{\lambda/\mu}}(X, Y) = \sum_{\rho,\nu} \mathfrak{G}_{\varpi_{\nu/\mu}}(X)\mathfrak{G}_{\varpi^{-1}_{\lambda/\rho}}(Y)$$

summed over all partitions ρ, ν with $\mu \subset \rho \subset \nu \subset \lambda$ such that ν/ρ is a rook strip.

Proof. The definition (1) of $\mathfrak{G}_\varpi$ implies that

$$\mathfrak{G}_\varpi(X,Y) = \sum_{u \circ v = \varpi} \mathfrak{G}_v(X)\mathfrak{G}_{u^{-1}}(Y)$$

summed over all Hecke factorizations $u \circ v$ of ϖ. The result follows immediately from this and Proposition 1. $\square$

Let λ and μ be any two partitions of length at most m with $\mu \subset \lambda$, and choose $n \geq 1$ such that $\varpi_\lambda \in S_n$. Let $\mathbf{P}$ denote the ordered alphabet

$$(n-1)' < \cdots < 1' < 1 < \cdots < n-1.$$

The symbols $(n-1)', \ldots, 1'$ are said to be *marked*, while the rest are *unmarked*. Given a subset A of $\mathbf{P}$, an any element of A will be called an *entry*. If A and B are two nonempty subsets of an ordered set of symbols, we write $A \leq B$ if $\max(A) \leq \min(B)$.

Definition 1. A *set-valued m-bitableau U* of shape λ/μ is a filling of the boxes in λ/μ with finite nonempty subsets of $\mathbf{P}$ which is weakly increasing along each row and down each column, such that (i) the marked (resp. unmarked) entries are strictly increasing each down each column (resp. along each row), and (ii) the entries in row i lie in the interval $[(\mu_i+m+1-i)', \lambda_i+m-i]$ for each $i \in [1,m]$. We let $|U|$ be the total number of symbols of $\mathbf{P}$ which appear in U, including their multiplicities, and define

$$(xy)^U := \prod_i x_i^{n'_i} \prod_i y_i^{n_i}$$

where n'_i (resp. n_i) denotes the number of times that i' (resp. i) appears in U.

Theorem 1. *For the skew permutation $\varpi := \varpi_{\lambda/\mu}$ in S_n, we have*

$$\mathfrak{G}_\varpi(X,Y) = \sum_U \beta^{|U|-|\lambda/\mu|}(xy)^U \tag{5}$$

summed over all set-valued m-bitableaux U of shape λ/μ.

Proof. It follows from formula (1) that we have

$$\mathfrak{G}_\varpi(X,Y) = \sum \beta^{\ell(u,v,\varpi)} y_{n-1}^{\ell(v_{n-1})} \cdots y_1^{\ell(v_1)} x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})}$$

where the sum is over all Hecke factorizations $v_{n-1} \circ \cdots \circ v_1 \circ u_1 \circ \cdots \circ u_{n-1}$ of ϖ such that v_p is increasing up from p and u_p is decreasing down to p for each $p \in [1, n-1]$, while $\ell(u,v,\varpi) := \sum_i \ell(u_i) + \sum_i \ell(v_i) - \ell(\varpi)$. Proposition 1 shows that such factorizations correspond to pairs of sequences of partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2n-2} = \lambda \quad \text{and} \quad \mu = \nu^0, \nu^1, \ldots, \nu^{2n-2} = \lambda$$

such that (i) $\mu \subset \nu^i \subset \lambda^i$ and λ^i/ν^i is a rook strip for each i, and (ii) λ^i/ν^{i-1} is a horizontal strip for $1 \leq i \leq n-1$ and a vertical strip for $n \leq i \leq 2n-2$. We have $u_{n-i} = \varpi_{\lambda^i/\nu^{i-1}}$, for $i \in [1, n-1]$, and $v_{i+1-n} = \varpi_{\lambda^i/\nu^{i-1}}$, for $i \in [n, 2n-2]$. Note that some factors in the product $v_{n-1} \circ \cdots \circ v_1 \circ u_1 \circ \cdots \circ u_{n-1}$ may be trivial, and in this case the associated skew diagram λ^i/ν^{i-1} is empty. We obtain a corresponding filling U of the boxes in λ/μ by including the entry $(n-i)'$ in each box of λ^i/ν^{i-1} for $1 \leq i \leq n-1$ and the entry $i+1-n$ in each box of λ^i/ν^{i-1} for $n \leq i \leq 2n-2$. The required bounds on these entries are established exactly as in the proof of [T6, Thm. 1], showing that U is a set-valued m-bitableau of shape λ/μ such that

$(xy)^U = y_{n-1}^{\ell(v_{n-1})} \cdots y_1^{\ell(v_1)} x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})}$ and $|U| = \sum_i \ell(u_i) + \sum_i \ell(v_i)$. Since the sum in equation (5) is over all such U, the result follows. $\square$

Example 1. For each positive integer m, the permutation s_m of length one can be viewed as the m-Grassmannian element ϖ_λ associated to the partition $\lambda = 1$. Theorem 1 therefore gives

$$\mathfrak{G}_{s_m}(X,Y) = \sum_E \beta^{|E|-1} x_1^{e_1} \cdots x_m^{e_m} y_1^{e_{m+1}} \cdots y_m^{e_{2m}}$$

summed over all $4^m - 1$ vectors $E = (e_1, \ldots, e_{2m})$ with $e_i \in \{0,1\}$ for each i and $|E| := e_1 + \cdots + e_{2m} > 0$. This corrects an error in [BKTY, Example 1].

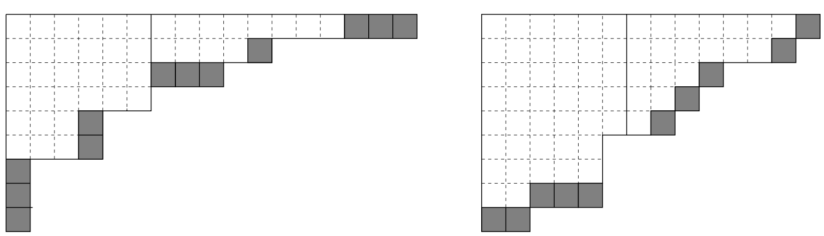
Example 2. Let $\varpi_{\lambda/\mu}$ be a skew element of S_∞. Extend the alphabet $\mathbf{P}$ to include all marked and unmarked integers, and omit the bounds on the entries of the m-bitableaux found in Definition 1. Then the right hand side of equation (5) gives a tableau formula for the *skew stable Grothendieck polynomial* $G_{\lambda/\mu}(X,Y)$ of Fomin and Kirillov [FK1]. This result was indicated by Buch in [B, Remark 3.3].

3. Tableau formula for type C skew Grothendieck polynomials

In this section, we assume that all Grassmannian and skew elements are taken with respect to a fixed nonnegative integer k.

Let λ and μ be any two k-strict partitions such that (λ, μ) is a compatible pair and let $w_{\lambda/\mu} = w_\lambda w_\mu^{-1}$ be the corresponding skew element of W_∞. According to [T1, Prop. 5], $w_{\lambda/\mu}$ is unimodal if and only if λ/μ is a k-horizontal strip, and in this case we have $n(w_{\lambda/\mu}) = n(\lambda/\mu)$. If $w_{\lambda/\mu}$ is decreasing down to 0 (resp. increasing up from 0), then we say that the k-horizontal strip λ/μ is a *z-strip* (resp. *$\overline{z}$-strip*). If we furthermore have $w_{\lambda/\mu} \in S_\infty$, we also say that λ/μ is an *x-strip* (resp. *y-strip*). A z-strip (resp. $\overline{z}$-strip) is an x-strip (resp. y-strip) if and only if $\ell_k(\lambda) = \ell_k(\mu)$.

Example 3. The shaded boxes in the Young diagrams below illustrate two k-horizontal strips with $k = 6$. The one on the left is a z-strip and the one on the right is both a $\overline{z}$-strip and a y-strip.



The z- and $\overline{z}$-strips are characterized among all k-horizontal strips as follows.

Proposition 2. *A k-horizontal strip λ/μ is a z-strip (resp. $\overline{z}$-strip) if and only if the left boxes in λ/μ form a vertical strip (resp. horizontal strip), and no two boxes in λ/μ are k'-related (resp. no two right boxes in λ/μ are in the same row).*

Proof. A signed permutation is decreasing down to 0 (resp. increasing up from 0) if and only if it has no reduced word which contains $i-1, i$ (resp. $i, i-1$) as a subword for some $i \geq 1$. Observe that in any reduced factorization $w_{\lambda/\mu} = us_{i-1}s_iv$ (resp. $w_{\lambda/\mu} = us_is_{i-1}v$), the elements u and v are also skew, and associated to k-horizontal strips which are substrips of λ/μ. Therefore we may assume that

$w_{\lambda/\mu} \in \{s_{i-1}s_i, s_is_{i-1}\}$ for some i and study the associated skew diagram λ/μ. For $i \geq 2$, the argument is the same as in the proof of [T6, Prop. 1], so it remains to examine the case when $i = 1$.

