Math 406 – Fall 2025 – Harry Tamvakis

PROBLEM SET 1 – Due September 11, 2025

Reading for this week: Appendix A (Proof by Induction, pp. 205-209).

Problems

From the textbook: Appendix A, Problems #1, 2, 4, 7, 11, 14. In addition, do the following problems:

A1) Suppose that x is variable. Show that for any natural number n, we have the identity

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

A2) Suppose that the numbers a_n are defined recursively by $a_1 := 1$, $a_2 := 2$, $a_3 := 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Use the strong induction principle to show that $a_n < 2^n$ for every natural number n.

Extra Credit Problems.

EC1) (a) Given the 2×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, use induction to prove that, for all integers $n \geq 1$, we have

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Here F_n is the n-th Fibonacci number, with the convention that $F_0 := 0$.

(b) Use part (a) to prove the identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

for any $n \geq 1$.

EC2) Prove that for every integer $n \geq 1$, the expression

$$(2 + \sqrt{3})^n + (2 - \sqrt{3})^n$$

is equal to a natural number.