TABLEAU FORMULAS FOR SKEW SCHUBERT POLYNOMIALS

HARRY TAMVAKIS

Abstract. The skew Schubert polynomials are those which are indexed by skew elements of the Weyl group, in the sense of [T1]. We obtain tableau formulas for the double versions of these polynomials in all four classical Lie types, where the tableaux used are fillings of the associated skew Young diagram. These are the first such theorems for symplectic and orthogonal Schubert polynomials, even in the single case. We also deduce tableau formulas for double Schur, double theta, and double eta polynomials, in their specializations as double Grassmannian Schubert polynomials. The latter results generalize the tableau formulas for symmetric (and single) Schubert polynomials due to Littlewood (in type A) and the author (in types B, C, and D).

0. Introduction

The double Schubert polynomials of Lascoux-Schützenberger [LS, L2] and Ikeda-Mihalcea-Naruse [IMN] represent the (stable) Schubert classes in the equivariant cohomology ring of complete flag manifolds, in each of the four classical Lie types. When the indexing Weyl group element is *skew*, in the sense of [T1, T3], we call these polynomials – which are really formal power series in types B, C, and D – *skew Schubert polynomials*. The aim here is to prove tableau formulas for the skew Schubert polynomials, in a type uniform manner. By definition, each skew signed permutation is associated with a pair of (typed) partitions $\lambda \supset \mu$, and our formulas are sums over tableaux which are fillings of the boxes in the skew diagram λ/μ.

The skew elements of the symmetric group coincide with the 321-avoiding or fully commutative permutations [BJS, S]. Their Schubert polynomials were named and studied by Lascoux and Chen-Yan-Yang [CYY], following the work of Wachs [W] and Billey-Jockusch-Stanley [BJS] in the single case. The results here are new even in type A, and provide an alternative to the formulas in [CYY], which extends readily to the symplectic and orthogonal Lie types.

Our theorems specialize to give the first tableau formulas for the single Schubert polynomials of Billey-Haiman [BH] indexed by skew signed permutations. The well known single skew Schur S-, P-, and Q-functions are not skew Schubert polynomials, and it has been a longstanding open problem to formulate a theory of such polynomials in types B, C, and D, even in the fully commutative case. We remark that the (type A and single) skew Schubert polynomials of Lenart-Sottile [LeS] are different from the ones found in [BJS, CYY] and the present paper.

The double *Grassmannian Schubert polynomials* are the images of the double Schur, double theta, and double eta polynomials of [KL, L1, TW, T5] in the ring of double Schubert polynomials of [LS, IMN] of the corresponding Lie type. Since the Grassmannian elements are the most important examples of skew elements, our

Date: March 21, 2023.
2010 *Mathematics Subject Classification.* Primary 05E05; Secondary 05E14, 14N15.

results also specialize to obtain formulas for Grassmannian Schubert polynomials. These expressions in turn generalize the tableau formulas for single Schur, theta, and eta polynomials found in [Li] and [T1, T3], respectively. The latter objects are the *symmetric Schubert polynomials*, where the symmetry means invariance under the action of the respective Weyl group, as explained in [T6].

The simple proof – which is new, and uniform across the four types – stems from the raising operator approach to tableau formulas and the ensuing theory of skew elements of the Weyl group, pioneered in [T1, T3]. We also employ the definition of double Schubert polynomials via the nilCoxeter algebra found in [T2], which originates in the work of Fomin-Stanley-Kirillov-Lam [FS, FK, La]. These ingredients combine in a harmonious way to yield straightforward arguments. In a sequel to this paper, we illustrate the power of these methods further by extending our results to *skew Grothendieck polynomials*.

This article is organized as follows. Section 1 contains preliminary material on the relevant Weyl groups and double Schubert polynomials. The following Sections 2, 3, and 4 deal in a parallel manner with tableau formulas for skew Schubert polynomials in the Lie types A, C, and D, respectively.

I thank the anonymous referee for a careful reading of the paper and detailed suggestions which helped to improve the exposition.

1. Weyl groups and Schubert polynomials

This section gathers background information on the double Schubert polynomials for the classical Lie groups due to Lascoux and Schützenberger [LS, L2] (in type A) and Ikeda, Mihalcea, and Naruse [IMN] (in types B, C, and D). We require the definition of these formal power series using the nilCoxeter algebra of the Weyl group, which originates in [FS, FK, La], and was used in [T1, T2, T3]. The precise way in which the Schubert polynomials studied here represent the stable equivariant Schubert classes on complete flag manifolds is explained in [IMN]; see also [T2, T4].

The Weyl group for the root system of type B_n or C_n is the *hyperoctahedral group* W_n, which consists of signed permutations on the set $\{1,\ldots,n\}$. The group W_n is generated by the transpositions $s_i=(i,i+1)$ for $1\leq i\leq n-1$ and the sign change $s_0(1)=\overline{1}$ (as is customary, we set $\overline{a}:=-a$ for any $a\geq 1$). The elements of W_n are written as n-tuples $(w_1,\ldots,w_n)$, where $w_i:=w(i)$ for each $i\in[1,n]$.

There is a natural embedding $W_n\hookrightarrow W_{n+1}$ defined by adding the fixed point $n+1$, and we let $W_\infty:=\cup_n W_n$. The *length* of an element $w\in W_\infty$, denoted $\ell(w)$, is the least integer r such that we have an expression $w=s_{a_1}\cdots s_{a_r}$. The word $a_1\cdots a_r$ is called a *reduced word* for w. The symmetric group S_n is the subgroup of W_n generated by $s_1,\ldots,s_{n-1}$, and we let $S_\infty:=\cup_n S_n$.

The nilCoxeter algebra $\mathbb{W}_n$ of W_n is the free unital associative algebra generated by the elements $\xi_0,\xi_1,\ldots,\xi_{n-1}$ modulo the relations

$$\begin{array}{rcll}
\xi_i^2 & = & 0 & i\geq 0\ ; \\
\xi_i\xi_j & = & \xi_j\xi_i & |i-j|\geq 2\ ; \\
\xi_i\xi_{i+1}\xi_i & = & \xi_{i+1}\xi_i\xi_{i+1} & i>0\ ; \\
\xi_0\xi_1\xi_0\xi_1 & = & \xi_1\xi_0\xi_1\xi_0. &
\end{array}$$

For every $w\in W_n$, define $\xi_w:=\xi_{a_1}\ldots\xi_{a_r}$, where $a_1\cdots a_r$ is any reduced word for w. The elements ξ_w for $w\in W_n$ form a free $\mathbb{Z}$-basis of $\mathbb{W}_n$. We denote the coefficient of $\xi_w\in\mathbb{W}_n$ in the expansion of the element $\alpha\in\mathbb{W}_n$ by $\langle\alpha,w\rangle$.

Let t be an indeterminate and define

$$\begin{aligned} A_i(t) &:= (1+t\xi_{n-1})(1+t\xi_{n-2})\cdots(1+t\xi_i)\ ; \\ \tilde{A}_i(t) &:= (1-t\xi_i)(1-t\xi_{i+1})\cdots(1-t\xi_{n-1})\ ; \\ C(t) &:= (1+t\xi_{n-1})\cdots(1+t\xi_1)(1+t\xi_0)(1+t\xi_0)(1+t\xi_1)\cdots(1+t\xi_{n-1}). \end{aligned}$$

Suppose that $X = (x_1, x_2, \ldots)$, $Y = (y_1, y_2, \ldots)$, and $Z = (z_1, z_2, \ldots)$ are three infinite sequences of commuting independent variables. For any $\varpi \in S_n$, the type A Schubert polynomial $\mathfrak{S}_\varpi$ is given by

$$\mathfrak{S}_\varpi(X,Y) := \left\langle \tilde{A}_{n-1}(y_{n-1})\cdots\tilde{A}_1(y_1)A_1(x_1)\cdots A_{n-1}(x_{n-1}), \varpi \right\rangle . \tag{1}$$

Let $C(Z) := C(z_1)C(z_2)\cdots$, and for $w \in W_n$, define the type C Schubert polynomial $\mathfrak{C}_w$ – which is a formal power series in the Z variables – by

$$\mathfrak{C}_w(Z;X,Y) := \left\langle \tilde{A}_{n-1}(y_{n-1})\cdots\tilde{A}_1(y_1)C(Z)A_1(x_1)\cdots A_{n-1}(x_{n-1}), w \right\rangle . \tag{2}$$

The type C Stanley function F_w of [BH, FK, La] is given by $F_w(Z) := \langle C(Z), w\rangle$. The polynomial $\mathfrak{C}_w$ is stable under the inclusion of W_n in W_{n+1}; it follows that $\mathfrak{C}_w$ and $\mathfrak{S}_\varpi$ are well defined for $w \in W_\infty$ and $\varpi \in S_\infty$, respectively.

For any $w \in W_\infty$, the type B Schubert polynomial $\mathfrak{B}_w$ satisfies $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$, where $s(w)$ denotes the number of indices i such that $w_i < 0$. We therefore omit any further discussion of type B, and will focus on the even orthogonal type D.