If $w_{\lambda/\mu} = s_0s_1$ then w_μ and w_λ have the form

$$w_\mu = (\cdots | \cdots \overline{1}2 \cdots), \ \ w_\lambda = (\cdots | \cdots \overline{2}\,\overline{1} \cdots)$$

so that λ/μ has two right boxes which are k'-related, or

$$w_\mu = (1 \cdots | \cdots 2 \cdots), \ \ w_\lambda = (2 \cdots | \cdots \overline{1} \cdots)$$

so that λ/μ has a left box and a right box which are k'-related (and in the same row). Here the vertical line $|$ lies between positions k and $k+1$. On the other hand, if $w_{\lambda/\mu} = s_1s_0$ then w_μ and w_λ have the form

$$w_\mu = (2 \cdots | \cdots 1 \cdots) \ \text{ or } \ w_\mu = (\cdots | \cdots 12 \cdots), \ \ w_\lambda = (\cdots | \cdots \overline{2} \cdots)$$

so that λ/μ has two right boxes which are in the same row (and in columns $k+1$ and $k+2$). The converse statements are proved by using the correspondence between ν and w_ν given in Section 1.3.2. This completes the proof. □

Definition 2. For any pair λ, μ of k-strict partitions with $\mu \subset \lambda$, we say that the skew Young diagram λ/μ is a *k-rook strip* if it contains no two boxes in the same row or column, and no two right boxes which are k-related.

Observe that the skew diagram λ/μ is a k-rook strip if and only if $\lambda \smallsetminus b$ is the diagram of a k-strict partition for each box b in λ/μ. The direct analogue of Proposition 1 does not hold in types B, C, or D, as the next example illustrates.

Example 4. Let $\lambda = \nu := (4,1)$, $\rho := 2$, and $\mu := \emptyset$, so that $\mu \subset \rho \subset \nu \subset \lambda$, $w_\lambda = s_1s_2s_1s_0s_1$, and $w_\rho = s_0s_1$. If $u := s_1s_2s_1$ and $v := w_\lambda$, then $w_{\lambda/\mu} = u{\circ}v$ and we have $u = w_{\lambda/\rho}$ and $v = w_{\nu/\mu}$. However, $\nu/\rho = (4,1)/2$ is not a k-rook strip. Notice that neither u nor v is fully commutative in the sense of [St].

In place of Proposition 1 and Corollary 1, we have the following weaker results.

Proposition 3. *Suppose that $w_{\lambda/\mu} = u{\circ}v$ for some skew element $w_{\lambda/\mu}$ of W_∞. Then there exist k-strict partitions ρ and ν such that $\mu \subset \rho \subset \lambda$ and $\mu \subset \nu \subset \lambda$ are compatible triples, $u = w_{\lambda/\rho}$, and $v = w_{\nu/\mu}$.*

Proof. Suppose that $a_1 \cdots a_r$ is a reduced word for u, so that $w_{\lambda/\mu} = s_{a_1}{\circ}\cdots{\circ}s_{a_r}{\circ}v$. Since $s_i{\circ}v$ is equal to v or to the reduced product s_iv for any i, it is clear that there exists a reduced factorization $w_{\lambda/\mu} = w'v$ for some w', and hence that $v = w_{\nu/\mu}$ for some k-strict partition ν with $\mu \subset \nu \subset \lambda$. Moreover, we have $w' = w_{\lambda/\nu}$ and the pairs (λ,ν) and (ν,μ) are compatible. Similarly, we see that there exists a reduced factorization $w_{\lambda/\mu} = uw''$ for some w'', and hence that $u = w_{\lambda/\rho}$ for some k-strict partition ρ with $\mu \subset \rho \subset \lambda$, such that the pairs (λ,ρ) and (ρ,μ) are compatible. □

Corollary 2. *For the skew element $w_{\lambda/\mu} \in W_\infty$, we have*

$$\mathfrak{G}^C_{w_{\lambda/\mu}}(Z;X,Y) = \sum_{\rho,\nu} \mathfrak{G}^C_{w_{\nu/\mu}}(Z;X)\mathfrak{G}_{w^{-1}_{\lambda/\rho}}(Y)$$

summed over all k-strict partitions ρ and ν such that $\mu \subset \rho \subset \lambda$ and $\mu \subset \nu \subset \lambda$ are compatible triples, $\ell_k(\rho) = \ell_k(\lambda)$, and $w_{\lambda/\mu} = w_{\lambda/\rho}{\circ}w_{\nu/\mu}$.

Proof. The definition (3) of $\mathfrak{G}^C_w$ implies that

$$\mathfrak{G}^C_w(Z;X,Y) = \sum_{u\circ v=w} \mathfrak{G}^C_v(Z;X)\mathfrak{G}_{u^{-1}}(Y)$$

summed over all Hecke factorizations $u\circ v$ of w with $u \in S_\infty$. The result follows immediately from this, Proposition 3, and the fact that $w_{\lambda/\rho} \in S_\infty$ if and only if $\ell_k(\rho) = \ell_k(\lambda)$. □

Suppose that λ is a k-strict partition and $i \geq 0$ is such that $\ell(s_i w_\lambda) = \ell(w_\lambda) - 1$. Then $s_i w_\lambda = w_\mu$ for a k-strict partition $\mu \subset \lambda$, so that λ/μ is a k-horizontal strip consisting of a single box b of λ. We then say that b is a *removable box* of λ. It follows that a box b of λ is a removable box of λ if and only if (i) $\lambda \smallsetminus b$ is a partition, and (ii) if b is a left box of λ, then b is not k'-related to two right boxes of λ (including boxes in row zero) which are both bottom boxes in their column.

Example 5. Let $k := 1$ and $\lambda := (3,1)$, so that $w_\lambda = s_1s_0s_2s_1$. Then the box of λ in the third column is removable, but the box in the second row is not removable. The diagram $(3,1)/3$ is a 1-rook strip but $((3,1),3)$ is not a compatible pair.

Similarly, suppose that (λ,μ) is a compatible pair of k-strict partitions and $i \geq 0$ is such that $\ell(s_i w_{\lambda/\mu}) = \ell(w_{\lambda/\mu}) - 1$. Then $s_i w_{\lambda/\mu} = w_{\nu/\mu}$ for a k-strict partition ν with $\mu \subset \nu \subset \lambda$, so that λ/ν is a k-horizontal strip consisting of a single box b of λ, which is not in μ. We then say that b is a *removable box* of λ/μ.

Example 6. Let $k := 1$, $\lambda := (4,1)$, and $\mu := 3$, so $w_\lambda = s_1s_2s_1s_0s_1 = s_2s_1s_2s_0s_1$, $w_\mu = s_1s_0s_1$, and $w_{\lambda/\mu} = s_1s_2$. Then λ has two removable boxes which are not in μ, but only the box $b := (4,1)/4$ in the second row is a removable box of λ/μ.

Proposition 4. *Suppose that (λ,μ) is a compatible pair of k-strict partitions, and b is a box of λ which is not in μ. Then b is a removable box of λ/μ if and only if b is a removable box of λ and $(\lambda \smallsetminus b, \mu)$ is a compatible pair.*

Proof. Suppose that b is a removable box of λ/μ and let $i \geq 0$ be such that $s_i w_{\lambda/\mu} = w_{\nu/\mu}$, with $\nu := \lambda \smallsetminus b$. Then (ν,μ) is a compatible pair and $s_i w_\lambda = s_i w_{\lambda/\mu} w_\mu = w_{\nu/\mu} w_\mu = w_\nu$, so b is a removable box of λ. Conversely, suppose that b is a removable box of λ, so that $s_i w_\lambda = w_\nu$ with $\nu := \lambda \smallsetminus b$, and assume that (ν,μ) is a compatible pair. Since (λ,ν) is also compatible, we have $w_{\lambda/\mu} = w_{\lambda/\nu} w_{\nu/\mu} = s_i w_{\nu/\mu}$, hence $s_i w_{\lambda/\mu} = w_{\nu/\mu}$ and so b is a removable box of λ/μ. □

The following proposition and its converse are the reason we can achieve tableau formulas for Grothendieck polynomials in types B, C, and D beyond the fully commutative Grassmannian examples of [GK, IN]. The point is that although Proposition 1 fails in these Lie types, an analogous result is true when the left factor u has a reduced word which is decreasing or increasing.

