Math 606 – Fall 2026 – Harry Tamvakis

PROBLEM SET 1 – Due September 10, 2026

1) Let $\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ and $\Lambda' = \mathbb{Z}\,\omega_1' + \mathbb{Z}\,\omega_2'$ be two lattices in $\mathbb{C}$. Show that $\Lambda = \Lambda'$ if and only if there exists a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

2) Let Λ, Λ' be two lattices in $\mathbb{C}$ and $X = \mathbb{C}/\Lambda$, $X' = \mathbb{C}/\Lambda'$ the corresponding complex tori. Prove that any holomorphic map $f : X \to X'$ is induced by a linear map $g : \mathbb{C} \to \mathbb{C}$ of the form $g(z) = \alpha z + \beta$, where $\alpha \in \mathbb{C}$ is such that $\alpha\Lambda \subset \Lambda'$. The map f is biholomorphic if and only if $\alpha\Lambda = \Lambda'$.

3) Show that every torus $X = \mathbb{C}/\Lambda$ is isomorphic to a torus of the form $X(\tau) := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\tau \in \mathbb{C}$ satisfies $\mathrm{Im}(\tau) > 0$. In addition, give a criterion for when two such tori $X(\tau)$ and $X(\tau')$ are biholomorphic.

Let $\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ be a lattice in $\mathbb{C}$. An *elliptic function* is a meromorphic function f on $\mathbb{C}$ such that $f(z + \omega) = f(z)$ for all $\omega \in \Lambda$ and every $z \in \mathbb{C}$ where f is holomorphic. For $z_0 \in \mathbb{C}$ let

$$P(z_0) := \{z_0 + t_1\omega_1 + t_2\omega_2 \mid 0 \le t_1 \le 1,\, 0 \le t_2 \le 1\}$$

and assume that the boundary $\partial P(z_0)$ does not contain any pole of f.

4) (a) Prove that $\displaystyle\int_{\partial P(z_0)} f(z)\, dz = 0$.

(b) If $\partial P(z_0)$ does not contain any zero of f, prove that the interior of $P(z_0)$ contains equally many zeroes as poles of f, all counted with their multiplicities.

5) Assume that $\partial P(z_0)$ does not contain any zero or pole of f, and let $z_1, \ldots, z_n$ (resp. $w_1, \ldots, w_n$) be the zeroes (resp. poles) of f in the interior of $P(z_0)$, counted with their multiplicities. Prove that $\displaystyle\sum_{i=1}^{n} (z_i - w_i) \in \Lambda$.

[Hint: Consider the integral $\displaystyle\int_{\partial P(z_0)} \frac{z f'(z)}{f(z)}\, dz$.]