Math 601 – Spring 2026 – Harry Tamvakis

PROBLEM SET 1 – Due February 12, 2026

A1) Let R be a commutative ring and A and B be two R-algebras. The tensor product $A \otimes_R B$ becomes an R-algebra in a natural way, by the multiplication rule $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. Prove that the $\mathbb{R}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ defined in this way is isomorphic to the product ring $\mathbb{C} \times \mathbb{C}$. [Hint: Look for two elements e_1 and e_2 in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ such that $e_1 e_2 = 0$, $e_1 + e_2 = 1$, $e_1^2 = e_1$, and $e_2^2 = e_2$. These should be the images of $(1, 0)$ and $(0, 1)$ under a ring isomorphism $\mathbb{C} \times \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.]

A2) Let V be any finite dimensional vector space and $e_1, \ldots, e_r \in V$.

(a) Prove that $e_1 \wedge \cdots \wedge e_r \neq 0$ if and only if the set $\{e_1, \ldots, e_r\}$ is linearly independent.

(b) Let $\omega \in \bigwedge^k V$ and suppose that $e_1 \wedge \cdots \wedge e_r \neq 0$. Prove that

$$\omega = \omega_1 \wedge e_1 \wedge \cdots \wedge e_r$$

for some $\omega_1 \in \bigwedge^{k-r} V$ if and only if $\omega \wedge e_i = 0$ for $1 \leq i \leq r$.

A3) Let $\omega := e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2n-1} \wedge e_{2n}$, where $e_1, \ldots, e_{2n}$ is a basis of a vector space. Let $\wedge^n \omega := \omega \wedge \omega \wedge \cdots \wedge \omega$, where ω occurs n times in the product. Prove that

$$\wedge^n \omega = n! \, e_1 \wedge \cdots \wedge e_{2n}.$$

A4) Prove that the formation of tensor, symmetric, and exterior powers is compatible with base change. That is, if $f : R \to S$ is a homomorphism of commutative rings, M is any R-module, and $n \geq 0$, then there is an isomorphism of S-modules

$$T^n(M \otimes_R S) \cong T^n M \otimes_R S$$

and similarly for the symmetric and exterior powers.

B1) (a) Let R be a commutative ring and M and N be two R-modules. Prove that for any $n \geq 0$, there is an isomorphism

$$\bigwedge^n (M \oplus N) \cong \bigoplus_{a+b=n} \bigwedge^a M \otimes_R \bigwedge^b N.$$

(b) Let $A = \{a_{ij}\}$ be an $n \times n$ matrix with entries in a commutative ring R. For each subset I of the set $\{1, \ldots, n\}$ with cardinality $|I| = p$, let A_I denote the matrix formed by the a_{ij} for which $i \in I$ and $1 \leq j \leq p$, and let A_I^c denote the 'complementary' matrix, formed by the a_{ij} for which $i \notin I$ and $p + 1 \leq j \leq n$. Finally, let $n(I)$ denote the number of ordered pairs

(i,j) such that $i \in I$, $j \notin I$ and $i > j$. Prove *Laplace's formula*

$$\det(A) = \sum_{|I|=p} (-1)^{n(I)} \det(A_I) \det(A_I^c)$$

the sum being over all *subsets* $I \subset \{1, \ldots, n\}$ such that $|I| = p$.

B2) Let R be a commutative ring and M be an R-module. The *dual* R-module M^* is defined to be $\mathrm{Hom}_R(M, R)$. Any R-linear map $T : M \to N$ induces a map $T^* : N^* \to M^*$, in the same way as for vector spaces. If M is free with basis $e_1, \ldots, e_n$, then M^* is also free, with the dual basis $e_1^*, \ldots, e_n^*$, defined by the equations $e_i^*(e_j) = \delta_{ij}$ for every i, j.

Construct, for any R-module M, a canonical ring homomorphism from $\bigwedge^k(M^*)$ to $(\bigwedge^k M)^*$, such that for any R-linear map $T : M \to N$, the diagram

$$\begin{array}{ccc} \bigwedge^k(N^*) & \longrightarrow & (\bigwedge^k N)^* \\ {\scriptstyle \bigwedge^k(T^*)} \downarrow & & \downarrow {\scriptstyle (\bigwedge^k T)^*} \\ \bigwedge^k(M^*) & \longrightarrow & (\bigwedge^k M)^* \end{array}$$

commutes. Show that if M is finitely generated and free, then the map $\bigwedge^k(M^*) \to (\bigwedge^k M)^*$ is an isomorphism.

B3) Suppose k is an integer with $1 \leq k \leq n$. The *Grassmannian* $G(k, n)$ is the set of k-dimensional linear subspaces of $\mathbb{R}^n$. If $k = 1$ then $G(1, n)$ is called *projective space* and is denoted $\mathbb{RP}^{n-1}$. The set $G(k, n)$ can be given the structure of a differentiable manifold in a natural way, although we will not need this. The dimension of the manifold $G(1, n) = \mathbb{RP}^{n-1}$ is $n - 1$; this explains the choice of superscript.

(a) Suppose V is a k-dimensional subspace of $\mathbb{R}^n$, so a point of $G(k, n)$. Choose a basis $v_1, \ldots, v_k$ of column vectors for V, and let $B = (v_1 \cdots v_k)$ be the corresponding $n \times k$ matrix. There are $d := \begin{pmatrix} n \\ k \end{pmatrix}$ k-element subsets I of $\{1, \ldots, n\}$; put them in some fixed order (say lexicographic): $I_1, \ldots, I_d$. For each subset I there is a corresponding $k \times k$ submatrix B_I of B, whose rows are the rows of B in the positions given by the numbers in I. We define a vector $P(B) \in \mathbb{R}^d$ by

$$P(B) := (\det B_{I_1}, \ldots, \det B_{I_d}).$$

Example 1. *In the case of* $G(2, 4)$ *suppose that*

$$B = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}$$

and that $(I_1, I_2, I_3, I_4, I_5, I_6) := (12, 13, 14, 23, 24, 34)$. *Then* $P(B) \in \mathbb{R}^6$ *is given by*

$$P(B) := \left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_4 & b_4 \end{vmatrix}, \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} \right).$$

Prove that if $B' = (v'_1 \ \dots \ v'_k)$ is another basis of W and $P(B') \in \mathbb{R}^d$ is defined by the same method, then $P(B')$ is a scalar multiple of $P(B)$. Deduce that we get a well defined map

$$P : G(k, n) \longrightarrow \mathbb{RP}^{d-1}.$$

(b) Prove that the map P above is injective. P is called the *Plücker embedding*.