The Weyl group $\widetilde{W}_n$ for the root system D_n is the subgroup of W_n consisting of all signed permutations with an even number of sign changes. The group $\widetilde{W}_n$ is an extension of S_n by $s_\Box := s_0s_1s_0$, an element which acts on the right by

$$(w_1, w_2, \ldots, w_n)s_\Box = (\overline{w}_2, \overline{w}_1, w_3, \ldots, w_n).$$

There is a natural embedding $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$ of Weyl groups defined by adjoining the fixed point $n+1$, and we let $\widetilde{W}_\infty := \cup_n \widetilde{W}_n$. The simple reflections in $\widetilde{W}_\infty$ are indexed by the members of the set $\mathbb{N}_\Box := \{\Box, 1, 2, \ldots\}$, and are used to define the length and reduced words of elements in $\widetilde{W}_\infty$ as above.

The nilCoxeter algebra $\widetilde{\mathbb{W}}_n$ of the group $\widetilde{W}_n$ is the free unital associative algebra generated by the elements $\xi_\Box, \xi_1, \ldots, \xi_{n-1}$ modulo the relations

$$\begin{array}{rcll} \xi_i^2 &=& 0 & i \in \mathbb{N}_\Box\ ; \\ \xi_\Box\xi_1 &=& \xi_1\xi_\Box & \\ \xi_\Box\xi_2\xi_\Box &=& \xi_2\xi_\Box\xi_2 & \\ \xi_i\xi_{i+1}\xi_i &=& \xi_{i+1}\xi_i\xi_{i+1} & i > 0\ ; \\ \xi_i\xi_j &=& \xi_j\xi_i & j > i+1, \text{ and } (i,j) \neq (\Box, 2). \end{array}$$

For any element $w \in \widetilde{W}_n$, choose a reduced word $a_1 \cdots a_r$ for w, and define $\xi_w := \xi_{a_1} \ldots \xi_{a_r}$. As before, denote the coefficient of $\xi_w \in \widetilde{\mathbb{W}}_n$ in the expansion of the element $\alpha \in \widetilde{\mathbb{W}}_n$ in the ξ_w basis by $\langle \alpha, w\rangle$. Following Lam [La], define

$$D(t) := (1+t\xi_{n-1})\cdots(1+t\xi_2)(1+t\xi_1)(1+t\xi_\Box)(1+t\xi_2)\cdots(1+t\xi_{n-1}).$$

Let $D(Z) := D(z_1)D(z_2)\cdots$, and for $w \in \widetilde{W}_n$, define the type D Schubert polynomial $\mathfrak{D}_w$ by

$$\mathfrak{D}_w(Z;X,Y) := \left\langle \tilde{A}_{n-1}(y_{n-1})\cdots\tilde{A}_1(y_1)D(Z)A_1(x_1)\cdots A_{n-1}(x_{n-1}), w \right\rangle . \tag{3}$$

The type D Stanley function E_w of [BH, La] is defined by $E_w(Z) := \langle D(Z), w\rangle$. The Schubert polynomial $\mathfrak{D}_w(Z;X,Y)$ is stable under the natural inclusions $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$, and hence is well defined for $w \in \widetilde{W}_\infty$.

Given any Weyl group elements $u_1,\ldots,u_r,w$, we say that the product $u_1\cdots u_r$ is a *reduced factorization* of w if $u_1\cdots u_r = w$ and $\ell(u_1)+\cdots+\ell(u_r) = \ell(w)$.

2. Tableau formula for type A skew Schubert polynomials

2.1. **Grassmannian permutations and partitions.** We recall here some standard definitions and notation. A *partition* $\lambda = (\lambda_1,\lambda_2,\ldots)$ is a weakly decreasing sequence of nonnegative integers with finite support. The length of λ is the number of non-zero parts λ_i. We identify a partition λ with its Young diagram of boxes, which are arranged in left justified rows, with λ_i boxes in the i-th row for each $i \geq 1$. An inclusion $\mu \subset \lambda$ of partitions corresponds to the containment relation of their respective diagrams; in this case, the skew diagram λ/μ is the set-theoretic difference $\lambda \smallsetminus \mu$. A skew diagram is called a *horizontal strip* (respectively, *vertical strip*) if it does not contain two boxes in the same column (respectively, row).

Fix an integer $m \geq 1$. An element $\varpi \in S_\infty$ is m-Grassmannian if $\ell(\varpi s_i) > \ell(\varpi)$ for all $i \neq m$. This is equivalent to the conditions

$$\varpi_1 < \cdots < \varpi_m \quad \text{and} \quad \varpi_{m+1} < \varpi_{m+2} < \cdots .$$

Every m-Grassmannian permutation $\varpi \in S_\infty$ corresponds to a unique partition λ of length at most m, called the *shape* of ϖ. When the shape λ and m are given, we denote $\varpi = \varpi(\lambda,m)$ by ϖ_λ, and have $\varpi_\lambda(i) = \lambda_{m+1-i}+i$ for $1 \leq i \leq m$.

2.2. **Skew permutations and main theorem.** A permutation $\varpi \in S_\infty$ is called *skew* if there exists an m-Grassmannian permutation ϖ_λ (for some m) and a reduced factorization $\varpi_\lambda = \varpi\varpi'$ in S_∞. In this case, the right factor ϖ' equals ϖ_μ for some m-Grassmannian permutation ϖ_μ, and we have $\mu \subset \lambda$. We say that (λ,μ) is a *compatible pair* and that ϖ is associated to the pair (λ,μ). There is a 1-1 correspondence between reduced factorizations uv of ϖ and partitions ν with $\mu \subset \nu \subset \lambda$ (this is a special case of [T1, Cor. 8]). It follows from [S, Thm. 4.2] that the skew permutations coincide with the fully commutative elements of S_∞.

Definition 1. We say that a permutation ϖ is *decreasing down to* p if ϖ has a reduced word $a_1\cdots a_r$ such that $a_1 > \cdots > a_r \geq p$. We say that ϖ is *increasing up from* p if ϖ has a reduced word $a_1\cdots a_r$ such that $p \leq a_1 < \cdots < a_r$.

If a permutation ϖ has a reduced word $a_1\cdots a_r$ such that $a_1 > \cdots > a_r$, then a_1 is the largest integer i such that $\ell(s_i\varpi) < \ell(\varpi)$. It follows by induction on $\ell(\varpi)$ that if ϖ is decreasing down to p or increasing up from p, then the decreasing (respectively, increasing) word $a_1\cdots a_r$ for ϖ in Definition 1 is uniquely determined.

Observe that an m-Grassmannian element $s_i\varpi \in S_n$ satisfies $\ell(s_i\varpi) > \ell(\varpi)$ if and only if $\varpi = (\cdots i \cdots | \cdots i+1 \cdots)$, where the vertical line $|$ lies between ϖ_m and ϖ_{m+1}. Using this and the relation between λ and ϖ_λ explained above, one sees that a skew permutation $\varpi = \varpi_\lambda\varpi_\mu^{-1}$ is decreasing down to 1 (respectively, increasing up from 1) if and only if λ/μ is a horizontal (respectively, vertical) strip.

Let λ and μ be any two partitions of length at most m with $\mu \subset \lambda$, and choose $n \geq 1$ such that $\varpi_\lambda \in S_n$. Let $\mathbf{P}$ denote the ordered alphabet

$$(n-1)' < \cdots < 1' < 1 < \cdots < n-1.$$

The symbols $(n-1)', \ldots, 1'$ are said to be *marked*, while the rest are *unmarked*.

Definition 2. An *m-bitableau* U of shape λ/μ is a filling of the boxes in λ/μ with elements of $\mathbf{P}$ which is weakly increasing along each row and down each column, such that (i) the marked (respectively, unmarked) entries are strictly increasing each down each column (respectively, along each row), and (ii) the entries in row i lie in the interval $[(\mu_i+m+1-i)', \lambda_i+m-i]$ for each $i \in [1,m]$. We define

$$(xy)^U := \prod_i x_i^{n'_i} \prod_i (-y_i)^{n_i}$$

where n'_i (respectively, n_i) denotes the number of times that i' (respectively, i) appears in U.

Theorem 1. *For the skew permutation $\varpi := \varpi_\lambda \varpi_\mu^{-1}$, we have*

$$\mathfrak{S}_\varpi(X,Y) = \sum_U (xy)^U \tag{4}$$

summed over all m-bitableaux U of shape λ/μ.

