Math 600 – Fall 2025 – Harry Tamvakis
PROBLEM SET 10 – Due November 20, 2025

A1) Let R be a commutative ring and suppose I is an ideal of R. The *radical* of I, denoted $\mathrm{rad}(I)$ or $\sqrt{I}$, is defined by

$$\mathrm{rad}(I) := \{x \in R \mid x^n \in I \text{ for some } n \geq 1\}.$$

(a) Prove that $\mathrm{rad}(I)$ is an ideal of R which contains I.

(b) Prove that $\mathrm{rad}(\mathrm{rad}(I)) = \mathrm{rad}(I)$.

(c) An ideal I of R is a *radical ideal* if $\mathrm{rad}(I) = I$. Prove that every prime ideal is a radical ideal. Find a necessary and sufficient condition on the natural number n so that $n\mathbb{Z}$ is a radical ideal of $\mathbb{Z}$.

A2) A commutative ring is called a *local* ring when it has but one maximal ideal. Consider the characteristic homomorphism $\chi : \mathbb{Z} \to R$, where R is a local ring, and suppose that the characteristic of R is not 0, i.e. $\mathrm{Ker}\chi \neq (0)$. Show that $\mathrm{Ker}\chi = (q^n)$ for some prime number q and integer n.

A3) Let p be a prime number and let F be the field $\mathbb{Z}/p\mathbb{Z}$. Prove that the polynomial ring $A[x]$ has infinitely many maximal ideals.

A4) Suppose that $f(x)$ is a polynomial of degree 2 with real coefficients. Prove that the quotient ring $\mathbb{R}[x]/(f(x))$ is isomorphic to one of the following: (i) $\mathbb{C}$, (ii) $\mathbb{R} \times \mathbb{R}$, or (iii) the ring $\mathbb{R}[x]/(x^2)$.

B1) Let A be a commutative ring. Consider all infinite sequences $\{a_i\}_{i=0}^{\infty}$, with the $a_i \in A$; add these termwise and multiply by the rule: $\{a_i\}\{b_j\} = \{c_k\}$, where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$. This makes a commutative ring. If x denotes the sequence $(0, 1, 0, 0, \ldots)$, we can identify our sequence ring with the ring of *formal power series* $\sum_{j=0}^{\infty} a_j x^j$ in the indeterminant x with coefficients in A. Denote this ring by $A[[x]]$.

(a) Write f for a formal power series $\sum_{j=0}^{\infty} a_j x^j$ and write $f(0)$ for the constant term, a_0, of f. Prove that the map $\Phi(f) = f(0)$ is a surjective ring homomorphism $A[[x]] \to A$.

(b) Show that if $f(0)$ is a unit of A, then f is a unit of $A[[x]]$.

(c) Deduce that A is a local ring if and only if $A[[x]]$ is local. Prove that the rings A and $A[[x]]$ have the same characteristic.

(d) Prove: if A is an integral domain then so is $A[[x]]$.

(e) Let K be a field. Define $K((x))$ to be the ring of formal power series in X (with coefficients in K) which involve as well a *finite* number of negative powers of x; i.e. of the form $\displaystyle\sum_{j\geq k}^{\infty} a_j x^j$ for some $k \in \mathbb{Z}$ (this is the ring of formal *Laurent series*.) Prove that $K((x))$ is the quotient field of $K[[x]]$.

(f) If A is a domain with quotient field K, it is not necessarily true that $K((x))$ is the quotient field of $A[[x]]$. For example, show that the quotient field of the power series ring $\mathbb{Z}[[x]]$ is *properly* contained in the field of Laurent series $\mathbb{Q}((x))$. [Hint: Consider the series for e^x.]

B2) Let $R := \mathbb{R}[\sin x, \cos x]$ be the ring of *trigonometric polynomials*, generated over the reals by the functions $\sin x$ and $\cos x$.

(a) Show that R consists of all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x) = a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \tag{1}$$

for some real numbers a_0, a_k, b_k and integer $n \geq 0$. Integrating $f(x)$, $f(x)\cos(kx)$, and $f(x)\sin(kx)$ on $[0, 2\pi]$ proves that the coefficients in (1) are uniquely determined by the function f.

(b) The *trigonometric degree* $\deg_{\text{tr}}(f)$ of a non-zero trigonometric polynomial is defined as the largest value of k for which a_k and b_k in (1) are not both zero. Prove that $\deg_{\text{tr}}(fg) = \deg_{\text{tr}}(f) + \deg_{\text{tr}}(g)$.

(c) Show that R is an integral domain, and determine the units in R. Prove that all elements of degree 1 in R are irreducible. Finally, deduce from the identity

$$\sin^2 x = (1 + \cos x)(1 - \cos x)$$

that R is not a unique factorization domain.

B3) If R is any ring, not necessarily commutative, write $J(R)$ for the intersection of all the maximal ideals of R.

(a) Show that $J(R)$ is a two-sided ideal in R. Also show that if $x \in J(R)$, then $1 - x$ is a unit in R.

(b) Suppose that I is a two-sided ideal of R and $J(R/I) = (0)$. Show that $J(R) \subset I$.

(c) Suppose $\phi : R \to S$ is a surjective ring homomorphism. Show that $\phi(J(R)) \subset J(S)$. If, in addition, $\text{Ker}\phi \subset J(R)$, show that $\phi(J(R)) = J(S)$.

(d) Deduce that $J(R/J(R)) = (0)$. Let F be any field, and compute $R/J(R)$ in each of the following two cases: (i) $R = M_n(F)$ and (ii) $R \subset M_n(F)$ the ring of upper triangular matrices in $M_n(F)$. This means, give a cogent description of the ring $R/J(R)$ which does not involve cosets.