For the first few weeks we will study linear algebra, so we will not be following the sequential order of the textbook. Chapter 11 contains some of the material that we will cover.

A1) Suppose that $V$ is a vector space over a field $F$, and let $u, v$ and $w$ be linearly independent vectors in $V$.

(a) If $F = \mathbb{R}$, prove that the vectors $u + v$, $v + w$ and $w + u$ are also linearly independent.

(b) Show that the conclusion of part (a) can fail if $F \neq \mathbb{R}$.

A2) Let $V$ be the real vector space of all functions $f : [0, 1] \rightarrow \mathbb{R}$. Prove that the functions $x^2$, $\sin x$, and $\cos x$ are linearly independent in $V$.

A3) Find a basis for the real subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 2, -1, 0)$, $(4, 8, -4, -3)$, $(0, 1, 3, 4)$, and $(2, 5, 1, 4)$.

A4) Let $V \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $Ax = 0$, where $A = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$. Find a basis for $V$.

A5) Let $F$ be any field. Which $n \times n$ matrices $A \in M_n(F)$ have the property that $AB = BA$ for all $n \times n$ matrices $B$? Prove your assertion.

B1) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying

$$f(x + y) = f(x) + f(y) , \forall x, y \in \mathbb{R}.$$ 

(a) Prove that for every $n \in \mathbb{N}$ and each $x \in \mathbb{R}$, $f(nx) = nf(x)$ and $f(x/n) = f(x)/n$.

(b) Prove that $f(qx) = qf(x)$, for any $q \in \mathbb{Q}$ and $x \in \mathbb{R}$.

(c) Show that there is a $\lambda \in \mathbb{R}$ such that $f(x) = \lambda x$, for all $x \in \mathbb{Q}$.

(d) If $f$ is continuous, prove that $f(x) = \lambda x$ for some $\lambda \in \mathbb{R}$.

(e) If $g : \mathbb{R} \rightarrow (0, +\infty)$ is a continuous function such that

$$g(x + y) = g(x)g(y) , \forall x, y \in \mathbb{R},$$

prove that $g(x) = e^{\lambda x}$ for some $\lambda \in \mathbb{R}$. 

B2) Let $V$ be a real vector space.

(a) Suppose $H_1$ and $H_2$ are subspaces of $V$. Prove that $H_1 \cup H_2$ is a subspace of $V$ if and only if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

(b) Let $H_1, \ldots, H_n$ be subspaces of $V$ such that $H_i \neq V$ for each $i$ (that is, the $H_i$ are proper subspaces of $V$). Prove that $H_1 \cup \cdots \cup H_n \neq V$.

(c) Suppose $H_1, H_2, \ldots, H_n, \ldots$ is a sequence of proper subspaces of $V$. Is it true that the union $\bigcup_{n=1}^{\infty} H_n$ can never equal $V$? Give a proof or find a counterexample.

Extra Credit Problems

C1) In problem B1 you showed that the only continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with the property

$$f(x + y) = f(x) + f(y), \ \forall x, y \in \mathbb{R}$$

are the functions $f(x) = \lambda x$. Are there any other (necessarily non-continuous) functions with this property? Justify your answer.

C2) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-continuous function such that

$$f(x + y) = f(x) + f(y), \ \forall x, y \in \mathbb{R}$$

(assuming that such an $f$ exists). Prove that the graph of $f$ is dense in the plane $\mathbb{R}^2$ (this means that every disk in $\mathbb{R}^2$ contains a point of the graph of $f$).