Proof. It follows from formula (1) and the remark after Definition 1 that

$$\mathfrak{S}_\varpi(X,Y) = \sum_{v_{n-1}\cdots v_1 u_1 \cdots u_{n-1} = \varpi} (-y_{n-1})^{\ell(v_{n-1})} \cdots (-y_1)^{\ell(v_1)} x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})} \tag{5}$$

where the sum is over all reduced factorizations $v_{n-1}\cdots v_1 u_1 \cdots u_{n-1}$ of ϖ such that v_p is increasing up from p and u_p is decreasing down to p for each $p \in [1, n-1]$. Since the elements u_p and v_p involved are all skew permutations, such factorizations correspond to sequences of partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{n-1} \subset \lambda^n \subset \cdots \subset \lambda^{2n-2} = \lambda$$

with λ^i/λ^{i-1} a horizontal strip for $1 \le i \le n-1$ and a vertical strip for $n \le i \le 2n-2$, defined by the equations $\varpi_{\lambda^i} = u_{n-i}\varpi_{\lambda^{i-1}}$, for $i \in [1, n-1]$, and $\varpi_{\lambda^i} = v_{i+1-n}\varpi_{\lambda^{i-1}}$, for $i \in [n, 2n-2]$. Note that some factors u_p or v_p in the product $v_{n-1}\cdots v_1 u_1 \cdots u_{n-1}$ may be trivial, and in this case, the associated skew diagram λ^i/λ^{i-1} is empty. We obtain a corresponding filling U of the boxes in λ/μ by placing the entry $(n-i)'$ in each box of λ^i/λ^{i-1} for $1 \le i \le n-1$ and the entry $i+1-n$ in each box of λ^i/λ^{i-1} for $n \le i \le 2n-2$.

Consider the left action of the reflections s_a, for a in the reduced word of $v_{n-1}\cdots v_1 u_1 \cdots u_{n-1}$, on the m-Grassmannian permutations going from ϖ_μ to ϖ_λ. Choose $j \in [1,m]$ and set $e := \varpi_\mu(m+1-j)$ and $f := \varpi_\lambda(m+1-j)$. The leftmost entry h_1 of U in row j of λ/μ was added by the reflection s_e, hence we must have $h_1 \ge e'$. Similarly, the rightmost entry h_2 in row j was added by s_{f-1}, therefore we must have $h_2 \le f-1$. Since the entries of U are clearly weakly increasing along row j, they all lie in the interval $[e', f-1]$. It follows that U is an m-bitableau of shape λ/μ such that $(xy)^U = (-y_{n-1})^{\ell(v_{n-1})} \cdots (-y_1)^{\ell(v_1)} x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})}$. Conversely, the m-bitableaux U of shape λ/μ correspond to reduced factorizations of ϖ as in (5). Since the sum in equation (4) is over all such U, the result follows. □

Example 1. For any $r \ge 1$, we let $X_r := (x_1, \ldots, x_r)$ and $Y_r := (y_1, \ldots, y_r)$. Let ϖ be an m-Grassmannian permutation and λ be the corresponding partition. The Schubert polynomial $\mathfrak{S}_\varpi(X,Y)$ is equal to the *double Schur polynomial* $s_\lambda(X_m, Y)$, while $s_\lambda(X_m) := s_\lambda(X_m,0)$ is the corresponding single Schur polynomial. Equation (4) in this case reads

$$s_\lambda(X_m,Y) = \sum_U (xy)^U \tag{6}$$

summed over all fillings U of the boxes in λ with elements of $\mathbf{P}$ which are weakly increasing along each row and down each column, such that the marked (respectively, unmarked) entries are strictly increasing each down each column (respectively, along each row), and the entries in row i lie in the interval $[m', \lambda_i+m-i]$ for $1\leq i\leq m$. The reader may compare (6) with the similar result in [M, Prop. 4.1].

Equation (6) implies the known formula (see e.g. [K, Prop. 4.1])

$$s_\lambda(X_m,Y) = \sum_{\mu\subset\lambda} s_\mu(X_m)\det\left(e_{\lambda_i-\mu_j-i+j}(-Y_{\lambda_i+m-i})\right)_{1\leq i,j\leq m},$$

where $e_p(-Y_r)$ denotes the p-th elementary symmetric polynomial in $-y_1,\ldots,-y_r$. Indeed, the marked entries in each m-bitableau U on λ form a filling of a diagram μ contained in λ. The map $i'\mapsto m+1-i$ shows that these fillings are in bijection with semistandard Young tableaux T of shape μ with entries in $[1,m]$. For each fixed partition $\mu\subset\lambda$, the corresponding monomials x^T sum to give $s_\mu(X_m)$, a polynomial which is *symmetric* in the variables X_m. The rest follows from the determinantal formula for flagged skew Schur functions given in [W, Thm. 3.5*].

3. Tableau formula for type C skew Schubert polynomials

3.1. **Grassmannian elements and k-strict partitions.** The main references for this subsection are [BKT1, BKT2, T1]. Fix a nonnegative integer k. An element $w\in W_\infty$ is k-Grassmannian if $\ell(ws_i)>\ell(w)$ for all $i\neq k$. This is equivalent to the conditions

$$0<w_1<\cdots<w_k \quad\text{and}\quad w_{k+1}<w_{k+2}<\cdots.$$

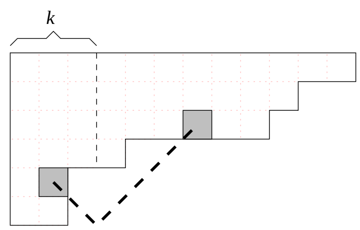
A partition λ is said to be *k-strict* if no part greater than k is repeated. The number of parts λ_i which are greater than k is denoted by $\ell_k(\lambda)$.

Each k-Grassmannian element w of W_∞ corresponds to a unique k-strict partition λ, called the *shape* of w. If the shape λ is given, then we denote the corresponding element $w=w(\lambda,k)$ by w_λ. To describe this bijection, let $\gamma_1\leq\cdots\leq\gamma_k$ be the lengths of the first k columns of λ, listed in increasing order. The sequence $(\gamma_1,\ldots,\gamma_k)$ is the *A-code* of w_λ, following [T6, Def. 2]. We then have

$$w_\lambda(j) = \gamma_j + j - \#\{p\in[1,\ell_k(\lambda)] \,:\, \lambda_p+p>\gamma_j+j+k\}$$

for $1\leq j\leq k$, while the equalities $w_\lambda(k+i)=k-\lambda_i$ for $1\leq i\leq\ell_k(\lambda)$ specify the negative entries of w_λ. For example, the 3-strict partition $\lambda=(8,4,2,1)$ satisfies $\ell_k(\lambda)=2$ and $(\gamma_1,\gamma_2,\gamma_3)=(2,3,4)$, therefore $w_\lambda=(2,4,7,\overline{5},\overline{1},3,6)$. The correspondence between w and its shape λ can be visualized using the notion of related and non-related diagonals; see [BKT2, Sec. 6.1].

We denote the box in row r and column c of a Young diagram by $[r,c]$. For any partition λ, we define $\lambda_0:=\infty$ and agree that the diagram of λ includes all boxes $[0,c]$ in row zero. The *rim* of λ is the set of boxes $[r,c]$ of its Young diagram such that box $[r+1,c+1]$ lies outside of the diagram of λ. We say that the boxes $[r,c]$ and $[r',c']$ are *k'-related* if $|c-k-\frac{1}{2}|+r=|c'-k-\frac{1}{2}|+r'$. For instance, the two grey boxes in the figure below are k'-related. We call the box $[r,c]$ a *left box* if $c\leq k$ and a *right box* if $c>k$.



Following [T1, Sec. 5.3], if $\mu \subset \lambda$ are two k-strict partitions, we let R (respectively, $\mathbb{A}$) denote the set of right boxes of μ (including boxes in row zero) which are bottom boxes of λ in their column and are (respectively, are not) k'-related to a left box of λ/μ. The pair $\mu \subset \lambda$ forms a *k-horizontal strip* λ/μ if (i) λ/μ is contained in the rim of λ, and the right boxes of λ/μ form a horizontal strip; (ii) no two boxes in R are k'-related; and (iii) if two boxes of λ/μ lie in the same column, then they are k'-related to exactly two boxes of R, which both lie in the same row.

Note that a k-horizontal strip λ/μ is a pair of partitions (λ, μ), and so, depends on λ and μ and not only on the difference $\lambda \smallsetminus \mu$. We say that two boxes in $\mathbb{A}$ are connected if they share a vertex or an edge, and let $n(\lambda/\mu)$ denote the number of connected components of $\mathbb{A}$ which do not have a box in column $k+1$.

3.2. **Skew elements and main theorem.** Following [T1, Sec. 6.3], an element $w \in W_\infty$ is called *skew* if there exists a k-Grassmannian element w_λ (for some k) and a reduced factorization $w_\lambda = ww'$ in W_∞. In this case, the right factor w' equals w_μ for some k-strict partition μ with $\mu \subset \lambda$. We say that (λ, μ) is a *compatible pair* and that w is associated to the pair (λ, μ). According to [T1, Cor. 8], there is a bijection between reduced factorizations uv of $w_\lambda w_\mu^{-1}$ and k-strict partitions ν with $\mu \subset \nu \subset \lambda$ such that (λ, ν) and (ν, μ) are compatible pairs. Moreover, any k-horizontal strip λ/μ is a compatible pair (λ, μ) of k-strict partitions.

Remark 1. The integer k, the compatible pair (λ, μ), and the skew shape λ/μ associated to a skew element $w \in W_\infty$ are not uniquely determined by w. For example, the 2-Grassmannian element $s_1 s_2 \in W_3$ is also a skew element when $k = 1$, and in the latter capacity is associated to both of the compatible pairs $((4,1),3)$ and $((4,3),(3,2))$.

