Math 600 – Fall 2025 – Harry Tamvakis
PROBLEM SET 1 – Due September 11, 2025

For the first few weeks we will study linear algebra, so we will not be following the sequential order of the textbook. Chapter 11 contains some of the material that we will cover.

A1) Suppose that V is a vector space over a field F, and let u, v and w be linearly independent vectors in V.

(a) If $F = \mathbb{R}$, prove that the vectors $u + v$, $v + w$ and $w + u$ are also linearly independent.

(b) Show that the conclusion of part (a) can fail if $F \neq \mathbb{R}$.

A2) Let $H_1 \subset \mathbb{R}^4$ be the subspace of vectors (x, y, z, w) such that $x-2y+4z-3w = 0$ and $H_2 \subset \mathbb{R}^4$ be the subspace of vectors (x, y, z, w) such that $2x - y - z + 6w = 0$. Find a basis of the subspace $H_1 \cap H_2$.

A3) Let V be the real vector space of all functions $f : [0, 1] \to \mathbb{R}$. Find (with proof) an infinite subset S of V that is linearly independent.

A4) Let F be any field. Which $n \times n$ matrices $A \in M_n(F)$ have the property that $AB = BA$ for all $n \times n$ matrices B? Prove your assertion.

A5) Let F be a finite field with q elements and $n \geq 1$.

(a) Determine the minimal value of $k \in \mathbb{N}$ such that any subset S of F^n with k elements must contain a basis of F^n.

(b) Determine the minimal value of $m \in \mathbb{N}$ such that any subset S of $M_n(F)$ with m elements must contain an invertible matrix.

B1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$f(x + y) = f(x) + f(y) \ , \ \forall x, y \in \mathbb{R}.$$

(a) Prove that for every $n \in \mathbb{N}$ and each $x \in \mathbb{R}$, $f(nx) = nf(x)$ and $f(x/n) = f(x)/n$.

(b) Prove that $f(qx) = qf(x)$, for any $q \in \mathbb{Q}$ and $x \in \mathbb{R}$.

(c) Show that there is a $\lambda \in \mathbb{R}$ such that $f(x) = \lambda x$, for all $x \in \mathbb{Q}$.

(d) If f is continuous, prove that $f(x) = \lambda x$ for some $\lambda \in \mathbb{R}$.

(e) Let $g : \mathbb{R} \to (0, +\infty)$ be a continuous function such that

$$g(x + y) = g(x)g(y) \quad \forall x, y \in \mathbb{R}.$$

Prove that $g(x) = e^{\lambda x}$ for some $\lambda \in \mathbb{R}$.

B2) Let V be a vector space over the field $\mathbb{R}$ of real numbers.

(a) Suppose H_1 and H_2 are subspaces of V. Prove that $H_1 \cup H_2$ is a subspace of V if and only if $H_1 \subset H_2$ or $H_2 \subset H_1$.

(b) Let $H_1, \ldots, H_n$ be subspaces of V such that $H_i \neq V$ for each i (that is, the H_i are *proper* subspaces of V). Prove that $H_1 \cup \cdots \cup H_n \neq V$.

(c) Suppose $H_1, H_2, \ldots, H_n, \ldots$ is a sequence of proper subspaces of V. Is it true that the union $\bigcup_{n=1}^{\infty} H_n$ can never equal V? Give a proof or find a counterexample.

Extra Credit Problems

C1) (a) In problem B1 you showed that the only continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property

$$f(x + y) = f(x) + f(y) \ , \ \forall x, y \in \mathbb{R}$$

are the functions $f(x) = \lambda x$. Are there any other (necessarily non-continuous) functions with this property? Justify your answer.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-continuous function such that

$$f(x + y) = f(x) + f(y) \ , \ \forall x, y \in \mathbb{R}$$

(assuming that such an f exists). Prove that the graph of f is dense in the plane $\mathbb{R}^2$ (this means that every disk in $\mathbb{R}^2$ contains a point of the graph of f).

C2) A school with 2025 students has n clubs such that

(i) each club has an odd number of members, and

(ii) any two clubs have an even number of members in common (zero included).

Determine the largest possible value of n.