Proposition 5. *Suppose that u is decreasing down to 0 or increasing up from 0 and $w_{\lambda/\mu} = u\circ v$ for some skew element $w_{\lambda/\mu}$. Then there exist k-strict partitions $\rho \subset \nu$ such that $\mu \subset \rho \subset \lambda$ and $\mu \subset \nu \subset \lambda$ are compatible triples, ν/ρ is a k-rook strip, $u = w_{\lambda/\rho}$, and $v = w_{\nu/\mu}$.*

Proof. Proposition 3 implies that $v = w_{\nu/\mu}$ for some k-strict partition ν such that $\mu \subset \nu \subset \lambda$ is a compatible triple. We argue by induction on $\ell(u)$. If $\ell(u) = 1$, then $u = s_i$ is a simple reflection. If the product $s_i w_{\nu/\mu}$ is reduced, then $u = w_{\lambda/\nu}$ and we can take ρ to be equal to ν. If the product $s_i w_{\nu/\mu}$ is not reduced, then $\nu = \lambda$

and we have $s_iw_{\lambda/\mu} = w_{\rho/\mu}$ for some k-strict partition $\rho \subset \lambda$, where $b := \lambda/\rho$ is a removable box of λ/μ. We are therefore done in both cases.

For the inductive step, we may assume that $u = s_iu'$ with $\ell(u) = \ell(u') + 1$ and u is decreasing down to 0. We have $u'\circ w_{\nu/\mu} = w_{\zeta/\mu}$ for some k-strict partition ζ with $\mu \subset \zeta \subset \lambda$. The inductive hypothesis implies that $u' = w_{\zeta/\rho}$ for some k-strict partition $\rho \subset \nu$ such that $\mu \subset \rho \subset \zeta$ and $\mu \subset \nu \subset \zeta$ are compatible triples and ν/ρ is a k-rook strip. If $s_iw_\zeta = w_\lambda$ then (λ, ζ) is a compatible pair. Hence (λ, ρ) is a compatible pair, $u = s_iw_{\zeta/\rho} = w_{\lambda/\rho}$, and we are done.

Otherwise, we have $\zeta = \lambda$ and so $u' = w_{\lambda/\rho}$, while $s_iw_\lambda = w_{\lambda\smallsetminus b}$ for some removable box b of λ which is not in μ. Now there is a reduced word $a_r \cdots a_1$ for $w_{\lambda/\rho}$ such that $i > a_r > \cdots > a_1 \geq 0$. Therefore λ/ρ is a z-strip and, using Propositions 2 and 3, we see that $u = w_{\lambda/\widetilde{\rho}}$ for some z-strip $\lambda/\widetilde{\rho}$ with $|\lambda/\widetilde{\rho}| = r + 1$ that contains λ/ρ as a substrip. As $\ell(s_iw_{\lambda/\widetilde{\rho}}) = \ell(w_{\lambda/\widetilde{\rho}}) - 1$, we deduce that b is a removable box of $\lambda/\widetilde{\rho}$ which is not in λ/ρ. Hence b is a box of ρ, $\widetilde{\rho} = \rho \smallsetminus b$, and the equation $w_{\lambda/\mu} = w_{\lambda/\widetilde{\rho}}\circ v$ implies that $\mu \subset \widetilde{\rho} \subset \lambda$ is a compatible triple. Since $\lambda \smallsetminus b$ is a k-strict partition, it follows that $\nu \smallsetminus b$ is a k-strict partition. As ν/ρ is a k-rook strip, we conclude that $\nu/\widetilde{\rho} = (\nu/\rho) \cup b$ is also a k-rook strip. Finally, the proof in case u is increasing up from 0 is similar. □

Remark 1. In the situation of Proposition 5, if u is decreasing down to 0 (resp. increasing up from 0), then λ/ρ and λ/ν are both z-strips (resp. $\overline{z}$-strips). To see this for λ/ν, note that if $a_r \cdots a_1$ is a decreasing (resp. increasing) reduced word of u, then $s_{a_r}\circ\cdots\circ s_{a_1}\circ w_{\nu/\mu} = w_{\lambda/\mu}$ implies that $s_{a_{j_p}} \cdots s_{a_{j_1}} w_{\nu/\mu} = w_{\lambda/\mu}$ for some subword $a_{j_p} \cdots a_{j_1}$ of $a_r \cdots a_1$, and hence $w_{\lambda/\nu} = s_{a_{j_p}} \cdots s_{a_{j_1}}$.

Example 7. Let $k := 1$, $\nu := (3, 1)$, and $\lambda := (4, 1)$, so that $w_\nu = 3\overline{2}1 = s_1s_2s_0s_1$ and $w_\lambda = 2\overline{3}1 = s_2w_\nu$. Let $u := s_1s_2$ and observe that $w_\lambda = u\circ w_\nu$. We then have $u = w_{\lambda/\rho}$ for $\rho := 3 \subset \nu$. Note that $b := \nu/\rho$ is a removable box of λ, but not a removable box of ν. This shows that (ν, ρ) need not be a compatible pair in Proposition 5. Next, let $v := s_2s_1$ and observe that again $w_\lambda = v\circ w_\nu$. We have $v = w_{\lambda/\widetilde{\rho}}$ for $\widetilde{\rho} := (2, 1) \subset \nu$. In this case $\widetilde{b} := \nu/\widetilde{\rho}$ is a removable box of ν, but not a removable box of λ.

The next result is a converse to Proposition 5.

Proposition 6. *Suppose that $\mu \subset \rho \subset \nu \subset \lambda$ are k-strict partitions such that $\mu \subset \rho \subset \lambda$ and $\mu \subset \nu \subset \lambda$ are compatible triples, λ/ρ is a z-strip or a $\overline{z}$-strip, and ν/ρ is a k-rook strip. Then we have a Hecke factorization $w_{\lambda/\mu} = w_{\lambda/\rho}\circ w_{\nu/\mu}$.*

Proof. Suppose that λ/ρ is a $\overline{z}$-strip and $a_r \cdots a_1$ is a reduced word for $w_{\lambda/\rho}$ with $r = |\lambda/\rho|$ and $0 \leq a_r < \cdots < a_1$. Since (ρ, μ) is a compatible pair, we can define a k-strict partition $\zeta_i \supset \rho$ for each $i \in [1, r]$ by the equation $s_{a_i} \cdots s_{a_1} w_{\rho/\mu} = w_{\zeta_i/\mu}$, and let b_i denote the box $\zeta_i \smallsetminus \zeta_{i-1}$. Define $\overline{\zeta}_i := \zeta_i \cup \nu$, $\zeta_0 := \rho$, $\overline{\zeta}_0 := \nu$, and set $w^{(i)} := s_{a_i}\circ\cdots\circ s_{a_1}\circ w_{\nu/\mu}$. As ν/ρ is a k-rook strip, it follows that $\overline{\zeta}_i$ is a k-strict partition. We will prove by induction on i that $(\overline{\zeta}_i, \nu)$ is a compatible pair and $w^{(i)} = w_{\overline{\zeta}_i/\mu}$ for each $i \in [1, r]$. Assume therefore that the pair $(\overline{\zeta}_{i-1}, \nu)$ is compatible, $w^{(i-1)} = w_{\overline{\zeta}_{i-1}/\mu}$, and consider the product $s_{a_i}\circ w_{\overline{\zeta}_{i-1}/\mu} = w^{(i)}$.

Suppose first that $b_i \notin \nu$, hence $b_i = \overline{\zeta}_i \smallsetminus \overline{\zeta}_{i-1}$. Since λ/ρ is a k-horizontal strip and (λ, ν) is compatible pair, we deduce that λ/ν is also a k-horizontal strip. Proposition 2 then implies that λ/ν is a $\overline{z}$-strip. Moreover, $a_r \cdots a_1$ has a subword

$a_{j_p}\cdots a_{j_1}$ which is a reduced word for $w_{\lambda/\nu}$. Since we have $i\in\{j_1,\ldots,j_p\}$, we deduce that the pair $(\overline{\zeta}_i,\overline{\zeta}_{i-1})$ is compatible. As the pair $(\overline{\zeta}_{i-1},\nu)$ is also compatible, it follows that $(\overline{\zeta}_i,\nu)$ is compatible. Therefore $(\overline{\zeta}_i,\mu)$ is compatible and we have $s_{a_i}\circ w_{\overline{\zeta}_{i-1}/\mu}=s_{a_i}w_{\overline{\zeta}_{i-1}/\mu}=w_{\overline{\zeta}_i/\mu}$, as required.