An element of W_∞ is called *unimodal* if it has a reduced word $a_1 \cdots a_r$ such that for some $q \in [0, r]$, we have $a_1 > a_2 > \cdots > a_q < a_{q+1} < \cdots < a_r$. Let λ and μ be any two k-strict partitions such that (λ, μ) is a compatible pair, choose an integer $n \geq 1$ such that $w_\lambda \in W_n$, and let $w := w_\lambda w_\mu^{-1}$ be the corresponding skew element of W_n. It was shown in [T1, Prop. 5] that w is unimodal if and only if λ/μ is a k-horizontal strip; we note that this can also be checked directly.

Definition 3. Suppose that $w = w_\lambda w_\mu^{-1}$ lies in S_n. If w is decreasing down to 1 (respectively, increasing up from 1), then we say that the k-horizontal strip λ/μ is an *x-strip* (respectively, *y-strip*).

The x- and y-strips are characterized among all k-horizontal strips as follows.

Proposition 1. *A k-horizontal strip λ/μ is an x-strip (respectively, y-strip) if and only if $\ell_k(\lambda) = \ell_k(\mu)$, the left boxes in λ/μ form a vertical strip (respectively, horizontal strip), and no two boxes in λ/μ are k'-related (respectively, no two right boxes in λ/μ are in the same row).*

Proof. Let $w := w_\lambda w_\mu^{-1}$ be the skew element of W_n associated to λ/μ. Then clearly $w \in S_n$ if and only if $s(w_\lambda) = s(w_\mu)$ if and only if $\ell_k(\lambda) = \ell_k(\mu)$. An element of S_n is decreasing down to 1 (respectively, increasing up from 1) if and only if it has no reduced word which contains $i-1, i$ (respectively, $i, i-1$) as a subword for some $i \geq 2$. Notice that in any reduced factorization $w = us_{i-1}s_iv$ (respectively, $w = us_is_{i-1}v$), the elements u and v are also skew, and associated to k-horizontal strips which are substrips of λ/μ. Therefore, by induction on the lengths of u and v, we may assume that $w \in \{s_{i-1}s_i, s_is_{i-1}\}$ for some i and study the associated skew diagram λ/μ. For $i \geq 1$, a k-Grassmannian element $s_iv \in W_n$ satisfies $\ell(s_iv) > \ell(v)$ if and only if v has one of the following three forms:

$$(\cdots i \cdots | \cdots i+1 \cdots), \quad (\cdots i+1 \cdots | \cdots \overline{i} \cdots), \quad (\cdots | \cdots \overline{i} \cdots i+1 \cdots)$$

where the vertical line $|$ lies between v_k and v_{k+1}. We deduce that if $w = s_{i-1}s_i$ for some $i \geq 2$, then w_μ and w_λ have the form

$$w_\mu = (\cdots i-1,\, i \cdots | \cdots i+1 \cdots), \;\; w_\lambda = (\cdots i,\, i+1 \cdots | \cdots i-1 \cdots)$$

so that λ/μ has two left boxes in the same row, or

$$w_\mu = (\cdots i+1 \cdots | \cdots \overline{i},\, \overline{i-1} \cdots), \;\; w_\lambda = (\cdots i-1 \cdots | \cdots \overline{i+1},\, \overline{i} \cdots)$$

so that λ/μ has two right boxes which are k'-related, or

$$w_\mu = (\cdots i \cdots | \cdots \overline{i-1},\, i+1 \cdots), \;\; w_\lambda = (\cdots i+1 \cdots | \cdots \overline{i},\, i-1 \cdots)$$

in which case λ/μ has a left box and a right box which are k'-related. On the other hand, if $w = s_is_{i-1}$, then we must have

$$w_\mu = (\cdots i-1 \cdots | \cdots i,\, i+1 \cdots), \;\; w_\lambda = (\cdots i+1 \cdots | \cdots i-1,\, i \cdots)$$

so that λ/μ has two left boxes in the same column, or

$$w_\mu = (\cdots | \cdots \overline{i-1} \cdots), \;\; w_\lambda = (\cdots | \cdots \overline{i+1} \cdots)$$

so λ/μ has two right boxes in the same row. Finally, the converse assertions are proved by using the correspondence between ν and w_ν given in Section 3.1. □

A *k-tableau* T of shape λ/μ is a sequence of k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^p = \lambda$$

such that λ^i/λ^{i-1} is a k-horizontal strip for $1 \leq i \leq p$. We represent T by a filling of the boxes in λ/μ with positive integers such that for each i, the boxes in T with entry i form the skew diagram λ^i/λ^{i-1}. For any k-tableau T we define $n(T) := \sum_i n(\lambda^i/\lambda^{i-1})$ and set $z^T := \prod_i z_i^{m_i}$, where m_i denotes the number of times that i appears in T. According to [T1, Thm. 6], the type C Stanley function $F_w(Z)$ of the skew signed permutation $w = w_\lambda w_\mu^{-1}$ satisfies the equation

$$F_w(Z) = \sum_T 2^{n(T)} z^T \tag{7}$$

summed over all k-tableaux T of shape λ/μ.

Let $\mathbf{Q}$ denote the ordered alphabet

$$(n-1)' < \cdots 2' < 1' < 1 < 2 < 3 < \cdots < 1'' < 2'' < \cdots < (n-1)''.$$

The single and double primed symbols in $\mathbf{Q}$ are said to be *marked*, while the rest are *unmarked*.

Definition 4. A *k-tritableau* U of shape λ/μ is a filling of the boxes in λ/μ with elements of $\mathbf{Q}$ which is weakly increasing along each row and down each column, such that (i) for each a in $\mathbf{Q}$, the boxes in λ/μ with entry a form a k-horizontal strip, which is an x-strip (respectively, y-strip) if $a \in [(n-1)', 1']$ (respectively, $a \in [1'', (n-1)'']$), and (ii) for $1 \leq i \leq \ell_k(\mu)$ (respectively, $1 \leq i \leq \ell_k(\lambda)$) and $1 \leq j \leq k$, the entries of U in row i are $\geq (\mu_i - k)'$ (respectively, $\leq (\lambda_i - k - 1)''$) and the entries in column $k+1-j$ lie in the interval $[(w_\mu(j))', (w_\lambda(j)-1)'']$. We define

$$n(U) := n(T) \quad \text{and} \quad (xyz)^U := z^T \prod_i x_i^{n'_i} \prod_i (-y_i)^{n''_i}$$

where T is the k-tableau formed by the unmarked entries in U, and n'_i and n''_i denote the number of times that i' and i'' appear in U, respectively.

Theorem 2. *For the skew element $w := w_\lambda w_\mu^{-1}$, we have*

$$\mathfrak{C}_w(Z; X, Y) = \sum_U 2^{n(U)} (xyz)^U \tag{8}$$

summed over all k-tritableaux U of shape λ/μ.

Proof. It is clear from formula (2) that

$$\mathfrak{C}_w(Z; X, Y) = \sum (-y_{n-1})^{\ell(v_{n-1})} \cdots (-y_1)^{\ell(v_1)} F_\sigma(Z) x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})} \tag{9}$$

where the sum is over all reduced factorizations $v_{n-1} \cdots v_1 \sigma u_1 \cdots u_{n-1}$ of w such that $v_p \in S_n$ is increasing up from p and $u_p \in S_n$ is decreasing down to p for each p. Such factorizations correspond to sequences of k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{n-1} \subset \lambda^n \subset \lambda^{n+1} \subset \cdots \subset \lambda^{2n-1} = \lambda$$

with λ^i/λ^{i-1} an x-strip for each $i \leq n-1$, $(\lambda^n, \lambda^{n-1})$ a compatible pair, and λ^i/λ^{i-1} a y-strip for each $i \geq n+1$. The partitions λ^i are determined by the equations $w_{\lambda^i} = u_{n-i} w_{\lambda^{i-1}}$, for $i \in [1, n-1]$, $w_{\lambda^n} = \sigma w_{\lambda^{n-1}}$, and $w_{\lambda^i} = v_{i-n} w_{\lambda^{i-1}}$, for $i \in [n+1, 2n-1]$. Note that some factors in the product $v_{n-1} \cdots v_1 \sigma u_1 \cdots u_{n-1}$ may be trivial, and in this case, the corresponding skew diagram λ^i/λ^{i-1} is empty. We extend each k-tableau T on λ^n/λ^{n-1} to a filling U of the boxes in λ/μ by placing the entry $(n-i)'$ in each box of λ^i/λ^{i-1} for $1 \leq i \leq n-1$ and $(i-n)''$ in each box of λ^i/λ^{i-1} for $n+1 \leq i \leq 2n-1$.

