Math 660 – Spring 2024 – Harry Tamvakis

PROBLEM SET 1 – Due February 8, 2024

1) Give an example of a continuous function $f : \mathbb{C} \to \mathbb{C}$ with the following three properties: (i) the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist everywhere, (ii) $\dfrac{\partial f}{\partial x}(0) = -i\,\dfrac{\partial f}{\partial y}(0)$, and (iii) f is *not* $\mathbb{C}$-differentiable at 0.

2) (a) Let $\Omega \subset \mathbb{C}$ be an open set, $a \in \Omega$, and $f : \Omega \to \mathbb{C}$ a function such that $\dfrac{\partial f}{\partial x}(a)$ and $\dfrac{\partial f}{\partial y}(a)$ both exist. Prove that $\dfrac{\partial f}{\partial \overline{z}}(a) = 0$ if and only if the differential $d_af : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear, that is, if $d_af(\lambda\zeta) = \lambda\, d_af(\zeta)$, for all $\lambda, \zeta \in \mathbb{C}$. In this case, we have $d_af(\zeta) = f'(a)\zeta$, for all $\zeta \in \mathbb{C}$.

(b) Exercise 47 in the textbook.

3) Let $\Omega \subset \mathbb{C}$ be a connected open set.

(a) Prove that if $f : \Omega \to \mathbb{C}$ has all its first partial derivatives defined at a point in Ω, then at that point we have $\dfrac{\partial}{\partial \overline{z}}\,\overline{f} = \overline{\dfrac{\partial}{\partial z}\,f}$.

(b) If $f_1, \dots, f_n$ are holomorphic functions on Ω such that $|f_1|^2 + \cdots + |f_n|^2$ is constant, prove that each function f_i is constant.

4) (a) Give examples of power series $\sum_{n=0}^{\infty} c_n z^n$ convergent for $|z| < 1$ and divergent for $|z| > 1$ such that

(i) the series converges for all z with $|z| \leq 1$.

(ii) the series diverges for all z with $|z| = 1$.

(iii) the series converges for some, but not all z with $|z| = 1$.

(b) Is the following statement true? Given an arbitrary subset $S \subset T := \{z \in \mathbb{C} \ : \ |z| = 1\}$, there exists a power series $\sum_{n=0}^{\infty} c_n z^n$ which converges for $z \in S$ and diverges for $z \in T \setminus S$.

5) If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is a holomorphic function on the unit disk $D = D(0,1)$, and f maps D one-to-one onto an open set Ω in $\mathbb{C}$, show that the area of Ω is $\pi \cdot \sum_{n=0}^{\infty} n|c_n|^2$. Your answer should include the possibility of this area being infinite.