Next, suppose that b_i is a box of the k-rook strip ν/ρ, hence $\overline{\zeta}_i=\overline{\zeta}_{i-1}$ and the pair $(\overline{\zeta}_i,\nu)$ is compatible. Using Proposition 2 again, we see that $(\overline{\zeta}_i,\rho)$ is a compatible pair and $a_r\cdots a_1$ has a subword which includes a_i and is a reduced word for $w_{\overline{\zeta}_i/\rho}$. This proves that b_i is a removable box of $\overline{\zeta}_i/\rho$ and hence $s_{a_i}\circ w_{\overline{\zeta}_i/\rho}=w_{\overline{\zeta}_i/\rho}$. Since $w_{\overline{\zeta}_i/\mu}=w_{\overline{\zeta}_i/\rho}w_{\rho/\mu}$ we deduce that $s_{a_i}\circ w_{\overline{\zeta}_{i-1}/\mu}=s_{a_i}\circ w_{\overline{\zeta}_i/\mu}=w_{\overline{\zeta}_i/\mu}$. As $\overline{\zeta}_r=\lambda$, we conclude that $w_{\lambda/\rho}\circ w_{\nu/\mu}=w^{(r)}=w_{\lambda/\mu}$. Finally, the argument when λ/ρ is a z-strip is similar. □

Example 8. The hypotheses that (λ,ν) and (ρ,μ) are compatible pairs in Proposition 6 are necessary. For instance, let $k:=1$, $\mu:=\emptyset$, $\rho:=2$, $\nu:=3$, and $\lambda:=(3,1)$. Then $\mu\subset\rho\subset\lambda$ is a compatible triple, (ν,μ) is a compatible pair, $w_{\lambda/\rho}=s_1s_2$ is a $\overline{z}$-strip, and ν/ρ is a 1-rook strip. However, we have

$$w_{\lambda/\rho}\circ w_\nu=(s_1s_2)\circ(s_1s_0s_1)=s_1s_2s_1s_0s_1\neq w_\lambda.$$

Moreover, if $k=1$, $\mu:=3$, $\rho:=(3,1)$, and $\nu=\lambda:=(4,1)$, then $\mu\subset\nu\subset\lambda$ is a compatible triple, (λ,ρ) is a compatible pair, and ν/ρ is a 1-rook strip. However, we compute that

$$w_{\lambda/\rho}\circ w_{\nu/\mu}=s_2\circ(s_1s_2)=s_2s_1s_2\neq w_{\lambda/\mu}.$$

Definition 3. A *set-valued k-tableau T* of shape λ/μ is a pair of sequences of k-strict partitions

$$\mu=\lambda^0\subset\lambda^1\subset\cdots\subset\lambda^{2r}=\lambda \quad\text{and}\quad \mu=\nu^0,\nu^1,\ldots,\nu^{2r}=\lambda$$

with the first one compatible, such that (i) $\mu\subset\nu^i\subset\lambda^i$, (ν^i,μ) is a compatible pair, and λ^i/ν^i is a k-rook strip for each i, and (ii) λ^i/ν^{i-1} is a $\overline{z}$-strip if i is odd and a z-strip if i is even, for $1\leq i\leq 2r$. We represent T by a filling of the boxes in λ/μ with finite nonempty subsets of barred and unbarred positive integers such that for each $h\in[1,r]$, the boxes in T which contain the entry $\overline{h}$ (resp. h) form the skew diagram $\lambda^{2h-1}/\nu^{2h-2}$ (resp. λ^{2h}/ν^{2h-1}). For any set-valued k-tableau T we let $|T|$ be the total number of barred and unbarred positive integers which appear in T, including their multiplicities, and set $z^T:=\prod_h z_h^{m_h}$, where m_h denotes the number of times that $\overline{h}$ or h appears in T.

Theorem 2. *Let $w=w_{\lambda/\mu}$ be the skew element associated to the compatible pair (λ,μ) of k-strict partitions. Then we have*

$$F_w^C(Z)=\sum_T\beta^{|T|-|\lambda/\mu|}\,z^T \tag{6}$$

where the sum is over all set-valued k-tableau T of shape λ/μ.

Proof. We deduce from the definition of F_w^C that

$$F_w^C(Z)=\sum_{w=\sigma_{2r}\circ\cdots\circ\sigma_1}\beta^{\sum_i\ell(\sigma_i)-\ell(w)}\prod_j z_j^{\ell(\sigma_{2j-1})+\ell(\sigma_{2j})} \tag{7}$$

where the inner sum is over all Hecke factorizations $\sigma_{2r}\circ\cdots\circ\sigma_1$ of w, for varying r, such that σ_{2j-1} is increasing up from 0 and σ_{2j} is decreasing down to 0 for all

$j \in [1,r]$. Propositions 5 and 6 imply that any such factorization $w = \sigma_{2r} \circ \cdots \circ \sigma_1$ corresponds to a pair of sequences of k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2r} = \lambda \quad \text{and} \quad \mu = \nu^0, \nu^1, \ldots, \nu^{2r} = \lambda$$

as in Definition 3, with $\sigma_i = w_{\lambda^i/\nu^{i-1}}$ for each $i \in [1, 2r]$. Moreover, if T is the associated set-valued k-tableau, then

$$|T| = \sum_{i=1}^{2r} |\lambda^i/\nu^{i-1}| = \sum_{i=1}^{2r} \ell(\sigma_i) \quad \text{and} \quad z^T = \prod_{j=1}^{r} z_j^{\ell(\sigma_{2j-1})+\ell(\sigma_{2j})}$$

while $\ell(w) = |\lambda/\mu|$. Equation (6) therefore follows immediately from (7). $\square$

Example 9. Let $k := 1$ and $\lambda := (5,3,1)$ so that $w_\lambda = 3\,\overline{4}\,\overline{2}\,1 = s_1s_3s_2s_0s_3s_1s_0s_2s_1$. The Hecke factorization

$$(s_3s_1s_0) \circ 1 \circ (s_3s_2) \circ 1 \circ (s_2s_0) \circ (s_2s_3) \circ (s_3s_1s_0) \circ (s_0s_2) \circ (s_2s_1) \circ 1$$

of w_λ as a product of $2r = 10$ factors σ_i in W_4 corresponds to the following set-valued 1-tableau T of shape λ:

$$\begin{matrix} 1 & \{\overline{2},2\} & 2 & \{\overline{3},3,4\} & \{4,5\} \\ \{1,\overline{2}\} & \{3,5\} & 5 & & \\ \{2,\overline{3}\} & & & & \end{matrix} \; .$$

Furthermore, we have $z^T = z_1^2z_2^5z_3^4z_4^2z_5^3$.

Example 10. Let (λ,μ) be a compatible pair of typed k-strict partitions. A *barred k-tableau* T of shape λ/μ is a sequence of k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2r} = \lambda$$

such that λ^i/λ^{i-1} is a $\overline{z}$-strip if i is odd and a z-strip if i is even, for $1 \leq i \leq 2r$. We represent T by a filling of the boxes in λ/μ with barred and unbarred positive integers such that for each $h \in [1,r]$, the boxes in T with entry $\overline{h}$ (resp. h) form the skew diagram $\lambda^{2h-1}/\lambda^{2h-2}$ (resp. $\lambda^{2h}/\lambda^{2h-1}$).

If we set $\beta := 0$, then the K-theoretic Stanley function $F_w^C(Z)$ specializes to the type C Stanley function $F_w(Z)$, introduced and studied in [BH, FK3, La]. If $w := w_{\lambda/\mu}$ then Theorem 2 gives

$$F_w(Z) = \sum_T z^T \tag{8}$$

summed over all barred k-tableaux T of shape λ/μ. The barred k-tableaux and equation (8) refine the k-tableaux of [T1, Def. 2] and [T6, Eqn. (7)] in the same way that the marked shifted tableaux of Sagan and Worley [Sa, W] and [Mac, III.(8.16′)] refine the shifted tableaux and [Mac, III.(8.16)].