Consider the left action of the reflections s_a for a in the reduced word of the product $v_{n-1} \cdots v_1 \sigma u_1 \cdots u_{n-1}$ on the k-Grassmannian signed permutations going from w_μ to w_λ. The left action of s_a on a k-Grassmannian element w_ν must be of the form $(\cdots a \cdots | \cdots a{+}1 \cdots) \mapsto (\cdots a{+}1 \cdots | \cdots a \cdots)$ or $(\cdots \overline{a} \cdots) \mapsto (\cdots \overline{a+1} \cdots)$; these add a left box or a right box to ν, respectively. It follows as in the proof of Theorem 1 that for $1 \leq i \leq \ell_k(\mu)$ (respectively, $1 \leq i \leq \ell_k(\lambda)$), the entries of U in row i are $\geq (\mu_i - k)'$ (respectively, $\leq (\lambda_i - k - 1)''$). Observe that when a right box is added to ν, at most one of the values $w_\nu(1), \ldots, w_\nu(k)$ will decrease by 1. We deduce that for $1 \leq j \leq k$, the top entry h_1 of U in column $k{+}1{-}j$ of λ/μ was added by a reflection s_a where $a \leq w_\mu(j)$, hence we must have $h_1 \geq (w_\mu(j))'$. Similarly, the bottom entry h_2 of U in column $k+1-j$ must satisfy $h_2 \leq (w_\lambda(j)-1)''$. Since the entries of U are clearly weakly increasing down column $k+1-j$, they must all be in the interval $[(w_\mu(j))', (w_\lambda(j)-1)'']$. It follows that the entries of U in the first k columns will lie within the intervals which are listed in Definition 4. We deduce that every filling U of λ/μ as above is a k-tritableau on λ/μ such that $(xyz)^U = (-y_{n-1})^{\ell(v_{n-1})} \cdots (-y_1)^{\ell(v_1)} z^T x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})}$. Conversely, the k-tritableaux of shape λ/μ correspond to reduced factorizations of w of the required form. Finally, by combining (9) with (7), we obtain (8). $\square$

Example 2. Following [TW], for any k-strict partition λ, there is a double theta polynomial $\Theta_\lambda(c\,|\,t)$, whose image $\Theta_\lambda(Z;X,Y)$ in the ring of type C Schubert polynomials is equal to the Grassmannian Schubert polynomial $\mathfrak{C}_{w_\lambda}(Z;X,Y)$. Formula (8) therefore gives

$$\Theta_\lambda(Z;X,Y) = \sum_U 2^{n(U)}(xyz)^U \tag{10}$$

summed over all k-tritableaux U of shape λ. Equation (10) extends [T1, Thm. 5] from single to double theta polynomials.

Example 3. We extend [T1, Example 7] to include k-tritableaux. Let $k := 1$, $\lambda := (3,1)$, and $Z_2 := (z_1,z_2)$. We have $w_\lambda = (3,\overline{2},1)$ and will compute $\Theta_{(3,1)}(Z_2;X,Y) = \mathfrak{C}_{3\overline{2}1}(Z_2;X,Y)$. Consider the alphabet $\mathbf{Q}_{1,2} = \{2' < 1' < 1 < 2 < 1'' < 2''\}$. The twelve 1-tritableaux of shape λ with entries in $\{2',1',1,2\}$ are listed in loc. cit. There are sixteen further 1-tritableaux of shape λ involved. The tritableau $U = \begin{matrix} 1\,2\,1'' \\ 2 \end{matrix}$ satisfies $n(U)=3$, the seven tritableaux

$$\begin{matrix}1\,1\,1'' \\ 2\end{matrix}\quad \begin{matrix}1\,2\,1'' \\ 1\end{matrix}\quad \begin{matrix}1'\,1\,1'' \\ 1\end{matrix}\quad \begin{matrix}1'\,2\,1'' \\ 1\end{matrix}\quad \begin{matrix}1'\,1\,1'' \\ 2\end{matrix}\quad \begin{matrix}1'\,2\,1'' \\ 2\end{matrix}\quad \begin{matrix}1\,2\,1'' \\ 1''\end{matrix}$$

satisfy $n(U)=2$, while the eight tritableaux

$$\begin{matrix}1\,1\,1'' \\ 1\end{matrix}\quad \begin{matrix}2\,2\,1'' \\ 2\end{matrix}\quad \begin{matrix}1'\,1\,1'' \\ 1'\end{matrix}\quad \begin{matrix}1'\,2\,1'' \\ 1'\end{matrix}\quad \begin{matrix}1\,1\,1'' \\ 1''\end{matrix}\quad \begin{matrix}2\,2\,1'' \\ 1''\end{matrix}\quad \begin{matrix}1'\,1\,1'' \\ 1''\end{matrix}\quad \begin{matrix}1'\,2\,1'' \\ 1''\end{matrix}$$

satisfy $n(U)=1$. We deduce from [T1, Example 7] and Theorem 2 that

$$\begin{aligned}\Theta_{3,1}(Z_2;X,Y) &= \Theta_{3,1}(Z_2;X) - y_1(2z_1^3+8z_1^2z_2+8z_1z_2^2+2z_2^3) \\ &\quad - y_1(4z_1^2+8z_1z_2+4z_2^2)x_1 - y_1(2z_1+2z_2)x_1^2 \\ &\quad + y_1^2(2z_1^2+4z_1z_2+2z_2^2) + y_1^2(2z_1+2z_2)x_1 \\ &= \Theta_{3,1}(Z_2;X) - y_1\Theta_{2,1}(Z_2;X) + y_1^2\Theta_2(Z_2;X).\end{aligned}$$

Remark 2. Suppose that w is a skew element of W_∞ associated to the compatible pair (λ,μ) of k-strict partitions. One may view the right hand side of (8) as a tableau formula for a 'double skew theta polynomial' $\Theta_{\lambda/\mu}(Z;X,Y)$ indexed by λ/μ. However, Wilson's double theta polynomials $\Theta_\lambda(c\,|\,t)$ from [TW], like their single versions $\Theta_\lambda(c)$ in [BKT2], are defined in terms of raising operators acting on monomials in a different set of variables. It remains an open question to determine an analogue of these raising operator formulas in the skew case. A similar remark applies to the theory of single and double eta polynomials found in [BKT3, T5]. See also Example 4 below, which examines Theorem 2 when $k=0$.

Example 4. Suppose that $k=0$ and let $w := w_\lambda w_\mu^{-1}$ be the fully commutative skew element associated to a pair $\lambda \supset \mu$ of 0-strict partitions. According to [IMN, Thm. 6.6], the Schubert polynomial $\mathfrak{C}_{w_\lambda}(Z;X,Y)$ is equal to a double analogue $Q_\lambda(Z;Y)$ of Schur's Q-function introduced by Ivanov [I], and we also have $\mathfrak{C}_{w_\mu}(Z;X,Y) = Q_\mu(Z;Y)$. However, the skew Schubert polynomial $\mathfrak{C}_w$ will in general involve both the X and Y variables. For instance, assume that $\lambda = \delta_n := (n, n-1, \ldots, 1)$, so that $w_\lambda = w_{\delta_n} = (\overline{n}, \ldots, \overline{1})$ is the longest 0-Grassmannian element in W_n. Since $w_\mu^{-1} w^{-1} = w_{\delta_n}^{-1} = w_{\delta_n}$, we see that in this case $w^{-1} = w_{\mu^\vee}$ is the 0-Grassmannian element with shape given by the 0-strict partition $\mu^\vee$ whose parts complement the parts μ_i of μ in the set $\{1, \ldots, n\}$. Using the symmetry property of double Schubert polynomials [IMN, Thm. 8.1], we conclude that $\mathfrak{C}_w(Z; X, Y) = \mathfrak{C}_{w^{-1}}(Z; -Y, -X) = Q_{\mu^\vee}(Z; -X)$. It is an instructive exercise to deduce this equality from the tableau formula (8).

4. Tableau formula for type D skew Schubert polynomials

4.1. **Grassmannian elements and typed k-strict partitions.** The main references for this subsection are [BKT1, BKT3, T3, T7]. According to [T7, Def. 1], we say that $w \in \widetilde{W}_\infty$ has type 0 if $|w_1| = 1$, type 1 if $w_1 > 1$, and type 2 if $w_1 < -1$. Fix a positive integer k. An element $w \in \widetilde{W}_\infty$ is k-Grassmannian if $\ell(ws_i) > \ell(w)$ for all $i \neq k$, if $k > 1$, and for all $i > 1$, if $k = 1$. This is equivalent to the conditions

$$|w_1| < \cdots < w_k \quad \text{and} \quad w_{k+1} < w_{k+2} < \cdots$$

with the first condition being vacuous if $k = 1$. Following [T7], we regard the $\Box$-Grassmannian elements as a subset of the 1-Grassmannian elements.

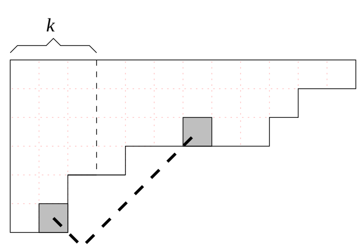
A *typed k-strict partition* is a pair consisting of a k-strict partition λ together with an integer $\mathrm{type}(\lambda) \in \{0, 1, 2\}$, which is positive if and only if $\lambda_i = k$ for some index i. There is a type-preserving bijection between the k-Grassmannian elements w of $\widetilde{W}_\infty$ and typed k-strict partitions λ. If the element w corresponds to the typed partition λ, then we denote $w = w(\lambda, k)$ by w_λ.

Given a typed k-strict partition λ, the bijection is determined as follows. Let $\ell_k(\lambda)$ and $\gamma_1 \leq \cdots \leq \gamma_k$ be defined as in Section 3.1. If $\mathrm{type}(\lambda) \neq 2$, then

$$w_\lambda(j) = \gamma_j + j - \#\{p \in [1, \ell_k(\lambda)] \,:\, \lambda_p + p \geq \gamma_j + j + k\}$$

for $1 \leq j \leq k$, while the equalities $w_\lambda(k+i) = k - 1 - \lambda_i$ for $1 \leq i \leq \ell_k(\lambda)$ give the negative entries of w_λ which are less than -1. If $\mathrm{type}(\lambda) = 2$ and λ' is the partition of type 1 with the same shape as λ, then w_λ is related to $w_{\lambda'} = (w_1', \ldots, w_n')$ by changing the sign of the first entry w_1' and of the entry w_p' with $|w_p'| = 1$. For example, the typed 3-strict partition $\lambda = (7, 4, 3, 2)$ of type 2 satisfies $\ell_k(\lambda) = 2$ and $(\gamma_1, \gamma_2, \gamma_3) = (3, 4, 4)$, therefore $w_\lambda = (\overline{3}, 6, 7, \overline{5}, \overline{2}, \overline{1}, 4, 8)$. We refer to [BKT3, Sec. 6.1] for a picture of the correspondence between w and λ which uses related and non-related diagonals.

We say that the boxes $[r, c]$ and $[r', c']$ in a Young diagram are $(k-1)$-*related* if $|c - k| + r = |c' - k| + r'$. For instance, the two grey boxes in the figure below are $(k-1)$-related. We call the box $[r, c]$ a *left box* if $c \leq k$ and a *right box* if $c > k$.



Following [T3, Sec. 2.6 and 3.4], if $\mu \subset \lambda$ are two typed k-strict partitions, we let R (respectively, $\mathbb{A}$) denote the set of right boxes of μ (including boxes in row zero) which are bottom boxes of λ in their column and are (respectively, are not) $(k-1)$-related to a left box of λ/μ. A pair $\mu \subset \lambda$ of typed k-strict partitions forms a *typed k'-horizontal strip* λ/μ if $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$ and (i) λ/μ is contained in the rim of λ, and the right boxes of λ/μ form a horizontal strip; (ii) no two boxes in R are $(k-1)$-related; and (iii) if two boxes of λ/μ lie in the same column, then they are $(k-1)$-related to exactly two boxes of R, which both lie in the same row. We define $n'(\lambda/\mu)$ to be one less than the number of connected components of $\mathbb{A}$.

4.2. **Skew elements and main theorem.** Following [T3, Sec. 4.3], an element $w \in \widetilde{W}_\infty$ is called *skew* if there exists a k-Grassmannian element w_λ (for some $k \geq 1$) and a reduced factorization $w_\lambda = ww'$ in $\widetilde{W}_\infty$. In this case, the right factor w' equals w_μ for some k-Grassmannian element w_μ, and we have $\mu \subset \lambda$. We say that (λ, μ) is a *compatible pair* and that w is associated to the pair (λ, μ). According to [T3, Cor. 2], there is a 1-1 correspondence between reduced factorizations uv of $w_\lambda w_\mu^{-1}$ and typed k-strict partitions ν with $\mu \subset \nu \subset \lambda$ such that (λ, ν) and (ν, μ) are compatible pairs. Any typed k'-horizontal strip λ/μ is an example of a compatible pair (λ, μ) of typed k-strict partitions.

Let λ and μ be any two typed k-strict partitions such that (λ, μ) is a compatible pair, choose an integer $n \geq 1$ such that $w_\lambda \in \widetilde{W}_n$, and let $w := w_\lambda w_\mu^{-1}$ be the corresponding skew element of $\widetilde{W}_n$. It was shown in [T3] that w is unimodal (in the sense of Section 3.2) if and only if λ/μ is a typed k'-horizontal strip.

Definition 5. We say that a typed k'-horizontal strip λ/μ is *extremal* if

$$(\ell_k(\lambda), \mathrm{type}(\lambda)) \neq (\ell_k(\mu), \mathrm{type}(\mu)).$$

For any typed k-strict partition ν, let $\epsilon(\nu) := \ell_k(\nu) + \mathrm{type}(\nu)$.

Definition 6. Suppose that $w = w_\lambda w_\mu^{-1}$ lies in S_n. If w is decreasing down to 1 (respectively, increasing up from 1), then we say that the typed k'-horizontal strip λ/μ is a *typed x-strip* (respectively, *typed y-strip*).

The typed x- and typed y-strips are characterized among all typed k'-horizontal strips by the next result.

Proposition 2. *A typed k'-horizontal strip λ/μ is a typed x-strip (respectively, typed y-strip) if and only if* (i) *the left boxes in λ/μ form a vertical strip (respectively, horizontal strip), and no two boxes in λ/μ are $(k-1)$-related (respectively, no two right boxes in λ/μ are in the same row), and* (ii) *if λ/μ is extremal then* $(\mathrm{type}(\lambda), \mathrm{type}(\mu)) \neq (0,0)$ *and the following condition holds: if $\epsilon(\mu)$ is odd, then $\epsilon(\lambda)$ is odd* and $\mathrm{type}(\mu) = 0$*, while if $\epsilon(\mu)$ is even, then $\epsilon(\lambda)$ is odd* or $\mathrm{type}(\mu) = 1$.

Proof. For $i \geq 1$, a k-Grassmannian element $s_i v \in \widetilde{W}_n$ satisfies $\ell(s_i v) > \ell(v)$ if and only if v has one of the following four forms:

$$(\cdots i \cdots i+1 \cdots), \quad (\cdots i+1 \cdots \overline{i} \cdots), \quad (\cdots \overline{i} \cdots i+1 \cdots), \quad (\overline{i+1} \cdots \overline{i} \cdots).$$

Moreover, a k-Grassmannian element $s_\square v$ satisfies $\ell(s_\square v) > \ell(v)$ if and only if v has one of the following four forms:

$$(\cdots 1 \cdots 2 \cdots), \quad (\overline{1} \cdots 2 \cdots), \quad (2 \cdots 1 \cdots), \quad (\overline{2} \cdots 1 \cdots).$$

TABLE 1. Statistics for the extremal pairs u and s_1u, v and $s_\Box v$

u	μ	$\ell_k(\mu)$	type(μ)	$\epsilon(\mu)$	s_1u	λ	$\ell_k(\lambda)$	type(λ)	$\epsilon(\lambda)$
1324	1	even	0	even	2314	2	even	1	odd
$\overline{2}3\overline{1}4$	2	even	2	even	$\overline{1}3\overline{2}4$	3	odd	0	odd
$\overline{3}4\overline{1}2$	(2,2)	even	2	even	$\overline{3}4\overline{2}1$	(3,2)	odd	2	odd
$\overline{1}4\overline{3}2$	(4,1)	odd	0	odd	$\overline{2}4\overline{3}1$	(4,2)	odd	2	odd
$24\overline{31}$	(4,2)	odd	1	even	$14\overline{32}$	(4,3)	even	0	even
v	μ	$\ell_k(\mu)$	type(μ)	$\epsilon(\mu)$	$s_\Box v$	λ	$\ell_k(\lambda)$	type(λ)	$\epsilon(\lambda)$
1324	1	even	0	even	$\overline{2}3\overline{1}4$	2	even	2	even
2314	2	even	1	odd	$\overline{1}3\overline{2}4$	3	odd	0	odd
3412	(2,2)	even	1	odd	$34\overline{21}$	(3,2)	odd	1	even
$\overline{1}4\overline{3}2$	(4,1)	odd	0	odd	$24\overline{31}$	(4,2)	odd	1	even
$\overline{2}4\overline{3}1$	(4,2)	odd	2	odd	$14\overline{32}$	(4,3)	even	0	even

Let $w := w_\lambda w_\mu^{-1}$ be the skew element of $\widetilde{W}_n$ associated to λ/μ. It is easy to verify that λ/μ is extremal if and only if $w(1) \neq 1$. In this case, we have $(\text{type}(\lambda), \text{type}(\mu)) \neq (0,0)$ if and only if no unimodal reduced word for w has $\Box 1$ as a subword. Now suppose that λ/μ is extremal, w lies in S_n, and w is decreasing down to 1 or increasing up from 1. A case-by-case analysis shows that if $\epsilon(\mu)$ is odd, then $\epsilon(\lambda)$ is odd *and* $\text{type}(\mu) = 0$, while if $\epsilon(\mu)$ is even, then $\epsilon(\lambda)$ is odd *or* $\text{type}(\mu) = 1$. On the other hand, for the skew element $s_0 w s_0$ of shape λ/μ, we observe that if $\epsilon(\mu)$ is odd, then $\epsilon(\lambda)$ is even *or* $\text{type}(\mu) = 1$, while if $\epsilon(\mu)$ is even, then $\epsilon(\lambda)$ is even *and* $\text{type}(\mu) = 0$. Indeed, since $w(1) \neq 1$, we are reduced to examining what happens when $w_\lambda = s_1 w_\mu$ and $w_\lambda = s_\Box w_\mu$, respectively. The $10 = 2 \cdot 5$ different cases are illustrated in Table 1 when $k = 2$ and $n = 4$, and the picture for other values of k and n follows the same pattern. The remainder of the argument is similar to the proof of Proposition 1. $\square$