For example, let $k = 1$, $\lambda = (3,1)$, $Z_2 := (1,2)$, and $\mathbf{Q}_{12}$ denote the alphabet $\{\overline{1} < 1 < \overline{2} < 2\}$. Following [T1, Example 7], there are four 1-tableaux T of shape λ with entries in $\{1,2\}$, each with multiplicity $2^{n(T)} = 4$:

$$\begin{matrix} 1\,2\,2 \\ 2 \end{matrix} \qquad \begin{matrix} 1\,1\,2 \\ 2 \end{matrix} \qquad \begin{matrix} 1\,2\,2 \\ 1 \end{matrix} \qquad \begin{matrix} 1\,1\,2 \\ 1 \end{matrix} \; .$$

These correspond to the following 16 barred 1-tableaux with entries in $\mathbf{Q}_{12}$:

$$\begin{matrix} \overline{1}\,2\,2 \\ \overline{2} \end{matrix} \quad \begin{matrix} \overline{1}\,\overline{2}\,2 \\ \overline{2} \end{matrix} \quad \begin{matrix} 1\,2\,2 \\ \overline{2} \end{matrix} \quad \begin{matrix} 1\,\overline{2}\,2 \\ \overline{2} \end{matrix} \quad \begin{matrix} \overline{1}\,\overline{1}\,\overline{2} \\ \overline{2} \end{matrix} \quad \begin{matrix} \overline{1}\,\overline{1}\,2 \\ \overline{2} \end{matrix} \quad \begin{matrix} \overline{1}\,1\,\overline{2} \\ \overline{2} \end{matrix} \quad \begin{matrix} \overline{1}\,1\,2 \\ \overline{2} \end{matrix}$$

$$\begin{array}{cccccccc} \overline{1}\,\overline{2}\,2 & \overline{1}\,2\,2 & 1\,\overline{2}\,2 & 1\,2\,2 & \overline{1}\,\overline{1}\,\overline{2} & \overline{1}\,\overline{1}\,2 & \overline{1}\,1\,\overline{2} & \overline{1}\,1\,2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}.$$

We deduce that $F_{w_\lambda}(Z_2) = 4z_1^3z_2 + 8z_1^2z_2^2 + 4z_1z_2^3$.

Let $\mathbf{Q}$ denote the ordered alphabet

$$(n-1)' < \cdots 2' < 1' < \overline{1} < 1 < \overline{2} < 2 < \cdots < 1'' < 2'' < \cdots < (n-1)''.$$

The symbols $\overline{1}, \overline{2}, \ldots$ (resp. $1, 2, \ldots$) are *barred integers* (resp. *unbarred integers*).

Definition 4. Consider a pair U of sequences of k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2n+2r-2} = \lambda \quad \text{and} \quad \mu = \nu^0, \nu^1, \ldots, \nu^{2n+2r-2} = \lambda$$

with the first one compatible, such that (i) $\mu \subset \nu^i \subset \lambda^i$, (ν^i, μ) is a compatible pair, and λ^i/ν^i is a k-rook strip for each i, and (ii) λ^i/ν^{i-1} is an x-strip for each $i \leq n-1$, a y-strip for each $i \geq n+2r$, and a $\overline{z}$-strip (resp. z-strip) if $i = n-1+j$ for some odd (resp. even) $j \in [1, 2r]$. We represent U by a filling of the boxes in λ/μ by including the entry $(n-i)'$ in each box of λ^i/ν^{i-1} for $1 \leq i \leq n-1$, the entry $(i-n-2r+1)''$ in each box of λ^i/ν^{i-1} for $n+2r \leq i \leq 2n+2r-2$, and the entry $\overline{j}$ (resp. j) in each box of $\lambda^{n-1+j}/\nu^{n-2+j}$ for all odd (resp. even) integers $j \in [1, 2r]$. We say that U is a *set-valued k-tritableau* of shape λ/μ if for $1 \leq i \leq \ell_k(\mu)$ (resp. $1 \leq i \leq \ell_k(\lambda)$) and $1 \leq j \leq k$, the entries of U in row i are $\geq (\mu_i - k)'$ (resp. $\leq (\lambda_i - k - 1)''$) and the entries in column $k+1-j$ lie in the interval $[(w_\mu(j))', (w_\lambda(j)-1)'']$. We let $|U|$ be the total number of symbols of $\mathbf{Q}$ which appear in U, including their multiplicities, and define

$$(xyz)^U := \prod_i x_i^{n'_i} \prod_i y_i^{n''_i} \prod_j z_j^{n_j}$$

where n'_i and n''_i denote the number of times that i' and i'' appear in U, respectively, for each $i \in [1, n-1]$, and n_j denotes the number of times that j or $\overline{j}$ appears in U, for each $j \geq 1$.

Theorem 3. *For the skew element $w := w_{\lambda/\mu}$ of W_n, we have*

$$\mathfrak{G}^C_w(Z; X, Y) = \sum_U \beta^{|U| - |\lambda/\mu|} (xyz)^U \tag{9}$$

summed over all set-valued k-tritableaux U of shape λ/μ.

Proof. It follows from formula (3) that we have

$$\mathfrak{G}^C_w(Z; X, Y) = \sum \beta^{\ell(u,v,\sigma,w)} y_{n-1}^{\ell(v_{n-1})} \cdots y_1^{\ell(v_1)} x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})} \prod_j z_j^{\ell(\sigma_{2j-1}) + \ell(\sigma_{2j})}$$

where the sum is over all Hecke factorizations

$$v_{n-1} \circ \cdots \circ v_1 \circ \sigma_{2r} \circ \cdots \circ \sigma_1 \circ u_1 \circ \cdots \circ u_{n-1}$$

of w, for varying $r \geq 0$, such that $v_p \in S_n$ is increasing up from p and $u_p \in S_n$ is decreasing down to p for each p, σ_i is increasing up from 0 (resp. decreasing down to 0) for all odd i (resp. all even i), and $\ell(u, v, \sigma, w) := \sum_p \ell(u_p) + \sum_p \ell(v_p) + \sum_i \ell(\sigma_i) - \ell(w)$. We deduce from Propositions 5 and 6 that such factorizations correspond to pairs of sequences of k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2n+2r-2} = \lambda \quad \text{and} \quad \mu = \nu^0, \nu^1, \ldots, \nu^{2n+2r-2} = \lambda$$

and an associated filling U of the the boxes in λ/μ as in Definition 4. The required bounds on the entries of U are established exactly as in the proof of [T6, Thm. 2], showing that U is a set-valued k-tritableau of shape λ/μ such that

$$(xyz)^U = y_{n-1}^{\ell(v_{n-1})}\cdots y_1^{\ell(v_1)}x_1^{\ell(u_1)}\cdots x_{n-1}^{\ell(u_{n-1})}\prod_j z_j^{\ell(\sigma_{2j-1})+\ell(\sigma_{2j})}$$

and $|U| = \sum_p \ell(u_p) + \sum_p \ell(v_p) + \sum_i \ell(\sigma_i)$. Since the sum in equation (9) is over all such U, the result follows. $\square$

Example 11. Let $k := 1$ and $\lambda := (4,2)$ so that $w_\lambda = 2\,\overline{3}\,\overline{1} = s_2s_0s_1s_2s_0s_1$, and consider the alphabet $\mathbf{Q}_{ex} := \{1' < \overline{1} < 1 < \overline{2} < 1'' < 2''\}$. The set-valued 1-tritableau

$$\begin{array}{cccc} 1' & \{\overline{1},1\} & 1 & \{1,\overline{2},1''\} \\ \{1',\overline{1}\} & \overline{2} & & \end{array} \quad \text{and} \quad \begin{array}{cccc} \{1',\overline{1}\} & \{\overline{1},1\} & 1 & \{1,\overline{2},1'',2''\} \\ \overline{2} & \overline{2} & & \end{array}$$

of shape λ with entries in $\mathbf{Q}_{ex}$ correspond to the Hecke factorizations

$$s_2\circ(s_0s_2)\circ(s_2s_1s_0)\circ(s_0s_2)\circ(s_2s_1) \quad \text{and} \quad s_2\circ s_2\circ(s_0s_1s_2)\circ(s_2s_1s_0)\circ(s_0s_1)\circ s_1$$

of w_λ and map to the monomials $x_1^2z_1^5z_2^2y_1$ and $x_1z_1^5z_2^3y_1y_2$, respectively.

Example 12. For each integer $k \geq 0$, the signed permutation s_k of length one can be viewed as the k-Grassmannian element w_λ associated to $\lambda = 1$. When k is positive, Theorem 3 therefore gives

$$\mathfrak{G}^C_{s_k}(Z;X,Y) = \sum_H \beta^{|H|-1}x_1^{e_1}\cdots x_k^{e_k}y_1^{e_{k+1}}\cdots y_k^{e_{2k}}\prod_j z_j^{f_j+g_j}$$

summed over all sequences $H = (e_1,\ldots,e_{2k},f_1,g_1,f_2,g_2,\ldots)$ with finite support such that $e_i,f_j,g_j \in \{0,1\}$ for each i,j and $|H| := e_1+\cdots+e_{2k}+\sum_j(f_j+g_j) > 0$. Furthermore, we have

$$\mathfrak{G}^C_{s_0}(Z;X,Y) = \sum_D \beta^{|D|-1}\prod_j z_j^{f_j+g_j}$$

summed over all sequences $D = (f_1,g_1,f_2,g_2,\ldots)$ with finite support such that $f_j,g_j \in \{0,1\}$ for each j and $|D| := \sum_j(f_j+g_j) > 0$.