Let $\mathbf{R}$ denote the ordered alphabet

$$(n-1)' < \cdots 2' < 1' < 1, 1^\circ < 2, 2^\circ < 3, 3^\circ < \cdots < 1'' < 2'' < \cdots < (n-1)''.$$

The single and double primed symbols in $\mathbf{R}$ are said to be *marked*, while the rest are *unmarked*. A *typed k'-tableau* T of shape λ/μ is a sequence of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^p = \lambda$$

such that λ^i/λ^{i-1} is a typed k'-horizontal strip for $1 \leq i \leq p$. We represent T by a filling of the boxes in λ/μ with unmarked symbols of $\mathbf{R}$ such that for each i, the boxes in T with entry i or i° form the skew diagram λ^i/λ^{i-1}, and we use i (respectively, i°) if and only if $\text{type}(\lambda^i) \neq 2$ (respectively, $\text{type}(\lambda^i) = 2$), for each $i \in [1,p]$. For any typed k'-tableau T we define $n(T) := \sum_i n'(\lambda^i/\lambda^{i-1})$ and set $z^T := \prod_i z_i^{n_i}$, where n_i denotes the number of times that i or i° appears in T. According to [T3, Thm. 4], the type D Stanley function $E_w(Z)$ satisfies the equation

$$E_w(Z) = \sum_T 2^{n(T)} z^T \tag{11}$$

summed over all typed k'-tableaux T of shape λ/μ.

Definition 7. A *typed k'-tritableau* U of shape λ/μ is a filling of the boxes in λ/μ with elements of $\mathbf{R}$ which is weakly increasing along each row and down each column, such that (i) for each a in $\mathbf{R}$, the boxes in λ/μ with entry a form a typed k'-horizontal strip, which is a typed x-strip (respectively, typed y-strip) if $a \in [(n-1)', 1']$ (respectively, $a \in [1'', (n-1)'']$) and non-extremal if $a \leq 2'$ (respectively, $a \geq 2''$), (ii) the unmarked entries of U form a typed k'-tableau T, and (iii) for $1 \leq i \leq \ell_k(\mu)$ (respectively, $1 \leq i \leq \ell_k(\lambda)$) and $1 \leq j \leq k$, the entries of U in row i are $\geq (\mu_i - k + 1)'$ (respectively, $\leq (\lambda_i - k)''$) and the entries in column $k+1-j$ lie in the interval $[|w_\mu(j)|', |w_\lambda(j)-1|'']$. Let

$$n(U) := n(T) \quad \text{and} \quad (xyz)^U := z^T \prod_i x_i^{n'_i} \prod_i (-y_i)^{n''_i}$$

where n'_i and n''_i denote the number of times that i' and i'' appear in U, respectively.

Theorem 3. *For the skew element $w := w_\lambda w_\mu^{-1}$, we have*

$$\mathfrak{D}_w(Z;X,Y) = \sum_U 2^{n(U)} (xyz)^U \tag{12}$$

summed over all typed k'-tritableaux U of shape λ/μ.

Proof. We deduce from formula (3) that

$$\mathfrak{D}_w(Z;X,Y) = \sum (-y_{n-1})^{\ell(v_{n-1})} \cdots (-y_1)^{\ell(v_1)} E_\tau(Z) x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})} \tag{13}$$

where the sum is over all reduced factorizations $v_{n-1}\cdots v_1 \tau u_1 \cdots u_{n-1}$ of w such that $v_p \in S_n$ is increasing up from p and $u_p \in S_n$ is decreasing down to p for each p. Such factorizations correspond to sequences of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{n-1} \subset \lambda^n \subset \lambda^{n+1} \subset \cdots \subset \lambda^{2n-1} = \lambda$$

with λ^i/λ^{i-1} a typed x-strip for each $i \leq n-1$, a typed y-strip for each $i \geq n+1$, and $(\lambda^n, \lambda^{n-1})$ a compatible pair. We extend each typed k'-tableau T on λ^n/λ^{n-1} to a filling U of the boxes in λ/μ by placing the entry $(n-i)'$ in each box of λ^i/λ^{i-1} for $1 \leq i \leq n-1$ and $(i-n)''$ in each box of λ^i/λ^{i-1} for $n+1 \leq i \leq 2n-1$. As in the proof of Theorem 2, referring this time to Section 4.1, one checks that the marked entries of U are restricted in accordance with Definition 7. We deduce that every such filling U of λ/μ is a typed k'-tritableau on λ/μ such that $(xyz)^U = (-y_{n-1})^{\ell(v_{n-1})} \cdots (-y_1)^{\ell(v_1)} z^T x_1^{\ell(u_1)} \cdots x_{n-1}^{\ell(u_{n-1})}$. Conversely, the typed k'-tritableaux of shape λ/μ correspond to reduced factorizations of w of the required form. Finally, by combining (13) with (11), we obtain (12). □

Example 5. Following [T3], for any typed k-strict partition λ, there is a double eta polynomial $H_\lambda(c\,|\,t)$, whose image $H_\lambda(Z;X,Y)$ in the ring of type D Schubert polynomials is equal to the Grassmannian Schubert polynomial $\mathfrak{D}_{w_\lambda}(Z;X,Y)$. Formula (12) therefore gives

$$H_\lambda(Z;X,Y) = \sum_U 2^{n(U)} (xyz)^U \tag{14}$$

summed over all typed k'-tritableaux U of shape λ. Equation (14) extends [T3, Thm. 3] from single to double eta polynomials.

Example 6. We extend [T3, Example 2] to include typed k'-tritableaux. Let $k := 1$, $\lambda := (3,1)$ of type 1, and $Z_2 := (z_1, z_2)$. We have $w_\lambda = (2, \overline{3}, \overline{1})$ and will compute $H_{(3,1)}(Z_2; X, Y) = \mathfrak{D}_{2\overline{3}\overline{1}}(Z_2; X, Y)$. Consider the alphabet $\mathbf{R}_{1,2} = \{2' < 1' < 1, 1^\circ < 2, 2^\circ < 1'' < 2''\}$. The thirteen typed $1'$-tritableaux of shape λ with entries in $\{2', 1', 1, 1^\circ, 2, 2^\circ\}$ are listed in loc. cit. There are fourteen further $1'$-tritableaux of shape λ involved. The two tritableaux $\begin{array}{l} 1'\,2\,1'' \\ 1 \end{array}$ and $\begin{array}{l} 1'\,2\,2'' \\ 1 \end{array}$ satisfy $n(U) = 1$, while the twelve tritableaux

$$\begin{array}{l} 1\,2\,1'' \\ 1 \end{array} \quad \begin{array}{l} 1\,2\,2'' \\ 1 \end{array} \quad \begin{array}{l} 1\,2\,1'' \\ 2 \end{array} \quad \begin{array}{l} 1\,2\,2'' \\ 2 \end{array} \quad \begin{array}{l} 1'\,1\,1'' \\ 1 \end{array} \quad \begin{array}{l} 1'\,1\,2'' \\ 1 \end{array}$$

$$\begin{array}{l} 1'\,2\,1'' \\ 2 \end{array} \quad \begin{array}{l} 1'\,2\,2'' \\ 2 \end{array} \quad \begin{array}{l} 1'\,1\,1'' \\ 1' \end{array} \quad \begin{array}{l} 1'\,1\,2'' \\ 1' \end{array} \quad \begin{array}{l} 1'\,2\,1'' \\ 1' \end{array} \quad \begin{array}{l} 1'\,2\,2'' \\ 1' \end{array}$$

satisfy $n(U) = 0$. We deduce from [T3, Example 2] and Theorem 3 that

$$\begin{aligned} H_{3,1}(Z_2; X, Y) = H_{3,1}(Z_2; X) &- (y_1 + y_2)(z_1^2 z_2 + z_1 z_2^2) \\ &- (y_1 + y_2)(z_1^2 + 2z_1 z_2 + z_2^2)x_1 - (y_1 + y_2)(z_1 + z_2)x_1^2 \\ = H_{3,1}(Z_2; X) &- (y_1 + y_2)H_{2,1}(Z_2; X). \end{aligned}$$