3.1. **Tableau formula for type B skew Grothendieck polynomials.** Given that the root systems of types B_n and C_n share a common Weyl group W_n, and the similarity between equations (2) and (3), it is easy to modify the definitions and theorems in this section to obtain set-valued k-tableau formulas for the type B Grothendieck polynomials $\mathfrak{G}^B_w(Z;X,Y)$ and K-theoretic Stanley functions $F^B_w(Z)$ indexed by skew elements $w = w_{\lambda/\mu}$. Indeed, it suffices to change the definition of a $\overline{z}$-strip in type B to be the same as that of a y-strip. With this modification, the main results of this section then apply verbatim to the setting of type B.

Example 13. The type B Grothendieck polynomials for the elements s_k of length one in S_∞ satisfy $\mathfrak{G}^B_{s_k}(Z;X,Y) = \mathfrak{G}^C_{s_k}(Z;X,Y)$. Furthermore, we have

$$\mathfrak{G}^B_{s_0}(Z;X,Y) = \sum_{D'} \beta^{|D'|-1}\prod_j z_j^{f_j}$$

summed over all sequences $D' = (f_1,f_2,\ldots)$ with finite support such that $f_j \in \{0,1\}$ for each j and $|D'| := \sum_j f_j > 0$.

4. Tableau formula for type D skew Grothendieck polynomials

In this section, we assume that all Grassmannian and skew elements are taken with respect to a fixed positive integer k.

Let λ and μ be any two typed k-strict partitions such that (λ, μ) is a compatible pair and let $w_{\lambda/\mu}$ be the corresponding skew element of $\widetilde{W}_n$. It was shown in [T3] that $w_{\lambda/\mu}$ is unimodal if and only if λ/μ is a typed k'-horizontal strip, and in this case we have $n'(w_{\lambda/\mu}) = n'(\lambda/\mu)$. If $w_{\lambda/\mu}$ is decreasing down to 1, then we say that the typed k'-horizontal strip λ/μ is an *typed x-strip* or a *typed z-strip*. If $w_{\lambda/\mu}$ is increasing up from 1 (resp. $\Box$), then we say that λ/μ is a *typed y-strip* (resp. *typed $\overline{z}$-strip*). We say that a typed k'-horizontal strip λ/μ is *extremal* if

$$(\ell_k(\lambda), \mathrm{type}(\lambda)) \neq (\ell_k(\mu), \mathrm{type}(\mu)).$$

For any typed k-strict partition λ, let $\epsilon(\lambda) := \ell_k(\lambda) + \mathrm{type}(\lambda)$.

According to [T6, Prop. 2], a typed k'-horizontal strip λ/μ is a typed x-strip (resp. typed y-strip) if and only if (i) the left boxes in λ/μ form a vertical strip (resp. horizontal strip), and no two boxes in λ/μ are $(k-1)$-related (resp. no two right boxes in λ/μ are in the same row), and (ii) if λ/μ is extremal then $(\mathrm{type}(\lambda), \mathrm{type}(\mu)) \neq (0,0)$ and the following condition holds: if $\epsilon(\mu)$ is odd, then $\epsilon(\lambda)$ is odd *and* $\mathrm{type}(\mu) = 0$, while if $\epsilon(\mu)$ is even, then $\epsilon(\lambda)$ is odd *or* $\mathrm{type}(\mu) = 1$. Moreover, it was shown in the proof of loc. cit. that the typed $\overline{z}$-strips are characterized among all typed k'-horizontal strips by the next Proposition.

Proposition 7. *A typed k'-horizontal strip λ/μ is a typed $\overline{z}$-strip if and only if* (i) *the left boxes in λ/μ form a horizontal strip, and no two right boxes in λ/μ are in the same row, and* (ii) *if λ/μ is extremal then* $(\mathrm{type}(\lambda), \mathrm{type}(\mu)) \neq (0,0)$ *and the following condition holds: if $\epsilon(\mu)$ is odd, then $\epsilon(\lambda)$ is even* or $\mathrm{type}(\mu) = 1$*, while if $\epsilon(\mu)$ is even, then $\epsilon(\lambda)$ is even* and $\mathrm{type}(\mu) = 0$.

The following results are proved exactly as their analogues in type C.

Proposition 8. *Suppose that $w_{\lambda/\mu} = u \circ v$ for some skew element $w_{\lambda/\mu}$ of $\widetilde{W}_\infty$. Then there exist typed k-strict partitions ρ and ν such that $\mu \subset \rho \subset \lambda$ and $\mu \subset \nu \subset \lambda$ are compatible triples, $u = w_{\lambda/\rho}$, and $v = w_{\nu/\mu}$.*

Corollary 3. *For the skew element $w_{\lambda/\mu} \in \widetilde{W}_\infty$, we have*

$$\mathfrak{G}^D_{w_{\lambda/\mu}}(Z;X,Y) = \sum_{\rho,\nu} \mathfrak{G}^D_{w_{\nu/\mu}}(Z;X)\mathfrak{G}_{w^{-1}_{\lambda/\rho}}(Y)$$

summed over all typed k-strict partitions ρ and ν such that $\mu \subset \rho \subset \lambda$ and $\mu \subset \nu \subset \lambda$ are compatible triples, $w_{\lambda/\rho} \in S_\infty$, and $w_{\lambda/\mu} = w_{\lambda/\rho} \circ w_{\nu/\mu}$.

Suppose that λ is a typed k-strict partition and $i \in \mathbb{N}_\Box$ is such that $\ell(s_i w_\lambda) = \ell(w_\lambda) - 1$. Then $s_i w_\lambda = w_\mu$ for a typed k-strict partition $\mu \subset \lambda$, so that λ/μ is a typed k'-horizontal strip consisting of a single box b of λ. We then say that b is a *removable box* of λ. It follows that a box b of λ is a removable box of λ if and only if (i) $\lambda \smallsetminus b$ is a partition and (ii) if b is a left box of λ, then b is not $(k-1)$-related to two right boxes of λ (including boxes in row zero) which are both bottom boxes in their column. Notice that the type of the typed k-strict partition $\mu = \lambda \smallsetminus b$ is determined by the condition $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$.

Similarly, suppose that (λ, μ) is a compatible pair of typed k-strict partitions and $i \in \mathbb{N}_\Box$ is such that $\ell(s_i w_{\lambda/\mu}) = \ell(w_{\lambda/\mu}) - 1$. Then $s_i w_{\lambda/\mu} = w_{\nu/\mu}$ for a typed k-strict partition ν with $\mu \subset \nu \subset \lambda$, so that λ/ν is a typed k-horizontal strip consisting of a single box b of λ, which is not in μ. We then say that b is a *removable box* of λ/μ, and we must have $\mathrm{type}(\lambda)+\mathrm{type}(\nu)\neq 3$. In this case, b is a removable box of λ which is not in μ and such that $(\lambda \smallsetminus b, \mu)$ is a compatible pair.

Proposition 9. *Suppose that u is decreasing down to 1, increasing up from 1, or increasing up from $\Box$, and $w_{\lambda/\mu} = u \circ v$ for some skew element $w_{\lambda/\mu}$. Then there exist typed k-strict partitions $\rho \subset \nu$ such that $\mu \subset \rho \subset \lambda$ and $\mu \subset \nu \subset \lambda$ are compatible triples, ν/ρ is a k-rook strip, $u = w_{\lambda/\rho}$, and $v = w_{\nu/\mu}$.*

Proof. The argument is the same as in the proof of Proposition 5. $\Box$

Remark 2. In the situation of Proposition 9, if u is decreasing down to 1 (resp. increasing up from 1 or increasing up from $\Box$), then λ/ν and λ/ρ are both typed z-strips (resp. typed y-strips or typed $\overline{z}$-strips).

As in the previous section, Proposition 9 admits a converse statement.

Proposition 10. *Suppose that $\mu \subset \rho \subset \nu \subset \lambda$ are typed k-strict partitions such that $\mu \subset \rho \subset \lambda$ and $\mu \subset \nu \subset \lambda$ are compatible triples, λ/ρ is a typed z-strip (resp. typed y-strip or typed $\overline{z}$-strip), and ν/ρ is a k-rook strip. Then we have a Hecke factorization $w_{\lambda/\mu} = w_{\lambda/\rho} \circ w_{\nu/\mu}$.*

Proof. The argument is the same as in the proof of Proposition 6. $\Box$

Let $\breve{\mathbf{R}}$ denote the ordered alphabet

$$\overline{1}, \overline{1}^{\circ} < 1, 1^{\circ} < \overline{2}, \overline{2}^{\circ} < 2, 2^{\circ} < \cdots .$$

The symbols $\overline{1}, \overline{1}^{\circ}, \overline{2}, \overline{2}^{\circ}, \ldots$ are *barred*, while the symbols $1, 1^{\circ}, 2, 2^{\circ}, \ldots$ are *unbarred*.

Definition 5. A *set-valued typed k'-tableau* T of shape λ/μ is a pair of sequences of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2r} = \lambda \quad \text{and} \quad \mu = \nu^0, \nu^1, \ldots, \nu^{2r} = \lambda$$

with the first one compatible, such that (i) $\mu \subset \nu^i \subset \lambda^i$, (ν^i, μ) is a compatible pair, and λ^i/ν^i is a k-rook strip for each i, and (ii) λ^i/ν^{i-1} is a typed $\overline{z}$-strip if i is odd and a typed z-strip if i is even, for $1 \leq i \leq 2r$. We represent T by a filling of the boxes in λ/μ with finite nonempty subsets of symbols of $\breve{\mathbf{R}}$ such that for each $h \in [1,r]$, the boxes in T which contain the entry $\overline{h}$ or $\overline{h}^{\circ}$ (resp. h or h°) form the skew diagram $\lambda^{2h-1}/\nu^{2h-2}$ (resp. λ^{2h}/ν^{2h-1}), and we use $\overline{h}^{\circ}$ (resp. h°) if and only if $\mathrm{type}(\lambda^{2h-1}) = 2$ (resp. $\mathrm{type}(\lambda^{2h}) = 2$), for each $h \in [1,r]$. For any set-valued typed k'-tableau T we let $|T|$ be the total number of symbols of $\breve{\mathbf{R}}$ which appear in T, including their multiplicities, and set $z^T := \prod_h z_h^{m_h}$, where m_h denotes the number of times that $\overline{h}$, h, $\overline{h}^{\circ}$, or h° appears in T.

Theorem 4. *Let $w = w_{\lambda/\mu}$ be the skew element associated to the compatible pair (λ, μ) of typed k-strict partitions. Then we have*

$$F_w^D(Z) = \sum_T \beta^{|T|-|\lambda/\mu|}\, z^T \tag{10}$$

where the sum is over all set-valued typed k'-tableau of shape λ/μ.

Proof. It is easy to see from the definition of F_w^D that

$$F_w^D(Z) = \sum_{w=\sigma_{2r}\circ\cdots\circ\sigma_1} \beta^{\sum_i \ell(\sigma_i)-\ell(w)} \prod_j z_j^{\ell(\sigma_{2j-1})+\ell(\sigma_{2j})} \tag{11}$$

where the inner sum is over all Hecke factorizations $\sigma_{2r}\circ\cdots\circ\sigma_1$ of w, for varying r, such that σ_{2j-1} is increasing up from $\Box$ and σ_{2j} is decreasing down to 1 for all $j \in [1,r]$. Propositions 9 and 10 imply that any such factorization $w = \sigma_{2r}\circ\cdots\circ\sigma_1$ corresponds to a pair of sequences of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2r} = \lambda \quad \text{and} \quad \mu = \nu^0, \nu^1, \cdots, \nu^{2r} = \lambda$$

as in Definition 5, with $\sigma_i = w_{\lambda^i/\nu^{i-1}}$ for each $i \in [1, 2r]$. Furthermore, if T is the associated set-valued typed k'-tableau, then

$$|T| = \sum_{i=1}^{2r} |\lambda^i/\nu^{i-1}| = \sum_{i=1}^{2r} \ell(\sigma_i) \quad \text{and} \quad z^T = \prod_{j=1}^{r} z_j^{\ell(\sigma_{2j-1})+\ell(\sigma_{2j})}$$

while $\ell(w) = |\lambda/\mu|$. Therefore, equation (10) follows directly from (11). $\Box$

Example 14. Given a compatible pair (λ,μ) of typed k-strict partitions, a *barred typed k'-tableau* T of shape λ/μ is a sequence of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2r} = \lambda$$

such that λ^i/λ^{i-1} is a typed $\overline{z}$-strip if i is odd and a typed z-strip if i is even, for $1 \le i \le 2r$. We represent T by a filling of the boxes in λ/μ with symbols in $\breve{\mathbf{R}}$ such that for each $h \in [1,r]$, the boxes in T with entry $\overline{h}$ or $\overline{h}^\circ$ (resp. h or h°) form the skew diagram $\lambda^{2h-1}/\lambda^{2h-2}$ (resp. $\lambda^{2h}/\lambda^{2h-1}$), and we use $\overline{h}^\circ$ (resp. h°) if and only if $\text{type}(\lambda^{2h-1}) = 2$ (resp. $\text{type}(\lambda^{2h}) = 2$), for each $h \in [1,r]$.

If we set $\beta := 0$, then the K-theoretic Stanley function $F_w^D(Z)$ specializes to the type D Stanley function $E_w(Z)$, introduced and studied in [BH, La]. If $w := w_{\lambda/\mu}$ then Theorem 4 gives

$$E_w(Z) = \sum_T z^T \tag{12}$$

summed over all barred typed k'-tableaux T of shape λ/μ. The barred typed k'-tableaux and equation (12) refine the typed k'-tableaux of [T3, Def. 3] and [T6, Eqn. (11)] in a manner directly analogous to Example 10.

For instance, let $k := 1$, $\lambda := (3,1)$ with $\text{type}(\lambda) := 1$ so that $w_\lambda = 2\,\overline{3}\,\overline{1}$, and $\breve{\mathbf{R}}_{12}$ denote the alphabet $\{\overline{1}, \overline{1}^\circ < 1, 1^\circ < \overline{2}, \overline{2}^\circ < 2, 2^\circ\}$. There are four barred typed $1'$-tableaux of shape λ with entries in $\breve{\mathbf{R}}_{12}$:

$$\begin{matrix} \overline{1}^\circ\,1\,1 \\ \overline{2} \end{matrix} \qquad \begin{matrix} \overline{1}^\circ\,1\,\overline{2} \\ \overline{2} \end{matrix} \qquad \begin{matrix} 1\,\overline{2}\,2 \\ 1 \end{matrix} \qquad \begin{matrix} 1\,\overline{2}\,2 \\ \overline{2} \end{matrix}$$

corresponding to the reduced factorizations $s_\Box \cdot s_2 s_1 \cdot s_\Box$, $s_\Box s_2 \cdot s_1 \cdot s_\Box$, $s_2 \cdot s_\Box \cdot s_2 s_1 \cdot 1$, and $s_2 \cdot s_\Box s_2 \cdot s_1 \cdot 1$ of w_λ, respectively. Moreover, let $\lambda^\circ := (3,1)$ with $\text{type}(\lambda^\circ) := 2$ so that $w_{\lambda^\circ} = \overline{2}\,\overline{3}\,1$. Then we find the following four barred typed $1'$-tableaux of shape λ° with entries in $\breve{\mathbf{R}}_{12}$:

$$\begin{matrix} \overline{1}^\circ\,1\,1 \\ 2^\circ \end{matrix} \qquad \begin{matrix} \overline{1}^\circ\,1\,\overline{2} \\ 2^\circ \end{matrix} \qquad \begin{matrix} \overline{1}^\circ\,2^\circ\,2^\circ \\ 1^\circ \end{matrix} \qquad \begin{matrix} \overline{1}^\circ\,2^\circ\,2^\circ \\ \overline{2}^\circ \end{matrix}$$

corresponding to the reduced factorizations $s_1\cdot 1\cdot s_2s_1\cdot s_\square$, $s_1\cdot s_2\cdot s_1\cdot s_\square$, $s_2s_1\cdot 1\cdot s_2\cdot s_\square$, and $s_2s_1\cdot s_2\cdot 1\cdot s_\square$ of w_{λ°, respectively. Setting $Z_2:=(z_1,z_2)$, we deduce that

$$E_{w_\lambda}(Z_2)=E_{w_{\lambda^\circ}}(Z_2)=z_1^3z_2+2z_1^2z_2^2+z_1z_2^3.$$

The reader may compare this with [T3, Example 2].

Let $\mathbf{R}$ denote the ordered alphabet

$$(n-1)'<\cdots 2'<1'<\overline{1},\overline{1}^\circ<1,1^\circ<\overline{2},\overline{2}^\circ<2,2^\circ<\cdots<1''<2''<\cdots<(n-1)''$$

which extends the alphabet $\breve{\mathbf{R}}$.