For $\lambda := (3,1)$ with $\mathrm{type}(\lambda) = 2$, we have $w_\lambda = (\overline{2}, \overline{3}, 1)$ and will compute $H'_{(3,1)}(Z_2; X, Y) = \mathfrak{D}_{\overline{2}\overline{3}1}(Z_2; X, Y)$. Here, as in op. cit., the prime in $H'_{(3,1)}$ indicates that the indexing partition has type 2. It is shown in [T3] that in this case there are six typed $1'$-tritableaux of shape λ with entries in $\{2', 1', 1, 1^\circ, 2, 2^\circ\}$. There are 23 further $1'$-tritableaux of the same shape λ. The three tritableaux

$$\begin{array}{l} 1\,1\,2 \\ 1'' \end{array} \quad \begin{array}{l} 1'\,1\,2 \\ 1'' \end{array} \quad \begin{array}{l} 1^\circ\,1''\,2'' \\ 2^\circ \end{array}$$

satisfy $n(U) = 1$, while the twenty tritableaux

$$\begin{array}{l} 1\,1\,1 \\ 1'' \end{array} \quad \begin{array}{l} 2\,2\,2 \\ 1'' \end{array} \quad \begin{array}{l} 1\,2\,2 \\ 1'' \end{array} \quad \begin{array}{l} 1^\circ\,2\,2 \\ 1'' \end{array} \quad \begin{array}{l} 1'\,1\,1 \\ 1'' \end{array} \quad \begin{array}{l} 1'\,2\,2 \\ 1'' \end{array} \quad \begin{array}{l} 1\,1\,1'' \\ 1'' \end{array} \quad \begin{array}{l} 2\,2\,1'' \\ 1'' \end{array}$$

$$\begin{array}{l} 1\,2\,1'' \\ 1'' \end{array} \quad \begin{array}{l} 1^\circ\,2\,1'' \\ 1'' \end{array} \quad \begin{array}{l} 1'\,1\,1'' \\ 1'' \end{array} \quad \begin{array}{l} 1'\,2\,1'' \\ 1'' \end{array} \quad \begin{array}{l} 1^\circ\,2^\circ\,1'' \\ 1^\circ \end{array} \quad \begin{array}{l} 1^\circ\,2^\circ\,2'' \\ 1^\circ \end{array}$$

$$\begin{array}{l} 1^\circ\,2^\circ\,1'' \\ 2^\circ \end{array} \quad \begin{array}{l} 1^\circ\,2^\circ\,2'' \\ 2^\circ \end{array} \quad \begin{array}{l} 1^\circ\,1''\,2'' \\ 1^\circ \end{array} \quad \begin{array}{l} 2^\circ\,1''\,2'' \\ 2^\circ \end{array} \quad \begin{array}{l} 1^\circ\,1''\,2'' \\ 1'' \end{array} \quad \begin{array}{l} 2^\circ\,1''\,2'' \\ 1'' \end{array}$$

satisfy $n(U) = 0$. We deduce from [T3, Example 2] and Theorem 3 that

$$\begin{aligned} H'_{3,1}(Z_2; X, Y) &= H'_{3,1}(Z_2; X) - y_1(z_1^3 + 2z_1^2 z_2 + 2z_1 z_2^2 + z_2^3 + (z_1^2 + 2z_1 z_2 + z_2^2)x_1) \\ &\quad - (y_1 + y_2)(z_1^2 z_2 + z_1 z_2^2) + y_1^2(z_1^2 + 2z_1 z_2 + z_2^2 + (z_1 + z_2)x_1) \\ &\quad + y_1 y_2(z_1^2 + 2z_1 z_2 + z_2^2) - y_1^2 y_2(z_1 + z_2) \\ &= H'_{3,1}(Z_2; X) - y_1 H_3(Z_2; X) - (y_1 + y_2)H'_{2,1}(Z_2; X) \\ &\quad + y_1^2 H_2(Z_2; X) + y_1 y_2 H'_{1,1}(Z_2; X) - y_1^2 y_2 H'_1(Z_2; X). \end{aligned}$$

The y-factors in the last equality are exactly the type A single Schubert polynomials $\mathfrak{S}_\varpi(-Y)$ for $\varpi \in S_3$. Since $w_\lambda = s_1 s_2 s_1 s_\square$, this is in agreement with [T5, Cor. 1].

Example 7. Let $w := w_\lambda w_\mu^{-1}$ be a skew element of W_∞ or $\widetilde{W}_\infty$. Extend the alphabets $\mathbf{Q}$ and $\mathbf{R}$ to include all primed and double primed positive integers, omit the bounds on the entries of the tritableaux found in Definitions 4 and 7, and the non-extremal condition in the latter. Then the right hand sides of equations (8) and (12) give tableau formulas for the type C *double mixed Stanley function* $J_w(Z;X,Y)$ of [T2, Ex. 3] and its type D analogue $I_w(Z;X,Y)$, respectively.

References

[BH] S. Billey and M. Haiman : *Schubert polynomials for the classical groups*, J. Amer. Math. Soc. **8** (1995), 443–482.

[BJS] S. Billey, W. Jockusch and R. P. Stanley : *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), 345–374.

[BKT1] A. S. Buch, A. Kresch, and H. Tamvakis : *Quantum Pieri rules for isotropic Grassmannians*, Invent. Math. **178** (2009), 345–405.

[BKT2] A. S. Buch, A. Kresch, and H. Tamvakis : *A Giambelli formula for isotropic Grassmannians*, Selecta Math. New Ser. **23** (2017), 869–914.

[BKT3] A. S. Buch, A. Kresch, and H. Tamvakis : *A Giambelli formula for even orthogonal Grassmannians*, J. reine angew. Math. **708** (2015), 17–48.

[CYY] W. Y. C. Chen, G.-G. Yan, and A. L. B. Yang : *The skew Schubert polynomials*, European J. Combin. **25** (2004), 1181–1196.

[FK] S. Fomin and A. N. Kirillov : *Combinatorial B_n-analogs of Schubert polynomials*, Trans. Amer. Math. Soc. **348** (1996), 3591–3620.

[FS] S. Fomin and R. P. Stanley : *Schubert polynomials and the nil-Coxeter algebra*, Adv. Math. **103** (1994), 196–207.

[IMN] T. Ikeda, L. C. Mihalcea, and H. Naruse : *Double Schubert polynomials for the classical groups*, Adv. Math. **226** (2011), 840–886.

[I] V. N. Ivanov : *Interpolation analogues of Schur Q-functions*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **307** (2004), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 10, 99–119, 281–282; translation in J. Math. Sci. (N. Y.) **131** (2005), 5495–5507 .

[KL] G. Kempf and D. Laksov : *The determinantal formula of Schubert calculus*, Acta Math. **132** (1974), 153–162.

[K] V. Kreiman : *Products of factorial Schur functions*, Electron. J. Combin. **15** (2008), Research Paper 84, 12 pp.

[La] T. K. Lam : *B and D analogues of stable Schubert polynomials and related insertion algorithms*, Ph.D. thesis, M.I.T., 1994; available at http://hdl.handle.net/1721.1/36537.

[L1] A. Lascoux : *Puissances extérieures, déterminants et cycles de Schubert*, Bull. Soc. Math. France **102** (1974), 161–179.

[L2] A. Lascoux : *Classes de Chern des variétés de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 393–398.

[LS] A. Lascoux and M.-P. Schützenberger : *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 447–450.

[LeS] C. Lenart and F. Sottile : *Skew Schubert polynomials*, Proc. Amer. Math. Soc. **131** (2003), 3319–3328.

[Li] D. E. Littlewood : *The construction of invariant matrices*, Proc. London Math. Soc. (2) **43** (1937), 226–240.

[M] Molev, A. I. : *Littlewood-Richardson polynomials*, J. Algebra **321** (2009), 3450–3468.

[S] J. R. Stembridge : *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), 353–385.

[T1] H. Tamvakis : *Giambelli, Pieri, and tableau formulas via raising operators*, J. reine angew. Math. **652** (2011), 207–244.

[T2] H. Tamvakis : *A Giambelli formula for classical G/P spaces*, J. Algebraic Geom. **23** (2014), 245–278.

[T3] H. Tamvakis : *A tableau formula for eta polynomials*, Math. Annalen **358** (2014), 1005–1029.

[T4] H. Tamvakis : *Giambelli and degeneracy locus formulas for classical G/P spaces*, Mosc. Math. J. **16** (2016), 125–177.

[T5] H. Tamvakis : *Double eta polynomials and equivariant Giambelli formulas*, J. London Math. Soc. **94** (2016), 209–229.

[T6] H. Tamvakis : *Schubert polynomials, theta and eta polynomials, and Weyl group invariants*, Mosc. Math. J. **21** (2021), 191–226.

[T7] H. Tamvakis : *Degeneracy locus formulas for amenable Weyl group elements*, arXiv:1909.06398.

[TW] H. Tamvakis and E. Wilson : *Double theta polynomials and equivariant Giambelli formulas*, Math. Proc. Cambridge Philos. Soc. **160** (2016), 353–377.

[W] M. L. Wachs : *Flagged Schur functions, Schubert polynomials, and symmetrizing operators*, J. Combin. Theory Ser. A **40** (1985), 276–289.

University of Maryland, Department of Mathematics, William E. Kirwan Hall, 4176 Campus Drive, College Park, MD 20742, USA

Email address: `harryt@umd.edu`