Definition 6. Consider a pair U of sequences of typed k-strict partitions

$$\mu=\lambda^0\subset\lambda^1\subset\cdots\subset\lambda^{2n+2r-2}=\lambda \quad\text{and}\quad \mu=\nu^0,\nu^1,\ldots,\nu^{2n+2r-2}=\lambda$$

with the first one compatible, such that (i) $\mu\subset\nu^i\subset\lambda^i$, (ν^i,μ) is a compatible pair, and λ^i/ν^i is a k-rook strip for each i, and (ii) λ^i/ν^{i-1} is a typed x-strip for each $i\leq n-1$ (non-extremal if $i\leq n-2$), a typed y-strip for each $i\geq n+2r$ (non-extremal if $i\geq n+2r+1$), and a typed $\overline{z}$-strip (resp. typed z-strip) if $i=n-1+j$ for some odd (resp. even) $j\in[1,2r]$. We represent U by a filling of the boxes in λ/μ by including the entry $(n-i)'$ in each box of λ^i/ν^{i-1} for $1\leq i\leq n-1$, the entry $(i-n-2r+1)''$ in each box of λ^i/ν^{i-1} for $n+2r\leq i\leq 2n+2r-2$, and the entry $\overline{j}$ or $\overline{j}^\circ$ (resp. j or j°) in each box of $\lambda^{n-1+j}/\nu^{n-2+j}$ for all odd (resp. even) integers $j\in[1,2r]$, so that the circled entries j° and $\overline{j}^\circ$ are used if and only if the ambient partition has type 2, as in Definition 5. We say that U is a *set-valued typed k'-tritableau* of shape λ/μ if for $1\leq i\leq \ell_k(\mu)$ (resp. $1\leq i\leq\ell_k(\lambda)$) and $1\leq j\leq k$, the entries of U in row i are $\geq(\mu_i-k+1)'$ (resp. $\leq(\lambda_i-k)''$) and the entries in column $k+1-j$ lie in the interval $[|w_\mu(j)|',|w_\lambda(j)-1|'']$. We let $|U|$ be the total number of symbols of $\mathbf{R}$ which appear in U, including their multiplicities, and define

$$(xyz)^U:=\prod_i x_i^{n'_i}\prod_i y_i^{n''_i}\prod_j z_j^{n_j}$$

where n'_i and n''_i denote the number of times that i' and i'' appear in U, respectively, for each $i\in[1,n-1]$, and n_j denotes the number of times that j, j°, $\overline{j}$, or $\overline{j}^\circ$ appears in U, for each $j\in\mathbb{N}_\square$.

Theorem 5. *For the skew element $w:=w_{\lambda/\mu}$ of $\widetilde{W}_n$, we have*

$$\mathfrak{G}^D_w(Z;X,Y)=\sum_U\beta^{|U|-|\lambda/\mu|}\,(xyz)^U \tag{13}$$

summed over all set-valued typed k'-tritableaux U of shape λ/μ.

Proof. We deduce from formula (4) that $\mathfrak{G}^D_w(Z;X,Y)$ is equal to

$$\sum\beta^{\ell(u,v,\sigma,w)}y_{n-1}^{\ell(v_{n-1})}\cdots y_1^{\ell(v_1)}x_1^{\ell(u_1)}\cdots x_{n-1}^{\ell(u_{n-1})}\prod_j z_j^{\ell(\sigma_{2j-1})+\ell(\sigma_{2j})}$$

where the sum is over all Hecke factorizations

$$v_{n-1}\circ\cdots\circ v_1\circ\sigma_{2r}\circ\cdots\circ\sigma_1\circ u_1\circ\cdots\circ u_{n-1}$$

of w, for varying $r\geq 0$, such that $v_p\in S_n$ is increasing up from p and $u_p\in S_n$ is decreasing down to p for each p, σ_i is increasing up from $\square$ (resp. decreasing down to 1) for all odd i (resp. all even i), and $\ell(u,v,\sigma,w):=\sum_p\ell(u_p)+\sum_p\ell(v_p)+$

$\sum_i \ell(\sigma_i) - \ell(w)$. Propositions 9 and 10 imply that such factorizations correspond to pairs of sequences of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{2n+2r-2} = \lambda \quad \text{and} \quad \mu = \nu^0, \nu^1, \ldots, \nu^{2n+2r-2} = \lambda$$

and an associated filling U of the the boxes in λ/μ as in Definition 6. The required bounds on the entries of U are established as in the proof of [T6, Thm. 3], showing that U is a set-valued typed k'-tritableau of shape λ/μ such that

$$(xyz)^U = y_{n-1}^{\ell(v_{n-1})} \cdots y_1^{\ell(v_1)} x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})} \prod_j z_j^{\ell(\sigma_{2j-1})+\ell(\sigma_{2j})}.$$

Conversely, the typed set-valued k'-tritableaux of shape λ/μ correspond to Hecke factorizations of w of the required form, and equation (13) follows. $\square$

Example 15. Let $k := 1$ and $\lambda := (4,2)$ so that $w_\lambda = 1\,\overline{4}\,\overline{2}\,3 = s_1 s_3 s_2 s_\square s_2 s_1$, and consider the alphabet $\mathbf{R}_{ex} := \{1' < \overline{1}, \overline{1}^\circ < 1, 1^\circ < \overline{2}, \overline{2}^\circ < 2, 2^\circ < 1'' < 2''\}$. The set-valued typed $1'$-tritableau

$$\begin{matrix} 1' & \{\overline{1},\overline{2}\} & 2 & \{2,1'',2''\} \\ \{1',\overline{1}\} & 1'' & & \end{matrix} \quad \text{and} \quad \begin{matrix} 1' & \{\overline{1},1\} & 1 & \{1,\overline{2},2\} \\ \overline{2} & \{2,1''\} & & \end{matrix}$$

of shape λ with entries in $\mathbf{R}_{ex}$ correspond to the Hecke factorizations

$$s_3 \circ (s_1 s_3) \circ (s_3 s_2) \circ s_\square \circ 1 \circ (s_\square s_2) \circ (s_2 s_1) \quad \text{and} \quad s_1 \circ (s_3 s_1) \circ (s_\square s_3) \circ (s_3 s_2 s_1) \circ s_\square \circ s_1$$

of w_λ and map to the monomials $x_1^2 z_1^2 z_2^3 y_1^2 y_2$ and $x_1 z_1^4 z_2^4 y_1$, respectively.

Example 16. Using Theorem 5, the type D Grothendieck polynomials for the elements s_r of length one in $\widetilde{W}_\infty$ are computed as follows. If $r \geq 2$, then we have

$$\mathfrak{G}^D_{s_r}(Z;X,Y) = \mathfrak{G}^C_{s_r}(Z;X,Y) = \mathfrak{G}^B_{s_r}(Z;X,Y).$$

For $r = 1$, we have

$$\mathfrak{G}^D_{s_1}(Z;X,Y) = \sum_{H'} \beta^{|H'|-1} x_1^{e_1} y_1^{e_2} \prod_j z_j^{f_j}$$

summed over all sequences $H' = (e_1, e_2, f_1, f_2, f_3, \ldots)$ with finite support such that $e_i, f_j \in \{0,1\}$ for each i,j and $|H'| := e_1 + e_2 + \sum_j f_j > 0$. Finally, for $r = \square$ we have

$$\mathfrak{G}^D_{s_\square}(Z;X,Y) = \sum_{D'} \beta^{|D'|-1} \prod_j z_j^{f_j}$$

summed over all sequences $D' = (f_1, f_2, \ldots)$ with finite support such that $f_j \in \{0,1\}$ for each j and $|D'| := \sum_j f_j > 0$.

Example 17. Let $w \in W_\infty$ (respectively $w \in \widetilde{W}_\infty$) and define the *$K$-theoretic double mixed Stanley functions* J^B_w, J^C_w, and (respectively) I^D_w by the equations

$$\begin{aligned} J^B_w(Z;X,Y) &:= \langle A'(Y)B(Z)A(X), w\rangle\,, \\ J^C_w(Z;X,Y) &:= \langle A'(Y)C(Z)A(X), w\rangle\,, \quad \text{and} \\ I^D_w(Z;X,Y) &:= \langle A'(Y)D(Z)A(X), w\rangle\,. \end{aligned}$$

These functions are analogues in connective K-theory of the double mixed Stanley functions $J_w(Z;X,Y)$ and $I_w(Z;X,Y)$ studied in [T2, Example 3] and [T4, Sections 3.5 and 4.6]. Now suppose that $w := w_{\lambda/\mu}$ is a skew element of W_∞ or $\widetilde{W}_\infty$. Extend the alphabets $\mathbf{Q}$ and $\mathbf{R}$ to include all primed and double primed positive integers, and modify Definition 4 and Definition 6 by omitting the bounds on the entries of the tritableaux found in both, and the non-extremal condition in the latter. Then the right hand sides of equations (9) and (13) give tableau formulas for the type C and type D functions $J^C_w(Z;X,Y)$ and $I^D_w(Z;X,Y)$, respectively.

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